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NONSMOOTH COORDINATION AND GEOMETRIC OPTIMIZATION VIA DISTRIBUTED DYNAMICAL SYSTEMS*

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Abstract. Emerging applications for networked and cooperative robots motivate the study of motion coordination for groups of agents. For example, it is envisioned that groups of agents will perform a variety of useful tasks including surveillance, exploration, and environmental monitoring. This paper deals with basic interactions among mobile agents such as "move away from the closest other agent" or "move toward the furthest vertex of your own Voronoi polygon." These simple interactions amount to distributed dynamical systems because their implementation requires only minimal information about neighboring agents. We characterize the close relationship between these distributed dynamical systems and the disk-covering and sphere-packing cost functions from geometric optimization. Our main results are: (i) we characterize the smoothness properties of these geometric cost functions, (ii) we show that the interaction laws are variations of the nonsmooth gradient of the cost functions, and (iii) we establish various asymptotic convergence properties of the laws. The technical approach relies on concepts from computational geometry, nonsmooth analysis, and nonsmooth stability theory.

Key words. distributed dynamical systems, coordination and cooperative control, geometric optimization, disk-covering problem, sphere-packing problem, nonsmooth analysis, Voronoi partitions

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1. Introduction.

Consider n points $(p_1, \dots, p_n)$ moving inside a convex polygon Q according to one of the following interaction laws: (i) each point moves away from the closest other point or polygon boundary, (ii) each point moves toward the furthest vertex of its own Voronoi polygon, or (iii) each point moves toward a geometric center (circumcenter, incenter, centroid, etc.) of its own Voronoi polygon. Recall that the Voronoi polygon of the ith point is the closed set of points $q \in Q$ closer to p_i than to any other p_j.

These and related interaction laws give rise to strikingly simple dynamical systems whose behavior remains largely unknown. What are the critical points of such dynamical systems? What is their asymptotic behavior? Are these systems optimizing any aggregate function? In what way do these local interactions give rise to distributed systems? Does any biological ensemble evolve according to these behaviors and are they of any engineering use in coordination problems? These are the questions that motivate this paper.

Coordination in robotics, control, and biology. Coordination problems are becoming increasingly important in numerous engineering disciplines. The deployment of large groups of autonomous vehicles is rapidly becoming possible because of technological advances in computing, networking, and miniaturization of electromechanical systems. These future multiple-vehicle networks will coordinate their actions to perform challenging spatially distributed tasks (e.g., search and recovery

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operations, exploration, surveillance, and environmental monitoring for pollution detection and estimation). This future scenario motivates the study of algorithms for autonomy, adaptation, and coordination of multiple-vehicle networks. It is also important to take into careful consideration all constraints on the behavior of the multiple-vehicle network. Coordination algorithms need to be adaptive and distributed in order for the resulting closed-loop network to be scalable, to comply with bandwidth limitations, to tolerate failures, and to adapt to changing environments, topologies, and sensing tasks. The interaction laws introduced above have these properties and, remarkably, they optimize network-wide performance measures for meaningful spatially distributed tasks.

FIG. 1.1. *Territories of male* Tilapia mossambica*. Some species of fish exhibit territorial behavior by globally partitioning the environment into nonoverlapping zones. In this top-view photograph, each territory is a pit dug in the sand by its occupant. The rims of the pits form a pattern of polygons known as a Voronoi partition. The breeding males are the black fish, which range in size from about* 15 *cm to* 20 *cm. The gray fish are the females, juveniles, and nonbreeding males. Photograph reprinted from [3] with permission from Elsevier.*

Coordinated group motions are also a widespread phenomenon in biological systems. Some species of fish spend their lives in schools as a defense mechanism against predators. Others travel as swarms in order to protect an area that they have claimed as their own. Flocks of birds are able to travel in large groups and act as one unit. Other animals exhibit remarkable collective behaviors when foraging and selecting food. Certain foraging behaviors include individual animals partitioning their environment in nonoverlapping individual zones whereas other species develop overlapping team areas. An example environment partition by fish is given in Figure 1.1. These biological network systems possess extraordinary dynamic capabilities without apparently following a group leader. Yet these complex coordinated behaviors emerge while each individual has no global knowledge of the network state and can only plan its motion according to the observation of its closest neighbors.

Facility location, nonsmooth stability analysis, and cooperative control. To analyze the interaction laws introduced above we rely on concepts and methods from various disciplines. Facility location problems play a prominent role in the field of geometric optimization [1, 4]. Facility location pervades a broad spectrum of scientific and technological areas, including resource allocation (where to place mailboxes in a city or cache servers on the internet), quantization and information theory, mesh and grid optimization methods, clustering analysis, data compression, and statistical pattern recognition. Smooth multicenter functions for so-called centroidal Voronoi configurations and smooth distributed dynamical systems are presented in [15, 18].

Multicenter functions are studied in resource allocation problems [17, 45] and in quantization theory [22, 29]. The role of Voronoi tessellations and computational geometry in facility location is discussed in [34, 39].

The notion and computational properties of the generalized gradient are thoroughly studied in nonsmooth analysis [11]. In particular, tools for establishing stability and convergence properties of nonsmooth dynamical systems are presented in [2, 19, 42, 36]. A survey of nonsmooth analysis and stability is given in [13] and some applications to robotics are discussed in [9].

With regards to distributed motion coordination algorithms, much progress has been made on collective pattern formation and flocking [46, 49, 40, 47], formation control [26, 23, 48], motion camouflage [24], self-assembly [25], swarm aggregation [20], rendezvous [28], cyclic pursuit [6, 31, 32], motion planning with collision avoidance [30, 37, 41], and cooperative boundary estimation [10, 8, 44]. Two recent surveys on consensus algorithms are [35, 38]. Since the publication of our original paper [14], others works have used related tools and concepts; examples include [21] on optimal network configurations for spatial estimation, [27] on nonuniform coverage, [33] on a broad collection of deployment laws, and [12] on nonsmooth stability for finite-time consensus. Finally, a recent text on distributed control and coordination is [7].

Statement of contributions. The aim of this work is to design distributed coordination algorithms for dynamic networks as well as to provide formal verifications of their asymptotic correctness. A key aspect of our treatment is the inherent complexity of studying networks whose communication topology changes along the system evolution, as opposed to networks with fixed communication topologies.

We consider two facility location functions from geometric optimization that characterize coverage performance criteria. A collection of locations provides optimal service to a domain of interest if (i) it minimizes the largest distance from any point in the domain to one of the locations, or (ii) it maximizes the minimum distance between any two locations. In other words, if $P = (p_1, \dots, p_n)$ are n points inside a convex polygon Q, we extremize the *multicenter functions*

$$\max_{q \in Q} \Big\{ \min_{i \in \{1,\dots,n\}} d(q, p_i) \Big\}, \quad \min_{i \neq j \in \{1,\dots,n\}} \big\{ \tfrac{1}{2} d(p_i, p_j), d(p_i, \partial Q) \big\},$$

where $d(p, q)$ and $d(p, \partial Q)$ are the Euclidean distances between p and q, and between p and the boundary of Q, respectively. (The role of the $\frac{1}{2}$ factor will become clear later.) We study the differentiable properties of these functions via nonsmooth analysis. We show the functions are globally Lipschitz and regular, we compute their generalized gradients, and we characterize their critical points. Under certain technical conditions, we show that the local minima of the first multicenter function are so-called circumcenter Voronoi configurations, and that these critical points correspond to the solutions of disk-covering problems. Similarly, under analogous technical conditions, we show that the local maxima of the second multicenter function are so-called incenter Voronoi configurations, and that these critical points correspond to the solutions of sphere-packing problems.

Next, we aim to design distributed algorithms that extremize the multicenter functions. Roughly speaking, by distributed we mean that the motion of each point depends at most on the location of its own Voronoi neighbors. We study the generalized gradient flows induced by the multicenter functions using nonsmooth stability analysis. Although these dynamical systems possess some convergence properties, they are not amenable to distributed implementations. Next, drawing connections with quantization theory, we consider two dynamical systems associated to each multicenter function. First, we consider a novel strategy based on the generalized gradient of the 1-center functions of each point, and second, we consider a geometric centering strategy similar to the well-known Lloyd algorithm [22, 29].

Remarkably, these strategies arising from the nonsmooth gradient information have natural geometric interpretations and are indeed the local interaction rule described earlier. For the first (respectively, second) multicenter function, the first strategy corresponds to the interaction law "move toward the furthest vertex of own Voronoi polygon" (respectively, "move away from the closest other point or polygon boundary"), and the second strategy corresponds to the interaction law "move toward circumcenter of own Voronoi polygon" (respectively, "move toward incenter of own Voronoi polygon"). We prove the uniqueness of the solutions of the resulting distributed dynamical systems and we analyze their asymptotic behavior using nonsmooth stability analysis, showing that the active point will approach the corresponding centers of their own Voronoi cells.

Two of our results are related to well-known conjectures in the locational optimization literature [17, 45]: (i) that the first multicenter problem is equivalent to a disk-covering problem (how to cover a region with possibly overlapping disks of equal minimum radius), and (ii) that the generalized Lloyd strategy "move toward circumcenter of own Voronoi polygon" converges to the set of circumcenter Voronoi configurations.

Organization. The paper is organized as follows. Section 2 provides the preliminary concepts on Voronoi partitions, nonsmooth analysis, stability analysis, and gradient flows, and introduces the multicenter problems. Section 3 presents a complete treatment on the functions analysis and algorithm design for the 1-center problems. Section 4 discusses the differentiable properties and the critical points of the multicenter functions. Section 5 introduces a number of dynamical systems (smooth and nonsmooth, distributed and non-distributed) and analyzes their asymptotic correctness. For this revised version, we decided to omit the proofs of some statements and we refer to the original work [14] for the complete treatment.

