Vehicle Placement to Intercept Moving Targets

Shaunak D. Bopardikar Stephen L. Smith Francesco Bullo

***Abstract*— We address optimal placement of vehicles with simple motion, to intercept a mobile target that arrives stochastically on a line segment. The optimality of vehicle placement is measured through a cost function associated with intercepting the target. With a single vehicle, we assume that the target either moves with fixed speed and in a fixed direction or moves to maximize the vertical height or intercept time. We show that each of the corresponding cost functions is convex, has smooth gradient and has a unique minimizing location, and so the optimal vehicle placement is obtained by any standard gradient-based optimization technique. With multiple vehicles, we assume that the target moves with fixed speed and in fixed direction. We present a discrete time partitioning and gradient-based algorithm, and characterize conditions under which the algorithm asymptotically leads the vehicles to a set of critical configurations of the cost function.**

I. INTRODUCTION

Vehicle placement to provide optimal coverage has received lot of recent attention due to potential applications in environmental monitoring, patrolling regions between nations and even in games like soccer. This work addresses vehicle placement to minimize a cost associated with intercepting a mobile target that appears randomly on a segment.

In static environments, vehicle placement problems are analogous to geometric location problems, wherein given a set of static points, the goal is to find supply locations that minimize a cost function of the distance from each point to its nearest supply location (cf. [1]). For a single vehicle, the average distance to a random point, generated according to a probability density function is given by the Weber or the continuous 1–median function, for which there exists a global minimizer as shown in [2], termed as the *median*. For multiple distinct vehicle locations, the expected distance between a random point generated according to a probability density and one of the locations is known in literature as the continuous Weber or the continuous multi-median function, e.g., see [3]. For more than one location, the multi-median function is non-convex, and thus determining locations that minimize the multi-median function is hard in the general case. It is of interest to characterize the set of critical points of the multi-median function. [4] have characterized the set of critical points for the problem of deploying a group of robots in a region to optimize a multi-median cost function. This work has been extended in [5] to enable robots to approximate the function from sensor measurements. More recently, [6] presented a coverage algorithm for vehicles in a river environment.

In mobile target scenarios, the cost for the vehicle is a function of relative locations, speeds and motion constraints considered. For an adversarial target, the optimal vehicle motion is obtained by solving a min-max pursuit-evasion game, in which the target seeks to maximize while the vehicle seeks to minimize a certain cost function. With constraints such as a wall in the playing space or non-zero capture distance, strategies with optimal intercept time have been derived in [7] and in [8].

We consider a line segment on which a mobile target appears via a known spatial probability density and one or multiple vehicles seek to intercept it. The goal is to determine vehicle placements that minimize a cost function associated with the target motion. This work is an extension of [9], where we introduced the placement problem for target motion with fixed speed and in fixed direction, and for a uniform target arrival density. We address single and multiple vehicle scenarios. With a single vehicle, we consider a class of cost functions and establish properties such as convexity, smoothness and the existence and uniqueness of a globally minimizing vehicle location. We show that the cost functions associated with the target moving with fixed speed and in a fixed direction, and with the target seeking to maximize the distance from the segment, fall in the class of cost functions that we have analyzed. The cost function for target motion that maximizes the intercept time is shown to be proportional to the continuous 1–median function. With multiple vehicles and the target moving with fixed speed and in a fixed direction, we first provide an algorithm to partition the line segment among the vehicles and characterize its properties. With the expected intercept time as the cost function, we propose a Lloyd descent algorithm in which every vehicle computes its partition and moves along the gradient of the expected time computed over its partition. We characterize conditions under which the vehicles asymptotically reach a set of critical configurations.

This paper is organized as follows. The problem is formulated in Section II. Single vehicle scenarios are addressed in Section III. The multiple vehicle scenario is analyzed in Section IV. Due to space constraints, complete proofs to all results have been provided in our online technical report [10].

II. PROBLEM STATEMENT

We consider vehicles with simple motion and speed upper bounded by unity. A target arrives at a random position $(x, 0)$ on the segment $G := [0, W] \times \{0\}$, termed the *generator*, via a specified probability density function $\phi : [0, W] \to \mathbb{R}_{>0}$. We assume that the density function ϕ is bounded, i.e.,

This material is based upon work supported in part by ARO-MURI Award W911NF-05-1-0219, ONR Award N00014-07-1-0721 and by the Institute for Collaborative Biotechnologies through the grant DAAD19-03-D-0004 from the U.S. Army Research Office.