2. Preliminaries and problem setup. Let $N \in \mathbb{N}$. We denote by $\|\cdot\|$ the Euclidean distance function on $\mathbb{R}^N$ and by $v \cdot w$ the scalar product of the vectors $v, w \in \mathbb{R}^N$. Let $\operatorname{vrs}(v)$ denote the unit vector in the direction of $0 \neq v \in \mathbb{R}^N$, i.e., $\operatorname{vrs}(v) = v/\|v\|$. Given a set S in $\mathbb{R}^N$, we denote its convex hull by $\operatorname{co}(S)$ and its interior set by $\operatorname{int}(S)$. If S is a convex set in $\mathbb{R}^N$, let $\operatorname{proj}_S : \mathbb{R}^N \to S$ denote the orthogonal projection onto S and let $\mathrm{D}_S : \mathbb{R}^N \to \mathbb{R}$ denote the distance function to S. For $R > 0$, let $\overline{B}_N(p, R) = \{q \in \mathbb{R}^N \mid \|p - q\| \leq R\}$ and $B_N(p, R) = \operatorname{int}(\overline{B}_N(p, R))$. A set $\{v_1, \dots, v_M\}$ of vectors in $\mathbb{R}^N$ *positively spans* $\mathbb{R}^N$ if any $w \in \mathbb{R}^N$ can be written as $w = \sum_{l=1}^{M} a_l v_l$, with $a_l \geq 0$, $l \in \{1, \dots, M\}$. The following simple lemma, e.g., see [9], characterizes this situation.

LEMMA 2.1 (Positive span). *Given a set $\{v_1, \dots, v_M\}$ of $M > N$ arbitrary vectors in $\mathbb{R}^N$, the following statements are equivalent:*

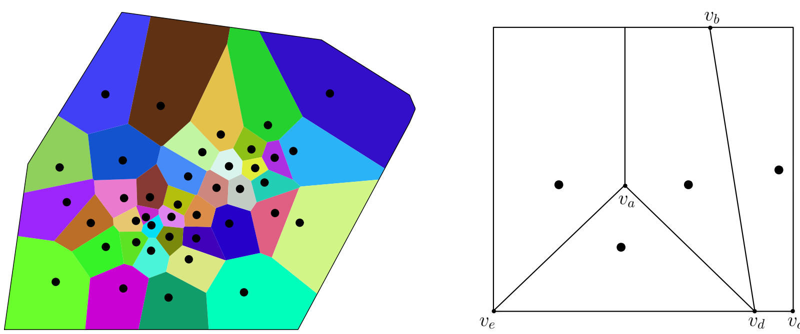
(i) $\{v_1, \dots, v_M\}$ *positively spans* $\mathbb{R}^N$*;*
(ii) $0 \in \operatorname{int}(\operatorname{co}\{v_1, \dots, v_M\})$*; and*
(iii) *for each* $w \in \mathbb{R}^N$*, there exists* v_i *such that* $w \cdot v_i > 0$.

Let Q be a convex simple polygon in $\mathbb{R}^2$. We denote by $\operatorname{Ed}(Q) = \{e_1, \dots, e_L\}$ and $\operatorname{Ve}(Q) = \{v_1, \dots, v_L\}$ the set of edges and vertexes of Q, respectively. Let $P = (p_1, \dots, p_n) \in Q^n \subset (\mathbb{R}^2)^n$ denote the location of n points (which we will call generators) in the space Q. Let $\pi_i : Q^n \to Q$ be the canonical projection onto the ith factor, $\pi_i(p_1, \dots, p_n) = p_i$. Note that this mapping is surjective, continuous, and open (the latter meaning that open sets of Q^n are mapped onto open sets of Q).

2.1. Voronoi partitions. We present here some relevant concepts on Voronoi diagrams and refer the reader to [16, 34] for comprehensive treatments. A *partition* of Q is a collection of n polygons $\mathcal{W} = \{W_1, \dots, W_n\}$ with disjoint interiors whose union is Q. Of course, more general types of partitions could be considered (as, for instance, continuous deformations of the previous ones), but these will be sufficient for our purposes. The *Voronoi partition* $\mathcal{V}(P) = (V_1(P), \dots, V_n(P))$ of Q generated by the points $(p_1, \dots, p_n)$ is defined by

$$V_i(P) = \{q \in Q \mid \|q - p_i\| \leq \|q - p_j\| \text{ for all } j \neq i\},$$

see Figure 2.1(i) for an illustration. For simplicity, we shall refer to $V_i(P)$ as V_i. Since Q is a convex polygon, the boundary of each V_i is the union of a finite number of segments. If V_i and V_j share an edge, i.e., $V_i \cap V_j$ is neither empty nor a singleton, then p_i is called a *(Voronoi) neighbor* of p_j (and vice versa). All Voronoi neighboring relations are encoded in the mapping $\mathcal{N} : Q^n \times \{1, \dots, n\} \to 2^{\{1,\dots,n\}}$, where $\mathcal{N}(P, i)$ is the set of indexes of the Voronoi neighbors of p_i. Of course, $j \in \mathcal{N}(P, i)$ if and only if $i \in \mathcal{N}(P, j)$. We will often omit P and instead write $\mathcal{N}(i)$.



(i) (ii)

Fig. 2.1. (i) *shows a Voronoi partition of a polygon.* (ii) *illustrates the notions of degenerate and nondegenerate vertexes. Vertexes v_a, v_b, and v_c are nondegenerate vertexes of type* (a), (b), *and* (c), *respectively. Vertexes v_d and v_e are degenerate.*

For $P \in Q^n$, the vertexes of the Voronoi partition $\mathcal{V}(P)$ are classified as follows: the vertex v is

- *of type* (a) if it is the center of the circle passing through three generators (say, p_i, p_j, and p_k),
- *of type* (b) if it is the intersection between an edge of Q and the bisector determined by two generators (say, e, p_i, and p_j), and
- *of type* (c) if it is a vertex of Q, i.e., it is determined by two edges of Q and by the generator of a cell containing it (say, e, f, and p_i).

Correspondingly, we shall write $v(i, j, k)$, $v(e, i, j)$, and $v(e, f, i)$, respectively, whenever we are interested in making explicit the elements defining the vertex v. The vertex $v \in \text{Ve}(V_i(P))$ is said to be *nondegenerate* if it is determined by exactly three elements (e.g., as described above, three generators, or an edge and two generators, or two edges and one generator), otherwise it is said to be *degenerate*. Further, the configuration P is said to be *nondegenerate at the ith generator* if all vertexes $v \in \text{Ve}(V_i(P))$

are nondegenerate, otherwise P is *degenerate at the ith generator*. Finally, a configuration P is said to be *nondegenerate* if all its vertexes are nondegenerate, otherwise it is said to be *degenerate*. These concepts are illustrated in Figure 2.1(ii).

For $P \in Q^n$, the edges of the Voronoi partition $\mathcal{V}(P)$ are classified as follows: the edge e is

- *of type* (a) if it is a segment of the bisector determined by two generators (say, p_i, p_j), and
- *of type* (b) if it is contained in the boundary of Q, i.e., it is a subset of an edge of Q and it belongs to a single cell (say, the cell of the generator p_i).

Correspondingly, we shall write $e(i,j)$ and $e(i)$, respectively, whenever we are interested in making explicit the elements defining the edge e. Further, when considering an edge of type (a), we let $n_{e(i,j)}$ denote the unit normal to $e(i,j)$ pointing toward $\operatorname{int}(V_i(P))$. When considering an edge of type (b), we let $n_{e(i)}$ denote the unit normal to $e(i)$ pointing toward $\operatorname{int}(Q)$.

2.2. The disk-covering and the sphere-packing problems. We are interested in the following locational optimization problems:

$$\min_{p_1,\dots,p_n} \left\{ \max_{q\in Q} \left\{ \min_{i\in\{1,\dots,n\}} \|q-p_i\| \right\} \right\}, \tag{2.1}$$

$$\max_{p_1,\dots,p_n} \left\{ \min_{\substack{i,j\in\{1,\dots,n\}\\ i\neq j,\, e\in\operatorname{Ed}(Q)}} \left\{ \tfrac{1}{2}\|p_i-p_j\|, \mathrm{D}_e(p_i) \right\} \right\}. \tag{2.2}$$

The optimization problem (2.1) is referred to as the p-center problem in [17, 45]. Throughout the paper, we will refer to it as the multi-circumcenter problem. In the context of coverage control of mobile sensor networks [15], the multi-circumcenter problem corresponds to considering the worst case scenario, in which no information is available on the distribution of the events taking place in the environment Q. The network therefore tries to minimize the largest possible distance of any point in Q to one of the generators' locations given by $p_1, \dots, p_n$, i.e., to minimize the function

$$\mathcal{H}_{\mathrm{DC}}(P) = \max_{q\in Q} \left\{ \min_{i\in\{1,\dots,n\}} \|q-p_i\| \right\} = \max_{i\in\{1,\dots,n\}} \left\{ \max_{q\in V_i} \|q-p_i\| \right\}.$$

It is conjectured in [45] that this problem can be restated as a disk-covering problem: how to cover a region with (possibly overlapping) disks of minimum radius. The disk-covering problem then reads

$$\min\{R \mid \cup_{i\in\{1,\dots,n\}} \overline{B}_2(p_i,R) \supseteq Q\}.$$

We shall present a proof of this statement in Theorem 4.7 below. Given a polytope W in $\mathbb{R}^N$, its circumcenter, denoted by $\mathrm{CC}(W)$, is the center of the minimum-radius sphere that contains W. The circumradius of W, denoted by $\mathrm{CR}(W)$, is the radius of this sphere. We will say that P is a *circumcenter Voronoi configuration* if $p_i = \mathrm{CC}(V_i(P))$, for all $i \in \{1,\dots,n\}$. We denote by $\mathrm{Ve}_{\mathrm{DC}}(\mathcal{V}(P))$ the set of vertexes of the Voronoi partition where the value $\mathcal{H}_{\mathrm{DC}}(P)$ is attained, i.e., $v \in \mathrm{Ve}_{\mathrm{DC}}(\mathcal{V}(P))$ if there exists i such that $v \in V_i(P)$ and $\|v-p_i\| = \mathcal{H}_{\mathrm{DC}}(P)$. In such cases, we will often refer to both the vertex v and the generator p_i as *active*.