S. D. Bopardikar and F. Bullo are with the Center for Control, Dynamical Systems and Computation, University of California at Santa Barbara, Santa Barbara, CA 93106, USA; email: {shaunak,bullo}@engineering.ucsb.edu. S. L. Smith is with the Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, Cambridge MA 02139, USA; email: slsmith@mit.edu

there exists an $M > 0$ such that $\phi(x) \leq M, \forall x \in [0, W]$. The target moves with bounded speed less than that of the vehicles, and is intercepted or captured if a vehicle and the target are at the same point. The goal is to determine vehicle placements that minimize a certain cost function based on the maneuvering abilities of the target. Specifically, we consider the following cases.

A. Single vehicle case

We seek a vehicle location $\mathbf{p} \in \mathbb{R} \times \mathbb{R}_{\geq 0}$ that minimizes

$$C_{\text{exp}}(\mathbf{p}) := \int_0^W C(\mathbf{p}, x)\phi(x)dx, \quad (1)$$

where $C : \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ is an appropriately defined cost of the vehicle position $\mathbf{p}$. In what follows, we seek to minimize the following different cost functions.

(i) *Expected constrained travel time:* We assume that the target arriving at $(x, 0)$ translates in the positive Y-direction with speed $v < 1$. From [9], the cost function C for this formulation is

$$T(\mathbf{p}, x) = \frac{\sqrt{(1 - v^2)(X - x)^2 + Y^2}}{1 - v^2} - \frac{vY}{1 - v^2}, \quad (2)$$

which is the time taken for the vehicle to intercept the constrained target.

(ii) *Expected vertical height:* The cost function C for this formulation is the vertical height $H(\mathbf{p}, x)$ which the target seeks to maximize before being intercepted.

(iii) *Expected intercept time*: The cost function C for this formulation is the intercept time $\mathbf{Ti}(\mathbf{p}, x)$ which the target seeks to maximize.

Explicit formulae for the quantities H and $\mathbf{Ti}$ are derived in Section III-B, and are illustrated in Figure 1.

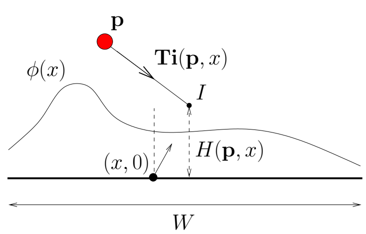

Fig. 1. Intercepting a target that seeks to maximize either the vertical height H or the time $\mathbf{Ti}$ until intercept.

B. Multiple vehicles case

We assume that the target translates in the positive Y-direction with speed $v < 1$. As shown in Figure 2, given $m \geq 2$ vehicles having complete communication, the goal is to determine vehicle locations $\mathbf{p}_i \in [0, W] \times \mathbb{R}_{\geq 0}$, for every $i \in \{1, \ldots, m\}$, that minimize the expected constrained travel time given by

$$T_{\text{exp}}(\mathbf{p}_1, \ldots, \mathbf{p}_m) := \int_0^W \min_{i \in \{1, \ldots, m\}} T(\mathbf{p}_i, x)\phi(x)dx, \quad (3)$$

where $T(\mathbf{p}_i, x)$ is given by Eq. (2).

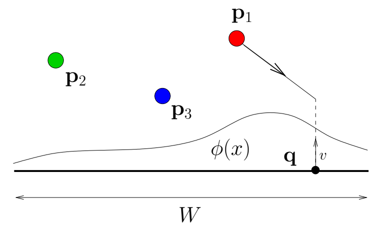

Fig. 2. Intercepting a target having constrained motion.

III. Single vehicle scenarios

We first analyze a class of cost functions. This form will appear in two distinct scenarios, the expected constrained travel time and the expected vertical height.

A. A class of cost functions

We assume that the cost function is given by Eq. (1), where the function C has the form

$$C(X, Y, x) := a\sqrt{b(X - x)^2 + Y^2} - cY, \quad (4)$$

and a, b, and c are positive constants, with $a > c$.