We will refer to the optimization problem (2.2) as the multi-incenter problem. In the context of applications, this problem corresponds to the situation where we are interested in maximizing the coverage of the area Q in such a way that the sensing radius of the generators do not overlap (in order not to interfere with each other) or leave the environment. We therefore consider the maximization of the function

$$\mathcal{H}_{\mathrm{SP}}(P) = \min_{\substack{i,j\in\{1,\dots,n\} \\ i\neq j,\, e\in \mathrm{Ed}(Q)}} \left\{ \tfrac{1}{2}\|p_i - p_j\|, \mathrm{D}_e(p_i) \right\} = \min_{i\in\{1,\dots,n\}} \left\{ \min_{q\notin \mathrm{int}(V_i)} \|q - p_i\| \right\}.$$

A similar conjecture to the one presented above is that the multi-incenter problem can be restated as a sphere-packing problem: how to maximize the coverage of a region with nonoverlapping disks (contained in the region) of maximum radius. The problem reads

$$\max\{R \mid \cup_{i\in\{1,\dots,n\}} \overline{B}_2(p_i, R) \subseteq Q,\ B_2(p_i,R)\cap B_2(p_j,R) = \emptyset\}.$$

In Theorem 4.8 we provide a positive answer to this question. Given a polytope W in $\mathbb{R}^N$, its incenter set (or Chebyshev center set; see [5]), denoted by $\mathrm{IC}(W)$, is the set of the centers of maximum-radius spheres contained in W. The inradius of W, denoted by $\mathrm{IR}(W)$, is the common radius of these spheres. We will say that $P \in Q^n$ is an *incenter Voronoi configuration* if $p_i \in \mathrm{IC}(V_i(P))$, for all $i \in \{1,\dots,n\}$. If P is an incenter Voronoi configuration and each Voronoi region $V_i(P)$ has a unique incenter, $\mathrm{IC}(V_i(P)) = \{p_i\}$, then we will say that P is a *generic incenter Voronoi configuration*. We denote by $\mathrm{Ed}_{\mathrm{SP}}(\mathcal{V}(P))$ the set of edges of the Voronoi partition where the value $\mathcal{H}_{\mathrm{SP}}(P)$ is attained; i.e., $e \in \mathrm{Ed}_{\mathrm{SP}}(\mathcal{V}(P))$ if there exists i such that $e \in \mathrm{Ed}(V_i(P))$ and $\mathrm{D}_e(p_i) = \mathcal{H}_{\mathrm{SP}}(P)$. In such cases, we will often refer to both the edge e and the generator p_i as *active*.

2.3. Nonsmooth analysis. The following facts on nonsmooth analysis [11] will be helpful in analyzing the properties of the locational optimization functions for the disk-covering and the sphere-packing problems, as well as the convergence of the distributed algorithms we will propose to extremize them.

We begin by recalling some basic notions. A function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be *locally Lipschitz at* $x \in \mathbb{R}^N$ if there exist positive constants L_x and ϵ such that $|f(y) - f(y')| \le L_x\|y - y'\|$ for all $y, y' \in B_N(x,\epsilon)$. The function f is said to be *locally Lipschitz on* $S \subset \mathbb{R}^N$ *if it is locally Lipschitz at* x*, for all* $x \in S$. Note that continuously differentiable functions at x are locally Lipschitz at x. On the other hand, a function $f : \mathbb{R}^N \to \mathbb{R}$ is said to be *regular at* $x \in \mathbb{R}^N$ if for all $v \in \mathbb{R}^N$ the right directional derivative of f at x in the direction of v, denoted by $f'(x;v)$, exists and coincides with the generalized directional derivative of f at x in the direction of v, denoted by $f^o(x;v)$. The interested reader is referred to [11] for the precise definition of these directional derivatives. Again, a continuously differentiable function at x is regular at x. Also, a locally Lipschitz function at x which is convex is regular (cf. Proposition 2.3.6 in [11]).

From Rademacher's theorem [11], we know that locally Lipschitz functions are differentiable almost everywhere (in the sense of Lebesgue measure). If Ω_f denotes the set of points in $\mathbb{R}^N$ at which f fails to be differentiable and S denotes any other set of measure zero, the *generalized gradient* of f is defined by

$$\partial f(x) = \mathrm{co}\left\{ \lim_{i\to+\infty} df(x_i) \mid x_i \to x,\ x_i \notin S \cup \Omega_f \right\}.$$

Note that this definition coincides with $df(x)$ if f is continuously differentiable at x. A point $x \in \mathbb{R}^N$ which verifies that $0 \in \partial f(x)$ is called a *critical point of* f. The following result corresponds to Proposition 2.3.12 in [11].

PROPOSITION 2.2. *Let $f_k : \mathbb{R}^N \to \mathbb{R}$, $k \in \{1, \dots, m\}$ be locally Lipschitz functions at $x \in \mathbb{R}^N$ and let $f(x') = \max\{f_k(x') \mid k \in \{1, \dots, m\}\}$. Then,*

(i) *f is locally Lipschitz at x,*

(ii) *if $I(x')$ denotes the set of indexes k for which $f_k(x') = f(x')$, we have*

$$\partial f(x) \subset \mathrm{co}\{\partial f_i(x) \mid i \in I(x)\}, \tag{2.3}$$

and if f_i, $i \in I(x)$, is regular at x, then equality holds and f is regular at x.

The extrema of Lipschitz functions are characterized by the following result.

PROPOSITION 2.3. *Let f be a locally Lipschitz function at $x \in \mathbb{R}^N$. If f attains a local minimum or a local maximum at x, then $0 \in \partial f(x)$, that is, x is a critical point.*

Let $\mathrm{Ln} : 2^{\mathbb{R}^N} \to 2^{\mathbb{R}^N}$ be the set-valued mapping that associates to each subset S of $\mathbb{R}^N$ the set of its least-norm elements $\mathrm{Ln}(S)$. If the set S is convex, then the set $\mathrm{Ln}(S)$ reduces to a singleton and we note the equivalence $\mathrm{Ln}(S) = \mathrm{proj}_S(0)$. In this paper, we shall only apply this function to convex sets. For a locally Lipschitz function f, we consider the *generalized gradient vector field* $\mathrm{Ln}(\partial f) : \mathbb{R}^N \to \mathbb{R}^N$ given by $x \mapsto \mathrm{Ln}(\partial f)(x) = \mathrm{Ln}(\partial f(x))$. The following theorem (cf. [11]) establishes an important feature of this vector field.

THEOREM 2.4. *Let f be a locally Lipschitz function at x. Assume $0 \notin \partial f(x)$. Then, there exists $T > 0$ such that*

$$f(x - t\,\mathrm{Ln}(\partial f)(x)) \le f(x) - \frac{t}{2}\|\mathrm{Ln}(\partial f)(x)\|^2, \quad 0 < t < T.$$

The vector $-\mathrm{Ln}(\partial f)(x)$ is called a direction of descent*.*

2.4. Stability analysis via nonsmooth Lyapunov functions. Throughout the paper, we will define the solutions of differential equations with discontinuous right-hand sides in terms of differential inclusions [19]. Let $F : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ be a set-valued map. Consider the differential inclusion

$$\dot{x} \in F(x). \tag{2.4}$$

A solution to this equation on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as an absolutely continuous function $x : [t_0, t_1] \to \mathbb{R}^N$ such that $\dot{x}(t) \in F(x(t))$ for almost all $t \in [t_0, t_1]$. Given $x_0 \in \mathbb{R}^N$, the existence of at least a solution with initial condition x_0 is guaranteed by the following lemma.

LEMMA 2.5. *Let the mapping F be upper semicontinuous with nonempty, compact, and convex values. Then, given $x_0 \in \mathbb{R}^N$, there exists a local solution of* (2.4) *with initial condition x_0.*

Now, consider the differential equation

$$\dot{x}(t) = X(x(t)), \tag{2.5}$$

where $X : \mathbb{R}^N \to \mathbb{R}^N$ is measurable and essentially locally bounded. There are various notions of solutions to discontinuous differential equations (see [13] for a comparative discussion among them). Here, we will understand the solution of this equation in the Filippov sense, which we define in the following. For each $x \in \mathbb{R}^N$, consider the set

$$K[X](x) = \bigcap_{\delta > 0} \bigcap_{\mu(S)=0} \mathrm{co}\{X(B_N(x, \delta) \setminus S)\},$$

where μ denotes the usual Lebesgue measure in $\mathbb{R}^N$. Alternatively, one can show [13] that there exists a set S_X of measure zero such that

$$K[X](x) = \text{co}\left\{ \lim_{i \to +\infty} X(x_i) \mid x_i \to x,\ x_i \notin S \cup S_X \right\},$$

where S is any set of measure zero. A Filippov solution of (2.5) on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as a solution of the differential inclusion $\dot{x} \in K[X](x)$. Since the multivalued mapping $K[X] : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is upper semicontinuous with nonempty, compact, convex values and locally bounded (cf. [19]), the existence of Filippov solutions of (2.5) is guaranteed by Lemma 2.5.

A set M is *weakly invariant* (respectively, *strongly invariant*) for (2.5) if, for each $x_0 \in M$, the set M contains a maximal solution (respectively, all maximal solutions) of (2.5). Given a locally Lipschitz function $f : \mathbb{R}^N \to \mathbb{R}$, the *set-valued Lie derivative of f with respect to X* at x is defined by

$$\widetilde{\mathcal{L}}_X f(x) = \{a \in \mathbb{R} \mid \exists v \in K[X](x) \text{ such that } \zeta \cdot v = a \text{ for all } \zeta \in \partial f(x)\}.$$

For each $x \in \mathbb{R}^N$, $\widetilde{\mathcal{L}}_X f(x)$ is a closed and bounded interval in $\mathbb{R}$, possibly empty. If f is continuously differentiable at x, then $\widetilde{\mathcal{L}}_X f(x) = \{df \cdot v \mid v \in K[X](x)\}$. If, in addition, X is continuous at x, then $\widetilde{\mathcal{L}}_X f(x)$ corresponds to the singleton $\{\mathcal{L}_X f(x)\}$, the usual Lie derivative of f in the direction of X at x. The importance of the set-valued Lie derivative stems from the next result [2].

THEOREM 2.6. *Let $x : [t_0, t_1] \to \mathbb{R}^N$ be a Filippov solution of* (2.5). *Let f be a locally Lipschitz and regular function. Then $\frac{d}{dt}(f(x(t)))$ exists a.e. and $\frac{d}{dt}(f(x(t))) \in \widetilde{\mathcal{L}}_X f(x(t))$ a.e.*

The following result is a generalization of the LaSalle Invariance Principle for differential equations of the form (2.5) with nonsmooth Lyapunov functions. The formulation is taken from [2] and slightly generalizes the one presented in [42].

THEOREM 2.7 (LaSalle Invariance Principle). *Let $f : \mathbb{R}^N \to \mathbb{R}$ be a locally Lipschitz and regular function. Let $x_0 \in \mathbb{R}^N$ and let $f^{-1}(\leq f(x_0), x_0)$ be the connected component of $\{x \in \mathbb{R}^N \mid f(x) \leq f(x_0)\}$ containing x_0. Assume the set $f^{-1}(\leq f(x_0), x_0)$ is bounded and assume either* $\max \widetilde{\mathcal{L}}_X f(x) \leq 0$ *or* $\widetilde{\mathcal{L}}_X f(x) = \emptyset$ *for all $x \in f^{-1}(\leq f(x_0), x_0)$. Then $f^{-1}(\leq f(x_0), x_0)$ is strongly invariant for* (2.5). *Let*

$$Z_{X,f} = \{x \in \mathbb{R}^N \mid 0 \in \widetilde{\mathcal{L}}_X f(x)\}.$$

Then, any solution $x : [t_0, +\infty) \to \mathbb{R}^N$ of (2.5) *starting from x_0 converges to the largest weakly invariant set M contained in $\overline{Z}_{X,f} \cap f^{-1}(\leq f(x_0), x_0)$. Furthermore, if the set M is a finite collection of points, then the limit of all solutions starting at x_0 exists and equals one of those points.*

The proof of the last fact in the theorem statement is the same as in the smooth case, since it only relies on the continuity of the trajectory. The next statement is based on Theorem 2 of [36].

PROPOSITION 2.8. *Under the same assumptions of Theorem* 2.7, *if* $\max \widetilde{\mathcal{L}}_X f(x) < -\epsilon < 0$ *a.e. on $\mathbb{R}^N \setminus Z_{X,f}$, then $Z_{X,f}$ is attained in finite time.*

Proof. Let $x : [t_0, +\infty) \to \mathbb{R}^N$ be a Filippov solution starting from x_0. We argue that there must exist T such that $x(T) \in Z_{X,f}$. Otherwise, we have

$$f(x(t)) = f(x(t_0)) + \int_{t_0}^{t} \frac{d}{ds} f(x(s)) ds < f(x(t_0)) - \epsilon(t - t_0) \overset{t \to +\infty}{\longrightarrow} -\infty,$$

contradicting the fact that $f^{-1}(\leq f(x_0), x_0)$ is strongly invariant and bounded. ∎

2.5. Nonsmooth gradient flows. Finally, we are in a position to present the nonsmooth analog of well-known results on gradient flows. Given a locally Lipschitz and regular function f, consider the following generalized gradient flow:

$$\dot{x}(t) = -\operatorname{Ln}(\partial f)(x(t)). \tag{2.6}$$

Theorem 2.4 guarantees that unless the flow is at a critical point, $-\operatorname{Ln}(\partial f)(x)$ is always a direction of descent at x. In general, the vector field $\operatorname{Ln}(\partial f)$ in (2.6) is discontinuous. We understand its solution in the Filippov sense. Note that since f is locally Lipschitz, $\operatorname{Ln}(\partial f) = df$ almost everywhere. An important observation in this setting is that $K[df](x) = \partial f(x)$ (cf. [36]). The following result, which is a generalization of the discussion in [2], guarantees the convergence of this flow to the set of critical points of f.

Proposition 2.9. *Let $x_0 \in \mathbb{R}^N$ and assume $f^{-1}(\leq f(x_0), x_0)$ is bounded. Then, any solution $x : [t_0, +\infty) \to \mathbb{R}^N$ of* (2.6) *starting from x_0 converges asymptotically to the set of critical points of f contained in $f^{-1}(\leq f(x_0), x_0)$.*

3. The 1-center problems. In this section we consider the disk-covering and the sphere-packing problems with a single generator, i.e., $n = 1$. This treatment will give us the necessary insight to tackle later the more involved multicenter version of both problems. When $n = 1$, the minimization of $\mathcal{H}_{\mathrm{DC}}$ simply consists of finding the center of the minimum-radius sphere enclosing the polygon Q. On the other hand, the maximization of $\mathcal{H}_{\mathrm{SP}}$ corresponds to determining the center of the maximum-radius sphere contained in Q. Let us therefore define the functions

$$\begin{aligned} \operatorname{lg}_Q(p) &= \max\{\|q-p\| \mid q \in Q\} = \max\{\|v-p\| \mid v \in \operatorname{Ve}(Q)\}, \\ \operatorname{sm}_Q(p) &= \min\{\|q-p\| \mid q \notin \operatorname{int}(Q)\} = \min\{\mathrm{D}_e(p) \mid e \in \operatorname{Ed}(Q)\}. \end{aligned} \tag{3.1}$$

When $n = 1$, we then have that $\mathcal{H}_{\mathrm{DC}} = \operatorname{lg}_Q : Q \to \mathbb{R}$ and $\mathcal{H}_{\mathrm{SP}} = \operatorname{sm}_Q : Q \to \mathbb{R}$.

3.1. Smoothness and critical points. We here discuss the smoothness properties and the critical points of the 1-center functions. Since the function lg_Q is the maximum of a (finite) set of convex functions in p, it is also a convex function [5]. Therefore, any local minimum of lg_Q is also global.

Lemma 3.1. *The function lg_Q has a unique global minimum, which is the circumcenter of the polygon Q.*

Proof. Let $F : \mathbb{R} \to \mathbb{R}$ be any continuous nondecreasing function. Then

$$F(\operatorname{lg}_Q(p)) = \max\{F(\|v-p\|) \mid v \in \operatorname{Ve}(Q)\}.$$

If we take $F(x) = x^2$, each function $\|v-p\|^2$ is strictly convex, and hence $F(\operatorname{lg}_Q(p))$ is also strictly convex. Therefore, this latter function has a single minimum on Q. Since any global minimum of lg_Q is also a global minimum of $F(\operatorname{lg}_Q(p))$, we conclude the result. ∎

The function sm_Q is the minimum of a (finite) set of affine (hence, concave) functions defined on the half-planes determined by the edges of Q, and hence it is also a concave function [5] on the intersection of their domains, which is precisely Q. Therefore, any local maximum of sm_Q is also global. However, this maximum is not unique in general.

Lemma 3.2. *The incenter set of the polygon Q is the set of maxima of the function sm_Q and it is a segment.*

Proof. It is clear that the set of maxima of sm_Q is $\mathrm{IC}(Q)$. As a consequence of the concavity of sm_Q over the convex domain Q, one deduces that $\mathrm{IC}(Q)$ is a convex set. Now, assume there are three points p_1, p_2, p_3 in $\mathrm{IC}(Q)$ which are not aligned. Since $B_2(q, \mathrm{IR}(Q)) \subset Q$ for all $q \in \mathrm{co}(p_1, p_2, p_3) \subset \mathrm{IC}(Q)$, and $\mathrm{co}(p_1, p_2, p_3)$ has a nonempty interior, there exist $q_0 \in Q$ and $r > \mathrm{IR}(Q)$ such that $B_2(q_0, r) \subset Q$, which is a contradiction. ∎

Note that the circumcenter of a polygon can be computed via the finite-step algorithm described in [43]. The incenter set of a polygon can be computed via the following linear program in q and r: maximize the radius r of the sphere centered at q subject to the constraints that the distance between q and each of the polygon edges is greater than or equal to r. Formally, the problem can be expressed as follows. For each $e \in \mathrm{Ed}(Q)$, select a point $q_e \in Q$ belonging to e. Then, we set

$$\begin{aligned} &\text{maximize } \ r, \\ &\text{subject to } \ (q - q_e) \cdot n_e \geq r, \quad \text{for all } e \in \mathrm{Ed}(Q). \end{aligned}$$

In what follows, we examine dynamical systems that compute these geometric centers.

PROPOSITION 3.3. *The functions* $\mathrm{lg}_Q(p)$, $-\mathrm{sm}_Q(p)$ *are locally Lipschitz and regular, and their generalized gradients are given by*

$$\partial \,\mathrm{lg}_Q(p) = \mathrm{co}\{\mathrm{vrs}(p - v) \mid v \in \mathrm{Ve}(Q),\ \mathrm{lg}_Q(p) = \|p - v\|\}, \tag{3.2}$$

$$\partial \,\mathrm{sm}_Q(p) = \mathrm{co}\{n_e \mid e \in \mathrm{Ed}(Q),\ \mathrm{sm}_Q(p) = \mathrm{D}_e(p)\}. \tag{3.3}$$

Moreover,

$$0 \in \partial \,\mathrm{lg}_Q(p) \Longleftrightarrow p = \mathrm{CC}(Q), \quad 0 \in \partial \,\mathrm{sm}_Q(p) \Longleftrightarrow p \in \mathrm{IC}(Q), \tag{3.4}$$

and, if $0 \in \mathrm{int}(\partial \,\mathrm{sm}_Q(p))$, *then* $\mathrm{IC}(Q) = \{p\}$.

Proof. Given the expressions in (3.1) and Proposition 2.2, we deduce that lg_Q and $-\mathrm{sm}_Q$ are locally Lipschitz and regular, and that their generalized gradients are given by (3.2) and (3.3), respectively. Concerning (3.4), the implications from right to left in (3.4) readily follow from Proposition 2.3. As for the other ones, note that it is sufficient to prove that p is a local minimum (respectively, that p is a local maximum). We prove the result for the function lg_Q. The proof for sm_Q is analogous. Assume that $0 \in \partial \,\mathrm{lg}_Q(p)$. Then there exist vertexes $v_{i_1}, \dots, v_{i_K}$ of Q with $\mathrm{lg}_Q(p) = \|v_{i_l} - p\|$, $l \in \{1, \dots, K\}$ such that $0 = \sum_{l \in \{1,\dots,K\}} \lambda_l \,\mathrm{vrs}(p - v_{i_l})$, where $\sum_{l \in \{1,\dots,K\}} \lambda_l = 1$, $\lambda_l \geq 0$, $l \in \{1, \dots, K\}$. Let U be a neighborhood of p and take $q \in U$. One can show that there must exist l^* such that $(p - v_{i_{l^*}}) \cdot (q - p) \geq 0$, since otherwise $0 = 0 \cdot (q - p) = (\sum_{l \in \{1,\dots,K\}} \lambda_l \,\mathrm{vrs}(p - v_{i_l})) \cdot (q - p) < 0$, which is a contradiction. Then

$$\|q - v_{i_{l^*}}\|^2 = \|q - p\|^2 + \|p - v_{i_{l^*}}\|^2 - 2(q - p) \cdot (v_{i_{l^*}} - p) \geq \|p - v_{i_{l^*}}\|^2.$$

Therefore, $\mathrm{lg}_Q(q) \geq \|p - v_{i_{l^*}}\| = \mathrm{lg}_Q(p)$, which shows that p is a local minimum. Finally, if $0 \in \mathrm{int}(\partial \,\mathrm{sm}_Q(p))$, then one can see that p is a strict local maximum. Furthermore, there cannot be any other local (hence global) maximum of sm_Q, as we now show. Assume $\bar{p} \in \mathrm{IC}(Q)$. By hypothesis, the sphere $B_2(\bar{p}, \mathrm{sm}_Q(p))$ centered at $\bar{p}$ of radius $\mathrm{sm}_Q(p)$ is contained in Q. Consider the vector $\bar{p} - p$. By Lemma 2.1, there exists $e \in \mathrm{Ed}(Q)$ with $\mathrm{D}_e(p) = \mathrm{sm}_Q(p)$ such that $(\bar{p} - p) \cdot n_e > 0$. Therefore, there are points of $B_2(\bar{p}, \mathrm{sm}_Q(p))$ which necessarily belong to the half-plane defined by e where Q is not contained, which is a contradiction. ∎

3.2. Convergence properties for nonsmooth gradient flows. Here we study the generalized gradient flows arising from the two 1-center functions. An immediate consequence of Propositions 2.9 and 3.3 is the following result: the gradient flows of the functions lg_Q and sm_Q,

$$\dot{x}(t) = -\,\mathrm{Ln}(\partial\,\mathrm{lg}_Q)(x(t)), \tag{3.5}$$

$$\dot{x}(t) = \mathrm{Ln}(\partial\,\mathrm{sm}_Q)(x(t)), \tag{3.6}$$

converge asymptotically to the circumcenter $\mathrm{CC}(Q)$ and the incenter set $\mathrm{IC}(Q)$, respectively. The following two propositions discuss the convergence properties of the gradient descents.