The partial derivatives of $C_{\text{exp}}(X, Y)$ with respect to X and Y are given by

$$\frac{\partial C_{\text{exp}}}{\partial X} = ab\int_0^W \frac{(X - x)\phi(x)}{\sqrt{b(X - x)^2 + Y^2}}dx, \quad (5)$$

$$\frac{\partial C_{\text{exp}}}{\partial Y} = aY\int_0^W \frac{\phi(x)}{\sqrt{b(X - x)^2 + Y^2}}dx - c. \quad (6)$$

Lemma III.1 (Convexity of expected cost) *The expected cost $C_{\text{exp}}(X, Y)$ is a convex function of X and Y over the domain $\mathbb{R} \times \mathbb{R}_{>0}$.*

The proof involves showing that the Hessian matrix of C with respect to X and Y is positive semi-definite.

Lemma III.2 (Existence of Minima) *There exists a vehicle location $(X^*, Y^*) \in]0, W[\times \mathbb{R}_{>0}$ that minimizes C_{exp}.*

Proof: [Sketch] We show that a minimizer cannot lie on the boundary of the region $[0, W] \times \mathbb{R}_{\geq 0}$. The first step is to show that Y^* exists and is finite by showing that $\liminf_{Y \to +\infty} C_{\text{exp}}(X, Y) = +\infty$. To show that $Y^* \neq 0$, we use Eq. (6) to upper bound and obtain $\partial C_{\text{exp}}/\partial Y \leq -c$. Thus, for Y near zero the gradient of C_{exp} points in the negative Y-direction, implying that $Y^* \neq 0$. Finally, we show that the partial derivative of C_{exp} with respect to X is strictly negative at $X = 0$ and is strictly positive at $X = W$, implying that a minima cannot lie at $X = 0$ nor $X = W$. ■

Lemma III.3 (Uniqueness) *There exists a unique vehicle location $(X^*, Y^*) \in]0, W[\times \mathbb{R}_{>0}$ that minimizes C_{exp}.*

Proof: [Sketch] We assume that there be two locations (X_1, Y_1) and (X_2, Y_2) that minimize the expected cost. From Lemma III.1, since the expected cost C_{exp} is convex in X and Y, a convex combination of (X_1, Y_1) and (X_2, Y_2) also minimizes the expected time. Thus, the necessary conditions for minima are satisfied by $(\bar{X}(\alpha), \bar{Y}(\alpha)) := (\alpha X_1 + (1-\alpha)X_2, \alpha Y_1 + (1-\alpha)Y_2)$, for every $\alpha \in [0,1]$. Since these conditions hold for every $\alpha \in [0,1]$, the partial derivatives of the above conditions evaluated at $\alpha = 0$, must equal zero. Thus, upon simplifying, we obtain the following equation.

$$\int_0^W f(X_2, Y_2, x)(Y_2(X_1 - X_2) - (X_2 - x)(Y_1 - Y_2))^2 dx = 0.$$

where f is a strictly positive function. Thus, $Y_2(X_1 - X_2) - (X_2 - x)(Y_1 - Y_2) = 0$, *for every* x. This is feasible only if $X_1 - X_2 = 0$ and $Y_1 - Y_2 = 0$. ■

Lemmas III.1, III.2 and III.3 lead to the following result.

Theorem III.4 (Minimizing expected cost) *From an initial location in $\mathbb{R} \times \mathbb{R}_{>0}$ and by using a gradient optimization technique, the vehicle reaches the unique point that minimizes the expected cost C_{exp}.*

Theorem III.4 answers the problem of minimizing the expected value of T, given by Eq. (2), with $a := 1/(1-v^2)$, $b := (1-v^2)$ and $c := v/(1-v^2)$, and $a > c$. In general, it is difficult to provide closed form expressions for the vehicle location that minimizes the expected time. A special case is described in Remark III.5.

Remark III.5 (Equal speeds) In this case, the optimal placement in the X variable is at the *centroid* of the distribution ϕ, with the optimal Y given by

$$X^* = \int_0^W \phi(x) x dx; \; Y^* = \sqrt{\int_0^W \phi(x)(X^* - x)^2 dx}. \quad \square$$

B. Optimal placement for adversarial target

We consider two types of cost functions that the evader tries to maximize; vertical height and intercept time.

1) Minimizing the expected vertical height: We first present the solution to the differential game with payoff equal to the vertical height. If the evader is slower than the pursuer, then the Appolonius circle is the boundary of the set of all points which the evader can reach without being captured. The following property is stated in [7].