PROPOSITION 3.4. *If* $0 \in \mathrm{int}(\partial\,\mathrm{lg}_Q(\mathrm{CC}(Q)))$*, then the flow* (3.5) *reaches* $\mathrm{CC}(Q)$ *in finite time.*

Note that if $0 \in \partial\,\mathrm{lg}_Q(\mathrm{CC}(Q)) \setminus \mathrm{int}(\partial\,\mathrm{lg}_Q(\mathrm{CC}(Q)))$, then convergence is generically achieved over an infinite time horizon.

PROPOSITION 3.5. *The flow* (3.6) *reaches the set* $\mathrm{IC}(Q)$ *in finite time.*

Proof. Let $p \notin \mathrm{IC}(Q)$. We know $\min \widetilde{\mathcal{L}}_{\mathrm{Ln}[\mathrm{sm}_Q]}\,\mathrm{sm}_Q(p) = \|\,\mathrm{Ln}[\mathrm{sm}_Q](p)\|^2$. Moreover, for all $p \notin \mathrm{IC}(Q)$, we have

$$\|\,\mathrm{Ln}[\mathrm{sm}_Q](p)\| \geq \epsilon = \min\big\{1, \{\|\,\mathrm{Ln}(\mathrm{co}\{n_e, n_f\})\| \mid e, f \in \mathrm{Ed}(Q), n_e \neq -n_f\}\big\} > 0.$$

Resorting to Proposition 2.8, we deduce the desired result. $\square$

Figure 3.1 shows an example of the implementation of the gradient descent (3.5) and (3.6). Note that if the circumcenter $\mathrm{CC}(Q)$ (respectively, the incenter set $\mathrm{IC}(Q)$) is first computed offline, then the strategy of directly going toward it would converge in a less "erratic" way. Note also that the move-toward-the-center strategy is exponentially fast.

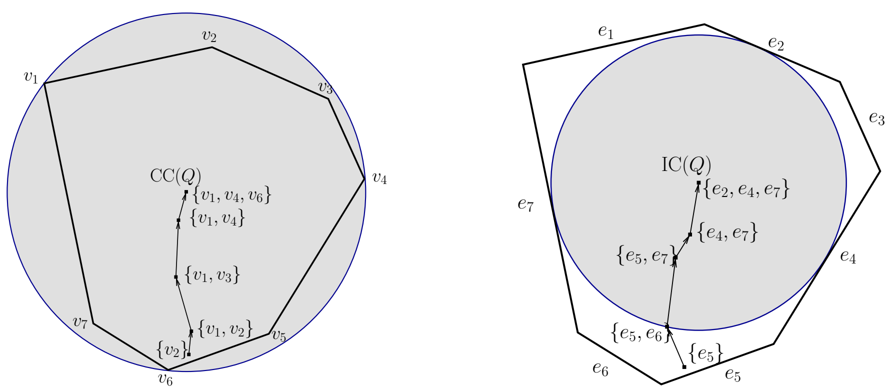


FIG. 3.1. *Illustration of the gradient descent of* lg_Q *and* sm_Q*. The points where the curve* $t \mapsto p(t)$ *fails to be differentiable correspond to points where there is a new vertex* v *of* Q *such that* $\|p(t) - v\| = \mathrm{lg}_Q(p(t))$ *(respectively, a new edge* e *of* Q *such that* $\mathrm{D}_e(p(t)) = \mathrm{sm}_Q(p(t))$*). The circumcenter and the incenter are attained in finite time according to Propositions* 3.4 *and* 3.5*.*

Finally, we conclude this section with four facts useful for later developments.

LEMMA 3.6. *Let* $q \in Q$*, let* $v(q)$ *be one of the vertexes of* Q *which is furthest away from* q*, and let* $e(q)$ *be one of the edges of* Q *which is nearest to* q*. Then*

(i) $\mathrm{Ln}[\mathrm{lg}_Q](q) \cdot (q - v(q)) \geq 0$*, and the inequality is strict if* $q \neq \mathrm{CC}(Q)$*,*

(ii) $(q - \mathrm{CC}(Q)) \cdot (q - v(q)) \geq \|q - \mathrm{CC}(Q)\|^2/2$*,*

(iii) $\mathrm{Ln}[\mathrm{sm}_Q](q) \cdot n_e \geq 0$, *and the inequality is strict if* $q \notin \mathrm{IC}(Q)$, *and*
(iv) $(x-q) \cdot n_e \geq \mathrm{IR}(Q) - \mathrm{D}_e(q) \geq 0$ *for any* $x \in \mathrm{IC}(Q)$, *and the second inequality is strict if* $q \notin \mathrm{IC}(Q)$.

4. Analysis of the multicenter functions. Here we study the locational optimization functions $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$ for the disk-covering and sphere-packing problems. We characterize their smoothness properties, generalized gradients, and critical points for arbitrary numbers of generators.

4.1. Smoothness and generalized gradients. We start by providing some alternative expressions and useful quantities. We write

$$\mathcal{H}_{\mathrm{DC}}(P) = \max_{i \in \{1,\dots,n\}} G_i(P), \quad \mathcal{H}_{\mathrm{SP}}(P) = \min_{i \in \{1,\dots,n\}} F_i(P),$$

where

$$G_i(P) = \max_{q \in V_i(P)} \|q - p_i\|, \quad F_i(P) = \min_{q \notin \mathrm{int}(V_i(P))} \|q - p_i\|.$$

Note that $G_i(P) = \mathrm{lg}_{V_i(P)}(p_i)$ and $F_i(P) = \mathrm{sm}_{V_i(P)}(p_i)$, where, for $i \in \{1,\dots,n\}$,

$$\mathrm{lg}_{V_i} : V_i \to \mathbb{R}, \quad \mathrm{sm}_{V_i} : V_i \to \mathbb{R}.$$

Proposition 3.3 provides an explicit expression for the generalized gradients of lg_{V_i} and sm_{V_i} when the Voronoi cell V_i is held fixed. Despite the slight abuse of notation, it is convenient to let $\partial \,\mathrm{lg}_{V_i(P)}(p_i)$ denote $\partial \,\mathrm{lg}_V(p_i)_{|V=V_i(P)}$ and let $\partial \,\mathrm{sm}_{V_i(P)}(p_i)$ denote $\partial \,\mathrm{sm}_V(p_i)_{|V=V_i(P)}$.

In contrast to this analysis at fixed Voronoi partition, the properties of the functions G_i and F_i are strongly affected by the dependence on the Voronoi partition $\mathcal{V}(P)$. We endeavor to characterize these properties in order to study $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$.

PROPOSITION 4.1. *The functions* $G_i, -F_i : Q^n \to \mathbb{R}$ *are locally Lipschitz and regular. As a consequence, the locational optimization functions* $\mathcal{H}_{\mathrm{DC}}, -\mathcal{H}_{\mathrm{SP}} : Q^n \to \mathbb{R}$ *are locally Lipschitz and regular.*

Proof. Here we only prove that G_i is locally Lipschitz and regular and we refer to [14] for the corresponding result for $-F_i$. The definition of the function G_i admits the following alternative expression:

$$G_i(P) = \max_{v \in \mathrm{Ve}(V_i)} \|p_i - v\|. \tag{4.1}$$

Let P_0 be nondegenerate at the ith generator. Then there exists a neighborhood U of P_0 where the set $\mathcal{N}(i)$ does not change. Let $\{v_1,\dots,v_{M_1}\}$, $\{w_1,\dots,w_{M_2}\}$, $\{z_1,\dots,z_{M_3}\}$ be the vertexes of V_i of types (a), (b), and (c), respectively. Then G_i can be locally written as

$$G_i(P) = \max\left\{ \max_{\ell \in \{1,\dots,M_1\}} \|v_\ell - p_i\|, \max_{\ell \in \{1,\dots,M_2\}} \|w_\ell - p_i\|, \max_{\ell \in \{1,\dots,M_3\}} \|z_\ell - p_i\| \right\}$$

for all $P \in U$. Therefore, G_i restricted to U coincides with the function $\mathcal{G}_{\mathcal{N}(i)} : Q^n \to \mathbb{R}$ defined by

$$\mathcal{G}_{\mathcal{N}(i)}(P) = \max\left\{ \max_{\ell \in \{1,\dots,M_1\}} \|v_\ell - p_i\|, \max_{\ell \in \{1,\dots,M_2\}} \|w_\ell - p_i\|, \max_{\ell \in \{1,\dots,M_3\}} \|z_\ell - p_i\| \right\}. \tag{4.2}$$

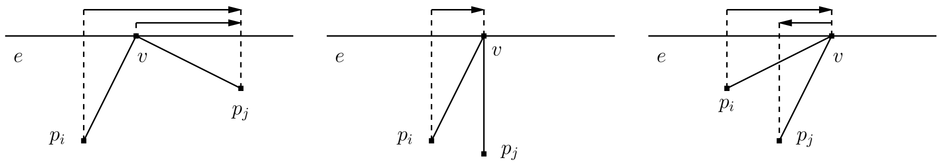


FIG. 4.1. *To illustrate* (4.3) *we draw the vectors* $\mathrm{proj}_e(p_j - v(e,i,j))$ *and* $\mathrm{proj}_e(p_j - p_i)$ *for various locations of* p_i*,* p_j*, and* e*. The left, center, and right figures correspond to* $\lambda(e,i,j) > 0$*,* $\lambda(e,i,j) = 0$*,* $\lambda(e,i,j) < 0$*, respectively.*

The function $\mathcal{G}_{\mathcal{N}(i)}$ is the maximum of a fixed finite set of locally Lipschitz and regular functions and, consequently, locally Lipschitz and regular by Proposition 2.2. We conclude that G_i is both locally Lipschitz and regular at P_0.