Proposition III.6 (Appolonius circle during pursuit) *If the pursuer and the evader both travel straight toward a point U on the Appolonius circle, then any new such circle, obtained from a pair of simultaneous intermediate positions of the pursuer and the evader, is tangent to the original circle at U, and is contained in the original circle.*

The optimal pursuit strategy (cf. [7]) is to choose its velocity vector such that the line joining the pursuer and the evader remains parallel at all times, while reducing the distance. So for optimal placement, it suffices to determine the optimal evader strategy. Algorithm 1 summarizes our evader strategy, shown in Figure 3.

The following result is immediate from Proposition III.6.

Algorithm 1: **Move towards top-most**

Assumes: Pursuer at (X, Y). Evader at $(x, 0)$.

1: Compute center and radius of the Appolonius circle:

$$O := (O_x, O_y) = \Big(\frac{x - v^2 X}{1 - v^2}, \frac{-v^2 Y}{1 - v^2}\Big),$$
$$R := \frac{v}{1 - v^2}\sqrt{(X - x)^2 + Y^2}.$$

2: Move towards the point $(O_x, O_y + R)$ with speed v.

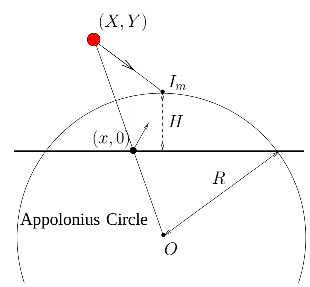

Fig. 3. Move towards top-most strategy for the evader.

Lemma III.7 (Move towards top-most is optimal) *The strategy* move towards top-most *is the evader's optimal strategy and the resulting optimal vertical height of the intercept point is*

$$H(X, Y, x) = \frac{v}{1 - v^2}\sqrt{(X - x)^2 + Y^2} - \frac{v^2 Y}{1 - v^2}.$$

Comparing the expression for H given by Lemma III.7 with the definition of C in Eq. (4), we have $a := v/(1-v^2)$, $b := 1$ and $c := v^2/(1-v^2)$, and $a > c$ since $v < 1$. Thus, by applying Theorem III.4, we obtain the following result.

Theorem III.8 (Minimizing expected height) *From an initial location in $\mathbb{R} \times \mathbb{R}_{>0}$, by using a gradient optimization technique, the vehicle reaches the unique point that minimizes the expected height H_{exp}.*

2) Minimizing the expected intercept time: This formulation assumes that the evader is constrained to remain above or on the X-axis. Thus, the underlying differential game in this set up is the classic *wall pursuit* game, proposed and solved in [7]. We present the main result for completeness.

Lemma III.9 (Wall pursuit game) *The evader strategy that maximizes the intercept time is to move towards the furthest point of the Appolonius circle on the X-axis.*

This optimal evader strategy is illustrated in Figure 4.

Now, given a convex region $\mathcal{Q} \subset \mathbb{R}$ and a density function $\psi : \mathcal{Q} \to \mathbb{R}_{\geq 0}$, the *median* (cf. [2]) p_{med} is the unique global minimizer of

$$\int_{\mathcal{Q}} |p - z| \psi(z) dz.$$

We now present the main result of this section.

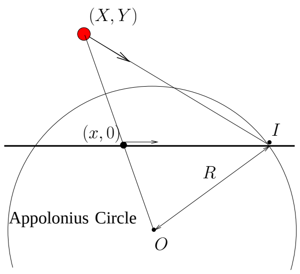

Fig. 4. Illustrating Lemma III.9.

Theorem III.10 (Optimal point is the median) *The median point of the region* $[0, W] \times \{0\}$ *with the density function* ϕ *uniquely minimizes the expected intercept time.*

IV. The case of multiple vehicles

We now address the multi-vehicle placement problem.

A. Dominance region partition

We introduce a generator partitioning procedure by defining dominance regions between each pair of vehicles.

Definition IV.1 (Pairwise dominance region) *For* $i, j \in \{1, \dots, m\}$*, the pairwise dominance region* $U_{ij} \subseteq [0, W]$ *of* $\mathbf{p}_i$ *with respect to* $\mathbf{p}_j$ *is the set of target locations for which vehicle* $\mathbf{p}_i$ *takes* lesser *time to intercept the target than* $\mathbf{p}_j$*:*

$$U_{ij} := \{x \in [0, W] \,|\, T(\mathbf{p}_i, x) \leq T(\mathbf{p}_j, x)\}.$$

A procedure to determine U_{ij} is summarized in Algorithm 2. The central idea is to use the relation $T(\mathbf{p}_i, x) \leq T(\mathbf{p}_j, x)$ to characterize the set of all target locations $(x, 0)$ as a function of (X_i, Y_i) and (X_j, Y_j) for which $\mathbf{p}_i$ can reach a target sooner than $\mathbf{p}_j$. One case of Algorithm 2 is illustrated in Figure 5.

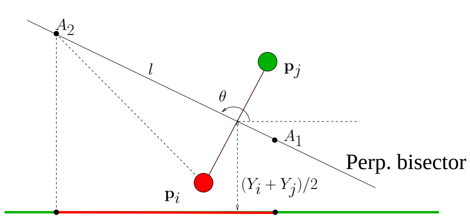

Fig. 5. To determine pairwise dominance regions.

The following property is established for Algorithm 2.

Proposition IV.2 (Pairwise dominance region) *Given distinct locations* $\mathbf{p}_i = (X_i, Y_i)$, $\mathbf{p}_j = (X_j, Y_j)$*, if a target arrives at* $(x, 0)$*, where* $x \in U_{ij}$ *generated by Algorithm 2, then* $T(\mathbf{p}_i, x) \leq T(\mathbf{p}_j, x)$*.*

Similar to pairwise dominance regions, we introduce the concept of *dominance region* $\mathcal{V}_i \in \mathcal{P}([0, W])$ for the ith vehicle, for every $i \in \{1, \dots, m\}$, which is the set of X-coordinates of target locations for which $\mathbf{p}_i$ takes the *minimum* time to intercept of all vehicles.

Assuming complete communication between vehicles, Algorithm 2 is extended to determine the dominance region

Algorithm 2: Pairwise dominance region

Assumes: Distinct $\mathbf{p}_i = (X_i, Y_i)$, $\mathbf{p}_j = (X_j, Y_j)$.

if $Y_i = Y_j$, **then**

  $U_{ij} := \begin{cases} [0, (X_i + X_j)/2], & \text{if } X_i < X_j \\ [(X_i + X_j)/2, W], & \text{if } X_i > X_j \end{cases}$

else

  $\theta := \arctan_2(Y_j - Y_i, X_j - X_i) + \pi/2$

  $\ell_{1,2}$:= two roots of $0 = 4(\sin^2(\theta) - v^2)\ell^2$
   $+4(Y_i + Y_j)\sin(\theta)\ell + (Y_i + Y_j)^2 - v^2\|\mathbf{p}_i - \mathbf{p}_j\|^2$

  $y_{1,2} := (Y_i + Y_j)/2 + \sin(\theta)\ell_{1,2}$

  if $y_1 > 0$ and $y_2 > 0$ **then**

    $x_{1,2} := (X_i + X_j)/2 + \cos(\theta)\ell_{1,2}$

    $U_{ij} := \begin{cases} [0, W] \cap [x_1, x_2], & \text{if } Y_i < Y_j \\ [0, W] \backslash]x_1, x_2[, & \text{if } Y_i \geq Y_j. \end{cases}$

  else

    k := index in $\{1, 2\}$ for which $y_k > 0$

    $x := (X_i + X_j)/2 + \ell_k \cos(\theta)$

    $U_{ij} := \begin{cases} [0, W] \cap]-\infty, x], & \text{if } X_i < X_j \\ [0, W] \cap [x, +\infty[, & \text{if } X_i \geq X_j \end{cases}$

Algorithm 3: Dominance region

Assumes: Distinct locations $\{\mathbf{p}_1, \dots, \mathbf{p}_m\}$.

foreach vehicle $j \in \{1, \dots, m\} \setminus \{i\}$, **do**

  Determine U_{ij} using Algorithm 2.

$\mathcal{V}_i = \bigcap_{j=1,\dots,m, j\neq i} U_{ij}$.

for a vehicle by (i) determining pairwise dominance regions between vehicles and, (ii) taking intersection of all pairwise dominance regions, as presented in Algorithm 3.

The next result for Algorithm 3 follows due to disjoint interiors of dominance regions, and due to Proposition IV.2.

Proposition IV.3 (Optimality of dominance regions) *Given distinct vehicle positions and a target arrival,*

(i) *the dominance regions generated by Algorithm 3 form a partition of the generator.*

(ii) *The time taken to reach the target is minimized by the vehicle whose dominance region contains the target arrival location.*

It is possible for the dominance region of a vehicle to be empty. For instance, when one of the vehicles is very far from the generating line (cf. first part of Figure 8). However, one condition under which every vehicle has a non-empty dominance region is when all vehicles have the same Y-coordinate. For a general set of locations, Figure 6 shows a dominance region partition with three vehicles.