Let P_0 be degenerate at the ith generator. Then in any neighborhood U of P_0 there are different sets of neighbors of the ith generator. Indeed, because the number of generators, edges of the boundary Q, and vertexes of Q is finite, there is only a finite number of different sets of neighbors of the ith generator over U, say $\mathcal{N}^1(i), \dots, \mathcal{N}^L(i)$. This implies that G_i admits the alternative expression $G_i(P) = \min\left\{\mathcal{G}_{\mathcal{N}^1(i)}(P), \dots, \mathcal{G}_{\mathcal{N}^L(i)}(P)\right\}$ over U. From this expression, one can conclude that G_i is both locally Lipschitz and regular at P_0. ∎

Next, one can actually prove the following stronger result.

PROPOSITION 4.2. *The locational optimization functions* $\mathcal{H}_{\mathrm{DC}}, \mathcal{H}_{\mathrm{SP}} : Q^n \to \mathbb{R}$ *are globally Lipschitz, with Lipschitz constant equal to* 1*.*

Proof. Let P, P' be two configurations of the n generators. Without loss of generality, assume that $\mathcal{H}_{\mathrm{DC}}(P) \leq \mathcal{H}_{\mathrm{DC}}(P')$. Let i, j and q_0, $q_0' \in Q$ be such that $\mathcal{H}_{\mathrm{DC}}(P) = G_i(P) = \|q_0 - p_i\|$ and $\mathcal{H}_{\mathrm{DC}}(P') = G_j(P') = \|q_0' - p_j'\|$. Now consider the set $B_2(q_0', G_i(P))$. Then there exists a k such that $p_k \in \overline{B}_2(q_0', G_i(P))$ (otherwise, $\|q_0' - p_l\| > G_i(P)$, which contradicts the definition of the function $\mathcal{H}_{\mathrm{DC}}$). On the other hand, we necessarily have that $p_k' \notin B_2(q_0', G_j(P'))$, since otherwise $\|q_0' - p_k'\| < \|q_0' - p_j'\|$, which implies that $q_0' \notin V_j'$, a contradiction. Finally, we apply the triangle inequality to obtain $\|q_0' - p_k'\| \leq \|q_0' - p_k\| + \|p_k - p_k'\|$. Gathering the previous facts, we have

$$
\begin{aligned}
|\mathcal{H}_{\mathrm{DC}}(P') - \mathcal{H}_{\mathrm{DC}}(P)| &= G_j(P') - G_i(P) \\
&\leq \|q_0' - p_k'\| - \|q_0' - p_k\| \leq \|p_k - p_k'\| \leq \|P - P'\|.
\end{aligned}
$$

This concludes our proof that $\mathcal{H}_{\mathrm{DC}}$ is globally Lipschitz. ∎

We now introduce some quantities that are useful in characterizing the generalized gradient of the functions G_i. Given a vertex of type (b), $v = v(e,i,j)$, determined by the edge e and two generators p_i and p_j, we consider the scalar function $\lambda(e,i,j)$ defined by

$$
\mathrm{proj}_e(p_j - v(e,i,j)) = \lambda(e,i,j)\ \mathrm{proj}_e(p_j - p_i), \tag{4.3}
$$

where we recall that proj_e denotes the orthogonal projection onto the edge e; see Figure 4.1. One can see that $\lambda(e,i,j) + \lambda(e,j,i) = 1$. If e is a segment in the line $ax + by + c = 0$, $(\Delta x_{ij}, \Delta y_{ij}) = p_j - p_i$, $(x_m, y_m) = (p_i + p_j)/2$, then one can show

$$
\lambda(e,i,j) = \frac{1}{2} - \frac{(a\Delta x_{ij} + b\Delta y_{ij})(ax_m + by_m + c)}{(a\Delta y_{ij} - b\Delta x_{ij})^2}. \tag{4.4}
$$

Given a vertex of type (a), $v = v(i,j,k)$, determined by the three generators p_i, p_j, and p_k, we consider the scalar function $\mu(i,j,k)$ defined by

$$\mathrm{proj}_{e_{jk}}(p_\ell - v(i,j,k)) = \mu(i,j,k)\ \mathrm{proj}_{e_{jk}}(p_\ell - p_i), \tag{4.5}$$

where e_{jk} is the bisector of p_j and p_k and where $p_\ell = p_j$ if p_j belongs to the half-plane defined by e_{jk} containing p_i, and $p_\ell = p_k$ otherwise. One can see that $\mu(i,j,k) = \mu(i,k,j)$ and that $\mu(i,j,k) + \mu(j,k,i) + \mu(k,i,j) = 1$. From the expression for λ, one can obtain

$$\mu(i,j,k) = \frac{1}{2} + \frac{(\Delta x_{ij}\Delta x_{jk} + \Delta y_{ij}\Delta y_{jk})(\Delta x_{ik}\Delta x_{jk} + \Delta y_{ik}\Delta y_{jk})}{2(x_k \Delta y_{ij} - x_j \Delta y_{ik} + x_i \Delta y_{jk})^2}. \tag{4.6}$$

Note that, in general, λ and μ are not positive functions. Now we are ready to describe in detail the structure of the generalized gradient of the functions G_i and F_i.

PROPOSITION 4.3. *The generalized gradient of $G_i : Q^n \to \mathbb{R}$ at $P \in Q^n$ is*

$$\partial G_i(P) = \mathrm{co}\{\partial_v G_i(P) \in (\mathbb{R}^2)^n \mid v \in \mathrm{Ve}(V_i(P)) \text{ such that } G_i(P) = \|p_i - v\|\},$$

where we consider separately the following cases. If $v = v(i,j,k)$ is a nondegenerate vertex of type (a)*, then*

$$\begin{aligned}&\partial_{v(i,j,k)} G_i(P) = \partial_{v(k,i,j)} G_k(P) = \partial_{v(j,k,i)} G_j(P)\\ &= (0,\dots,\underbrace{\mu(i,j,k)\,\mathrm{vrs}(p_i - v)}_{i\text{th place}},\dots,\underbrace{\mu(j,k,i)\,\mathrm{vrs}(p_j - v)}_{j\text{th place}},\dots,\underbrace{\mu(k,i,j)\,\mathrm{vrs}(p_k - v)}_{k\text{th place}},\dots,0),\end{aligned}$$

where, without loss of generality, we let $i < j < k$. If $v = v(e,i,j)$ is a nondegenerate vertex of type (b)*, then*

$$\begin{aligned}\partial_{v(e,i,j)} G_i(P) &= \partial_{v(e,j,i)} G_j(P)\\ &= (0,\dots,\underbrace{\lambda(e,i,j)\,\mathrm{vrs}(p_i - v)}_{i\text{th place}},\dots,\underbrace{\lambda(e,j,i)\,\mathrm{vrs}(p_j - v)}_{j\text{th place}},\dots,0),\end{aligned}$$

where, without loss of generality, we let $i < j$. If $v = v(e,f,i)$ is a nondegenerate vertex of type (c), then

$$\partial_{v(e,f,i)} G_i(P) = (0,\dots,0,\underbrace{\mathrm{vrs}(p_i - v)}_{i\text{th place}},0,\dots,0).$$

Finally, if the vertex v is degenerate, i.e., if v is determined by $d > 3$ elements (generators or edges), then there are $\binom{d-1}{2}$ pairs of elements which determine the vertex v together with the generator p_i. In this case, $\partial_v G_i(P)$ is the convex hull of $\partial_{v(\alpha,\beta,\gamma)} G_i(P)$ for all $\binom{d-1}{2}$ such triplets (α,β,γ).

Note that, at all nondegenerate configurations P, the quantity $\partial_v G_i(P)$ is the generalized gradient of the function $(p_1,\dots,p_n) \mapsto \|p_i - v(i,j,k)\|$; however, this interpretation cannot be given when P is degenerate.

Proof. We present the proof for the expression for $\partial G_i(P)$. Let us consider first the case when P is a nondegenerate configuration for the ith generator. According to the proof of Proposition 4.1, G_i coincides with the function $\mathcal{G}_{\mathcal{N}(i)}$ over a neighborhood U of P. Hence, $\partial G_i(P) = \partial \mathcal{G}_{\mathcal{N}(i)}(P)$ which, according to (4.2) and Proposition 2.2, takes the form

$$\mathrm{co}\left\{\frac{\partial}{\partial P}\|v - p_i\| \mid v \in \mathrm{Ve}(V_i(P)) \text{ such that } \|v - p_i\| = G_i(P)\right\}.$$

If $v = v(i,j,k)$ is a nondegenerate vertex of type (a), then one can compute

$$\begin{aligned}
\frac{\partial}{\partial p_i}\|p_i - v(i,j,k)\| &= \operatorname{vrs}(p_i - v)\left(I_2 - \frac{\partial v}{\partial p_i}\right) = \mu(i,j,k)\operatorname{vrs}(p_i - v),\\
\frac{\partial}{\partial p_j}\|p_i - v(i,j,k)\| &= -\operatorname{vrs}(p_i - v)\left(\frac{\partial v}{\partial p_j}\right) = \mu(j,k,i)\operatorname{vrs}(p_j - v),\\
\frac{\partial}{\partial p_\ell}\|p_i - v(i,j,k)\| &= 0, \qquad \ell \neq i,j,k,
\end{aligned}$$

where in the first and second chain of equalities we have used the expression of μ given in (4.6). If $v = v(e,i,j)$ is a nondegenerate vertex of type (b), then one can compute

$$\begin{aligned}
\frac{\partial}{\partial p_i}\|p_i - v(e,i,j)\| &= \operatorname{vrs}(p_i - v)\left(I_2 - \frac{\partial v}{\partial p_i}\right) = \lambda(e,i,j)\operatorname{vrs}(p_i - v),\\
\frac{\partial}{\partial p_j}\|p_i - v(e,i,j)\| &= -\operatorname{vrs}(p_i - v)\left(\frac{\partial v}{\partial p_j}\right) = \lambda(e,j,i)\operatorname{vrs}(p_j - v),\\
\frac{\partial}{\partial p_\ell}\|p_i - v(e,i,j)\| &= 0, \qquad \ell \neq i,j,
\end{aligned}$$

where in the first and second chain of equalities we have used the expression of λ given in (4.4). If $v = v(e,f,i)$ is a nondegenerate vertex of type (c), then

$$\begin{aligned}
\frac{\partial}{\partial p_i}\|p_i - v(e,f,i)\| &= \operatorname{vrs}(p_i - v),\\
\frac{\partial}{\partial p_\ell}\|p_i - v(e,f,i)\| &= 0, \qquad \ell \neq i.
\end{aligned}$$

If P is a degenerate configuration at the ith generator, then, following the proof of Proposition 4.1, the generalized gradient of G_i can be expressed as the convex hull of the generalized gradients of each of the functions $\mathcal{G}_{\mathcal{N}^1(i)}, \dots, \mathcal{G}_{\mathcal{N}^L(i)}$. The claim now follows by reproducing the previous discussion for the generalized gradients of each of the functions $\mathcal{G}_{\mathcal{N}^\ell(i)}$, $\ell \in \{1,\dots,L\}$. $\square$

The expression for $\partial F_i(P)$ can be deduced in an analogous (and simpler) way because it is not necessary to establish any distinction between the degenerate and the nondegenerate configurations in its calculation. Accordingly, we state the following result without proof.