Let $\mathcal{E} := [0, W] \times \mathbb{R}_{\geq 0}$, $\mathcal{P}([0, W])$ be the set of all subsets of $[0, W]$, $\mathcal{B}(r)$ be the closed ball of radius r around the

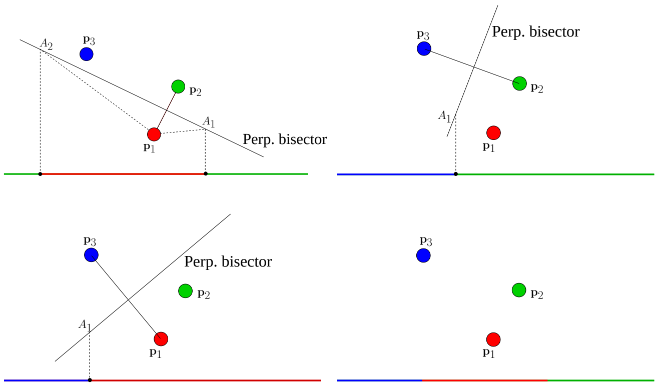

Fig. 6. Dominance region partition induced by three vehicles.

origin and $+$ denote the Minkowski sum of two sets. The domain of a set-valued map $F : X \rightrightarrows Z$ is the set of all $\mathbf{q} \in X$ such that $F(\mathbf{q}) \neq \emptyset$. F is said to be upper (resp. lower) semi-continuous in its domain if, for every $\mathbf{q}$ in its domain and for every $\epsilon > 0$, there exists a $\delta > 0$ such that for every $\mathbf{z} \in \mathbf{q} + \mathcal{B}(\delta)$, $F(\mathbf{z}) \subset F(\mathbf{q}) + B(\epsilon)$ (resp. $F(\mathbf{q}) \subset F(\mathbf{z}) + B(\epsilon)$). F is continuous in its domain if it is both upper and lower semi-continuous.

The pairwise dominance region between $\mathbf{p}_i$ and $\mathbf{p}_j$ is a set valued function $U_{ij} : \mathcal{E}^2 \setminus \mathcal{S}_{ij} \rightrightarrows \mathcal{P}([0,W])^2$, where $\mathcal{S}_{ij} \subset \mathcal{E}^2$ is the set of coincident locations for $\mathbf{p}_i$ and $\mathbf{p}_j$. Similarly, the dominance region partition for vehicle i is a set-valued map $\mathcal{V}_i : \mathcal{E}^m \setminus \mathcal{S}_i \rightrightarrows \mathcal{P}([0,W])^{2(m-1)}$, where $\mathcal{S} \subset \mathcal{E}^m$ is the set of vehicle locations in which at least one other vehicle is coincident with $\mathbf{p}_i$.

Proposition IV.4 (Continuity of dominance regions)

(i) *For every distinct i and j in the set $\{1,\dots,m\}$, the set valued map U_{ij} is continuous in $\mathcal{E}^2 \setminus \mathcal{S}_{ij}$.*

(ii) *For each vehicle $i \in \{1,\dots,m\}$, the set valued map $\mathcal{V}_i$ is continuous on its domain.*

B. Minimizing the expected constrained travel time

For distinct vehicle locations, Eq. (3) can be written as

$$T_{\text{exp}}(\mathbf{p}_1,\dots,\mathbf{p}_m) = \sum_{i=1}^{m} \int_{\mathcal{V}_i} T(\mathbf{p}_i, x)\phi(x)dx, \qquad (7)$$

where $\mathcal{V}_i$ is the dominance region of the ith vehicle. The gradient of T_{exp} is computed using the following formula.

Lemma IV.5 (Gradient computation) *For all vehicle configurations such that no two vehicles are at coincident locations, the gradient of the expected time with respect to vehicle location $\mathbf{p}_i$ is*

$$\frac{\partial T_{\text{exp}}}{\partial \mathbf{p}_i} = \int_{\mathcal{V}_i} \frac{\partial T}{\partial \mathbf{p}_i}(\mathbf{p}_i, x)\phi(x)dx.$$

Akin to similar results in [4], [11], [12], the proof involves writing the gradient as a sum of two contributing terms. The first is the final expression, while the second is a number of boundary terms which cancel out due to continuity of T at the boundaries of dominance regions.

For $\mathbf{z} \in \mathbb{R}^2$, define the function $\text{sat} : \mathbb{R}^2 \to \mathbb{R}^2$ denote the saturation function, i.e., if $\|\mathbf{z}\| \le 1$, then $\text{sat}(\mathbf{z}) = \mathbf{z}$,otherwise, $\text{sat}(\mathbf{z}) = \mathbf{z}/\|\mathbf{z}\|$. Inspired by the established Lloyd algorithm (cf. [12]), we present a discrete-time descent approach in Algorithm 4.

Algorithm 4: **Lloyd descent for vehicle i**

Assumes: Distinct locations $\{\mathbf{p}_1,\dots,\mathbf{p}_m\} \in \mathcal{E}^m$

1: **foreach** time $t \in \mathbb{N}$ **do**

2:   Compute $\mathcal{V}_i(t)$ by Algorithm 3 as a function of $\{\mathbf{p}_1(t),\dots,\mathbf{p}_m(t)\}$

3:   **if** $\mathcal{V}_i(t)$ is empty, **then**

4:     Move in unit time to $(X_i, Y_i - \min\{1, Y_i\})$

5:   **else**

6:     For $\tau \in [t, t+1]$, move according to

$$\dot{\mathbf{p}}_i(\tau) = -\,\text{sat}\left(\int_{\mathcal{V}_i(t)} \frac{\partial}{\partial \mathbf{p}_i} T(\mathbf{p}_i(\tau), x)\phi(x)dx\right)$$

We define the following vehicle configuration.

Definition IV.6 (Critical configuration) *A set of locations $\{\mathbf{p}_1,\dots,\mathbf{p}_m\}$ is a* critical dominance region configuration *if, for all $i \in \{1,\dots,m\}$,*

$$\mathbf{p}_i = \underset{\mathbf{z}\in\mathcal{E}}{\text{argmin}} \int_{\mathcal{V}_i} T(\mathbf{z}, x)\phi(x)dx,$$

where $\{\mathcal{V}_1,\dots,\mathcal{V}_m\}$ is the dominance region partition induced by $\{\mathbf{p}_1,\dots,\mathbf{p}_m\}$.

We now state the main result of this section.

Theorem IV.7 (Convergence of Lloyd descent) *Let $\gamma : \mathbb{N} \to \mathbb{R}^{2m}$ be the evolution of the m vehicles according to Algorithm 4 and assume that no two vehicle locations become coincident in finite time or asymptotically. The following statements hold:*

(i) *the expected travel time $t \mapsto T_{\text{exp}}(\gamma(t))$ is a non-increasing function of time;*

(ii) *if the dominance region $\mathcal{V}_i$ of any vehicle i is empty at some time, then $\mathcal{V}_i$ will be non-empty within a finite time; and*

(iii) *if there exists a time t such that every dominance region is non-empty for all times subsequent to t, then the vehicle locations converge to the set of critical dominance region configurations.*

Proof: [Sketch] Statement (i) follows because in steps **2:**, **4:** and **6:** of Algorithm 4, T_{exp} is non-increasing. Statement (ii) follows from the fact that whenever $\mathcal{V}_i = \emptyset$ for vehicle i, due to step **4:**, vehicle i reaches the generator after finite time and therefore has a non-empty $\mathcal{V}_i$.

For statement (iii), let $\mathcal{A} : \mathcal{X} \times \mathcal{P}([0,W]) \to \mathcal{X}$, be the flow map of the differential equation at step **6:** from time t to time $t+1$. Consider the discrete-time dynamical system given by the tuple $(\mathcal{X}, \mathcal{X}_0, \mathcal{A})$, where $\mathcal{X} = \mathcal{E}^m$ and $\mathcal{X}_0 \in \mathcal{E}^m$ is the

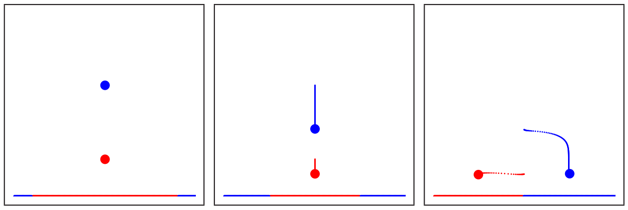

Fig. 7. Algorithm 4 for uniform arrival density. The vehicles first tend to an critical dominance region configuration (center figure). A perturbation to their positions makes them move to a stable configuration (third figure).