Proposition 4.4. *The generalized gradient of $F_i : Q^n \to \mathbb{R}$ at $P \in Q^n$ is*

$$\partial F_i(P) = \operatorname{co}\{\partial_e F_i(P) \in (\mathbb{R}^2)^n \mid e \in \operatorname{Ed}(V_i(P)) \text{ such that } F_i(P) = \mathrm{D}_e(p_i)\}$$

where, if $e = e(i,j)$ is an edge of type (a)*, then*

$$\partial_{e(i,j)} F_i(P) = \partial_{e(j,i)} F_j(P) = \frac{1}{2}(0,\dots,\underbrace{n_{e(i,j)}}_{i\text{th place}},\dots,\underbrace{-n_{e(i,j)}}_{j\text{th place}},\dots,0),$$

and if $e = e(i)$ is an edge of type (b)*, then*

$$\partial_{e(i)} F_i(P) = (0,\dots,\underbrace{n_{e(i)}}_{i\text{th place}},\dots,0).$$

Next, we give conditions under which the functions λ and μ take positive values.

Lemma 4.5. *Let $P \in Q^n$ and let $v \in \operatorname{Ve}_{\mathrm{DC}}(\mathcal{V}(P))$. Then*

(i) *if v belongs to an edge e of Q, then there exist generators p_i and p_j such that $\lambda(e,i,j)$ and $\lambda(e,j,i)$ are positive, and*

(ii) *if v belongs to $\mathrm{int}(Q)$, then there exist generators p_i, p_j, and p_k such that $\mu(i,j,k)$, $\mu(j,k,i)$, and $\mu(k,i,j)$ are positive.*

This completes our analysis of the generalized gradients of G_i and F_i and, with these results, we return to studying the generalized gradients of $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$. An immediate consequence of Propositions 2.2 and 4.1 is that

$$\begin{aligned} \partial\mathcal{H}_{\mathrm{DC}}(P) &= \mathrm{co}\{\partial G_i(P) \mid i \in I(P)\}, \\ \partial\mathcal{H}_{\mathrm{SP}}(P) &= \mathrm{co}\{\partial F_i(P) \mid i \in I(P)\}. \end{aligned} \tag{4.7}$$

Furthermore, we can provide the following more detailed characterization.

Proposition 4.6. *Let $P \in Q^n$. For each $i \in \{1,\dots,n\}$, the image by π_i of the generalized gradients of $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$ at P is given by*

$$\pi_i(\partial\mathcal{H}_{\mathrm{DC}}(P)) = \begin{cases} \pi_i(\partial G_i(P)) & \text{if } i \in I(P),\ \mathrm{Ve}_{\mathrm{DC}}(\mathcal{V}(P)) \subset \mathrm{Ve}(V_i(P)), \\ \mathrm{co}\{\pi_i(\partial G_i(P)), 0\} & \text{if } i \in I(P),\ \mathrm{Ve}_{\mathrm{DC}}(\mathcal{V}(P)) \not\subset \mathrm{Ve}(V_i(P)), \\ 0 & \text{if } i \notin I(P); \end{cases}$$

$$\pi_i(\partial\mathcal{H}_{\mathrm{SP}}(P)) = \begin{cases} \pi_i(\partial F_i(P)) & \text{if } i \in I(P),\ \mathrm{Ed}_{\mathrm{SP}}(\mathcal{V}(P)) \subset \mathrm{Ed}(V_i(P)), \\ \mathrm{co}\{\pi_i(\partial F_i(P)), 0\} & \text{if } i \in I(P),\ \mathrm{Ed}_{\mathrm{SP}}(\mathcal{V}(P)) \not\subset \mathrm{Ed}(V_i(P)), \\ 0 & \text{if } i \notin I(P). \end{cases}$$

Proof. From (4.7), if $i \notin I(P)$, then $\pi_i(\partial\mathcal{H}_{\mathrm{DC}}(P)) = 0$, $\pi_i(\partial\mathcal{H}_{\mathrm{SP}}(P)) = 0$. If $i \in I(P)$, then using Proposition 4.3 we deduce that the generators p_j such that ∂G_j has a nonzero entry in the ith place (and hence contributes to the projection by π_i of $\partial\mathcal{H}_{\mathrm{DC}}$) must share a vertex with the ith generator. Analogously, if $i \in I(P)$, then using Proposition 4.4 we deduce that the generators p_j such that ∂F_j has a nonzero entry in the ith place (and hence contributes to the projection by π_i of $\partial\mathcal{H}_{\mathrm{SP}}$) must satisfy $j \in \mathcal{N}(i)$. For the disk-covering function, if v is a common vertex of V_i and V_j, determined by i, j, and a third element α, then $\partial_{v(\alpha,j,i)} G_j = \partial_{v(\alpha,i,j)} G_i$, and the expression for $\pi_i(\partial\mathcal{H}_{\mathrm{DC}}(P))$ then follows. The argument for the expression of $\pi_i(\partial\mathcal{H}_{\mathrm{SP}}(P))$ is analogous. ∎

4.2. Critical points. Having characterized the generalized gradients of $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$, we now turn to studying their critical points.

Theorem 4.7 (Minima of $\mathcal{H}_{\mathrm{DC}}$). *Let $P \in Q^n$ be a nondegenerate configuration and $0 \in \mathrm{int}(\partial\mathcal{H}_{\mathrm{DC}}(P))$. Then P is a strict local minimum of $\mathcal{H}_{\mathrm{DC}}$, all generators are active, and P is a circumcenter Voronoi configuration.*

Proof. Since P is nondegenerate, note from Proposition 4.3 that $\partial_v G_i$ is a singleton for each $v \in \mathrm{Ve}(V_i(P))$, $i \in \{1,\dots,n\}$. Let $w \in (\mathbb{R}^2)^n$. We claim that moving the configuration of the generators from P in the direction w can only increase the cost. The hypothesis $0 \in \mathrm{int}(\partial\mathcal{H}_{\mathrm{DC}}(P))$ implies by Lemma 2.1 that there exists i and $v \in \mathrm{Ve}(V_i(P)) \cap \mathrm{Ve}_{\mathrm{DC}}(\mathcal{V}(P))$ such that $w \cdot \partial_v G_i(P) > 0$. Since P is nondegenerate, v will still belong to $V_i(P + \epsilon w)$ for sufficiently small $\epsilon > 0$, and consequently $\mathcal{H}_{\mathrm{DC}}(P + \epsilon w) \geq G_i(P + \epsilon w) > G_i(P) = \mathcal{H}_{\mathrm{DC}}(P)$. Therefore, P is a strict local minimum.

Since π_i is an open map, the set $\pi_i(\mathrm{int}(\partial\mathcal{H}_{\mathrm{DC}}(P)))$ is open for each $i \in \{1,\dots,n\}$. Therefore, $\pi_i(\mathrm{int}(\partial\mathcal{H}_{\mathrm{DC}}(P))) \neq 0$, and hence all generators are active, i.e., $I(P) = \{1,\dots,n\}$. Let us see that all generators must also be centered. Assume P is nondegenerate and consider the ith generator. Take $w \in \mathbb{R}^2$ and let $\overline{w} \in (\mathbb{R}^2)^n$ be the

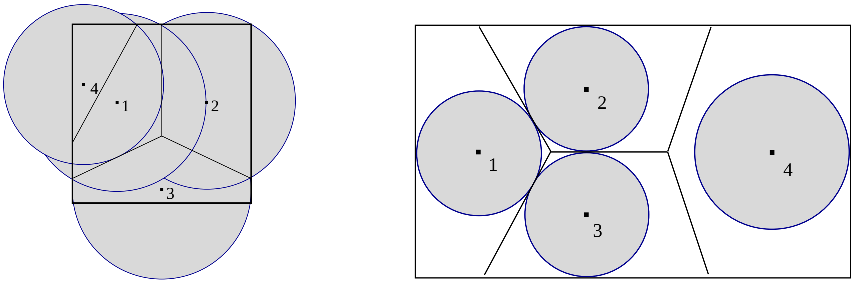


FIG. 4.2. *Local extrema of the disk-covering and the sphere-packing functions in a convex polygonal environment. The configuration on the left corresponds to a local minimum of $\mathcal{H}_{\mathrm{DC}}$ with $0 \in \partial\mathcal{H}_{\mathrm{DC}}(P)$ and $\mathrm{int}(\partial\mathcal{H}_{\mathrm{DC}}(P)) = \emptyset$. The configuration on the right corresponds to a local maximum of $\mathcal{H}_{\mathrm{SP}}$ with $0 \in \partial\mathcal{H}_{\mathrm{SP}}(P)$ and $\mathrm{int}(\partial\mathcal{H}_{\mathrm{SP}}(P)) = \emptyset$. In both configurations, the 4th generator is inactive and noncentered.*