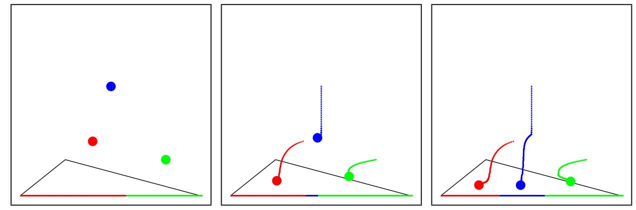

Fig. 8. Algorithm 4 for non-uniform arrival density (black line). Initially, the blue vehicle has no dominance region. The vehicles tend to a stable configuration.

set of initial vehicle positions. We now apply the discrete-time LaSalle Invariance Principle (Theorem 1.19 in [12]), for which we verify the four assumptions as follows.

1. Existence of a positively invariant set: The set $\mathcal{X}$ is invariant since every vehicle remains in $\mathcal{E}$ at all times.

2. Existence of a non-increasing function along $\mathcal{A}$: T_{exp} is non-increasing along $\mathcal{A}$, by statement (i) of this theorem.

3. Boundedness of all evolutions of $(\mathcal{X}, \mathcal{X}_0, \mathcal{A})$: If at least one vehicle, say vehicle k moves so that Y_k grows unbounded, then after finite time, the dominance region $\mathcal{V}_k$ becomes empty, contradicting the assumption of statement (iii) of this theorem. If the Y-coordinates of all vehicles grow unbounded, then T_{exp} grows unbounded, contradicting part (i) of this theorem.

4. Continuity of T_{exp} and $\mathcal{A}$: Continuity of T_{exp} follows from Eq.s (2) and (7). $\mathcal{A}$ is continuous as the integrand is continuous with respect to vehicle locations, and so is the region of integration $\mathcal{V}_i$ (cf. Proposition IV.4).

By LaSalle Invariance Principle, the evolutions of $(\mathcal{X}, \mathcal{X}_0, \mathcal{A})$ converge to a set of the form $T_{\text{exp}}^{-1}(\kappa) \cap \mathcal{M}$, where κ is a real constant and $\mathcal{M}$ is the largest positively invariant set in $\{x \in \mathcal{X} \,|\, T_{\text{exp}}(\mathcal{A}(x)) = T_{\text{exp}}(x)\}$. If a set of vehicle positions is not critical, then T_{exp} strictly decreases under the action $\mathcal{A}$, and therefore the set is not contained in a set of $T_{\text{exp}}^{-1}(\kappa) \cap \mathcal{M}$. Thus, the vehicle locations converge to the set of critical dominance region configurations. ■

C. Simulations

We now present some simulations of Algorithm 4.

[Examples of critical locations] We consider two vehicles, and a uniform probability density of target arrival, i.e., $\phi(x) = 1/W$. From initial locations such as in the leftmost of Figure 7 wherein both vehicles having the same X-coordinate of $W/2$, but different Y-coordinates, the vehicles asymptotically approach a set of locations shown in the center figure. However, a small perturbation to the positions leads the vehicles to positions in the rightmost figure. From most initial conditions, the vehicles converged to a critical configuration as in the rightmost figure.

[Non-uniform probability distribution] We consider three vehicles and the arrival probability density function,

$$\phi(x) = \begin{cases} \frac{8}{W^2} x, & \text{if } x \in [0, W/4], \\ \frac{2}{W} - \frac{8}{3W^2}(x - \frac{W}{4}), & \text{if } x \in]W/4, W]. \end{cases}$$

From most initial conditions, the vehicles converged to a critical configuration as in right-most part of Figure 8.

V. Conclusions and Future Directions

We addressed the problem of optimally placing vehicles having simple motion in order to intercept a mobile target that arrives stochastically on a line segment. For a single vehicle, we determined unique optimal placements when target motion was either constrained, i.e., with fixed speed and direction, or adversarial. For the multiple vehicle scenario and with constrained motion targets, we characterized conditions under which a partition and gradient based algorithm that takes the vehicles asymptotically to a set of critical locations of the cost function.

A natural future direction is to consider adversarial targets in the multiple vehicles scenario. Another direction is to consider stochasticity in the motion of the target.

Acknowledgments

The authors would like to thank Prof. Jorge Cortés from University of California, San Diego for insightful discussions that lead to the proof of Lemma III.3.

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