vector which has w in the ith place and 0 otherwise. By Lemma 2.1, there exist j and $v \in \mathrm{Ve}(V_j(P)) \cap \mathrm{Ve}_{\mathrm{DC}}(\mathcal{V}(P))$ such that $\overline{w} \cdot \partial_v G_j > 0$. Since $\overline{w} \cdot \partial_v G_j = w \cdot \pi_i(\partial_v G_j) > 0$, then necessarily $\pi_i(\partial_v G_j) \neq 0$, and therefore $v \in V_i(P)$ and $\pi_i(\partial_v G_j) = \pi_i(\partial_v G_i)$. The vertex v is determined by p_i, p_j and a third element, say α. Depending on whether α corresponds to an edge or to another generator, we have that $\pi_i(\partial_v G_i)$ is equal to $\lambda(\alpha, i, j)\,\mathrm{vrs}(p_i - v)$ or $\mu(\alpha, i, j)\,\mathrm{vrs}(p_i - v)$. In any case, from Lemma 4.5, we deduce that $\lambda(\alpha, i, j)$ (respectively, $\mu(\alpha, i, j)$) belongs to the interval $(0, 1)$. Therefore, $w \cdot \pi_i(\partial_v G_i) > 0$ implies $w \cdot \mathrm{vrs}(p_i - v) > 0$. Since $\mathrm{vrs}(p_i - v) \in \partial\,\mathrm{lg}_{V_i(P)}(p_i) = \partial\,\mathrm{lg}_V(p_i)_{|V=V_i(P)}$ (cf. (3.2)), we conclude from Lemma 2.1 that $0 \in \mathrm{int}(\partial\,\mathrm{lg}_{V_i(P)}(p_i))$. By Proposition 3.3, this implies that $p_i = \mathrm{CC}(V_i)$. Hence, P is a circumcenter Voronoi configuration. ∎

THEOREM 4.8 (Maxima of $\mathcal{H}_{\mathrm{SP}}$). *Let $P \in Q^n$ and $0 \in \mathrm{int}(\partial\mathcal{H}_{\mathrm{SP}}(P))$. Then P is a strict local maximum of $\mathcal{H}_{\mathrm{SP}}$, all generators are active, and P is a generic incenter Voronoi configuration.*

Proof. The proof of this result is analogous to the proof of Theorem 4.7. Note that $0 \in \mathrm{int}(\partial\,\mathrm{sm}_{V_i(P)}(p_i))$ implies, by Proposition 3.3, that $\mathrm{IC}(V_i(P)) = \{p_i\}$, and hence P is a generic incenter Voronoi configuration. ∎

Remark 4.9. Theorems 4.7 and 4.8 precisely provide the interpretation of the multicenter problems that we gave in Section 2.2: since all generators are active, they share the same radius. If one drops the hypothesis that 0 belongs to the generalized gradient of the locational optimization function, then one can think of simple examples where P is a local minimum of $\mathcal{H}_{\mathrm{DC}}$ (respectively, local maximum of $\mathcal{H}_{\mathrm{SP}}$), and there are generators which are inactive and noncentered; see Figure 4.2.

5. Dynamical systems for the multicenter problems. In this section, we describe three algorithms that (locally) extremize the multicenter functions for the disk-covering and the sphere-packing problems. We first examine the gradient flow descent associated with the locational optimization functions $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$. This flow is guaranteed to find a local critical point, but it has the drawback of being centralized, as we describe later. Then we propose two decentralized flows for each problem. One roughly consists of a distributed implementation of the gradient descent. As we show, it is very much in the spirit of behavior-based robotics. The other one follows the logical strategy given the results in Theorems 4.7 and 4.8: each generator moves toward the circumcenter (alternatively, incenter set) of its own Voronoi polygon. We call them Lloyd flows, since they resemble the original Lloyd algorithm for vector quantization problems, where each quantizer moves toward the centroid or center of mass of its own Voronoi region, see [18, 22, 29]. We present continuous-time versions of the algorithms and discuss their convergence properties. In our setting, the generators' location obeys a first-order dynamical behavior described by

$$\dot{p}_i = u_i(p_1, \dots, p_n), \quad i \in \{1, \dots, n\}. \tag{5.1}$$

The dynamical system (5.1) is said to be (strongly) *centralized* if there exists at least an $i \in \{1, \dots, n\}$ such that $u_i(p_1, \dots, p_n)$ cannot be written as a function of the form $u_i(p_i, p_{i_1}, \dots, p_{i_m})$, with $m < n - 1$. The dynamical system (5.1) is said to be *Voronoi-distributed* if each $u_i(p_1, \dots, p_n)$ can be written as a function of the form $u_i(p_i, p_{i_1}, \dots, p_{i_m})$, with $i_k \in \mathcal{N}(P, i)$, $k \in \{1, \dots, m\}$. Finally, the dynamical system (5.1) is said to be *nearest-neighbor-distributed* if each $u_i(p_1, \dots, p_n)$ can be written as a function of the form $u_i(p_i, p_{i_1}, \dots, p_{i_m})$, with $\|p_i - p_{i_k}\| \le \|p_i - p_j\|$ for all $j \in \{1, \dots, n\}$ and $k \in \{1, \dots, m\}$. A nearest-neighbor-distributed dynamical system is also Voronoi-distributed.

It is well known that there are at most $3n - 6$ neighborhood relationships in a planar Voronoi diagram [34, section 2.3]. Therefore, the number of Voronoi neighbors of each point is on average less than or equal to 6. (Recall that points are Voronoi-neighbors if they share an edge, not just a vertex.) We refer to [15] for more details on the distributed character of Voronoi neighborhood relationships.

Note that the set of indexes $\{i_1, \dots, i_m\}$ for a specific generator p_i of a Voronoi-distributed or a nearest-neighbor-distributed dynamical system is not the same for all possible configurations P. In other words, the identity of both the Voronoi neighbors and the nearest neighbors might change along the evolution; i.e., the topology of the dynamical system is *dynamic*.

5.1. Nonsmooth gradient dynamical systems. Consider the (signed) generalized gradient descent flow (2.6) for the locational optimization functions $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$,

$$\dot{P} = -\operatorname{Ln}(\partial\mathcal{H}_{\mathrm{DC}})(P), \quad \dot{P} = \operatorname{Ln}(\partial\mathcal{H}_{\mathrm{SP}})(P).$$

Alternatively, we may write the following for each $i \in \{1, \dots, n\}$:

$$\dot{p}_i = -\pi_i(\operatorname{Ln}(\partial\mathcal{H}_{\mathrm{DC}})(p_1, \dots, p_n)), \tag{5.2}$$

$$\dot{p}_i = \pi_i(\operatorname{Ln}(\partial\mathcal{H}_{\mathrm{SP}})(p_1, \dots, p_n)). \tag{5.3}$$

As noted in Section 2.4, these vector fields are discontinuous, and therefore we understand their solution in the Filippov sense. Equation (4.7) and Propositions 4.3 and 4.4 provide an expression of the generalized gradients at P, $\partial\mathcal{H}_{\mathrm{DC}}(P)$ and $\partial\mathcal{H}_{\mathrm{SP}}(P)$. One needs to first compute the generalized gradient, then compute the least-norm element, and finally project it to each of the n components; therefore, the expressions in Proposition 4.6 are not helpful. Note that the least-norm element of convex sets can be computed efficiently, see [5], however closed-form expressions are not available in general.

One can see that the compact set Q^n is strongly invariant for both vector fields $-\operatorname{Ln}(\partial\mathcal{H}_{\mathrm{DC}})$ and $\operatorname{Ln}(\partial\mathcal{H}_{\mathrm{SP}})$. Indeed, the components for each generator of both vector fields point always toward Q. Regarding $-\operatorname{Ln}(\partial\mathcal{H}_{\mathrm{DC}})$, this is a consequence of Proposition 4.3 and of Lemma 4.5. Regarding $\operatorname{Ln}(\partial\mathcal{H}_{\mathrm{SP}})$, this is a consequence of Proposition 4.4.

Proposition 5.1. *For the dynamical system* (5.2) *(respectively,* (5.3)*), the generators' location* $P = (p_1, \dots, p_n)$ *converges asymptotically to the set of critical points of* $\mathcal{H}_{\mathrm{DC}}$ *(respectively, of* $\mathcal{H}_{\mathrm{SP}}$*).*

Proof. From Propositions 4.1 and 4.2, $\mathcal{H}_{\mathrm{DC}}$ and $-\mathcal{H}_{\mathrm{SP}}$ are globally Lipschitz and regular over Q^n. The result follows from Proposition 2.9 considering the dynamical system restricted to the strongly invariant and compact domain Q^n. ∎

Remark 5.2. The gradient dynamical systems enjoy convergence guarantees, but their implementation is centralized for two reasons. First, all functions $G_i(P)$ (respectively, $F_i(P)$) need to be compared in order to determine which generator is active. Second, the least-norm element of the generalized gradients depends on the relative position of the active generators with respect to each other and to the environment.

Remark 5.3. As illustrated in Figure 5.1 the evolution of the gradient dynamical systems may not leave fixed the generators that are already centers (circumcenters or incenters).

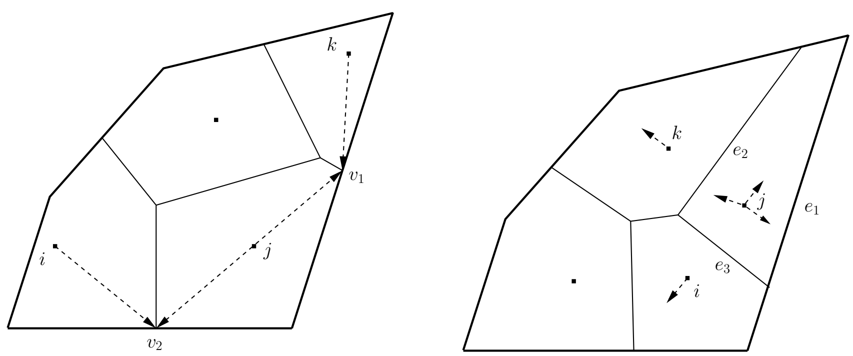


Fig. 5.1. *Illustration of the gradient descent. In the left figure, the only active vertexes at the given configuration are* v_1 *and* v_2*. Although the* j*th generator is in the circumcenter of its own Voronoi region, the control law* (5.2) *will drive it toward the vertex* v*. In the right figure, the only active edges at the given configuration are* e_1*,* e_2*, and* e_3*. Although the* j*th generator is in the incenter of its own Voronoi region, the control law* (5.3) *will drive it away from the edge* e_1*.*

5.2. Nonsmooth dynamical systems based on distributed gradients. In this section, we propose a distributed implementation of the previous gradient dynamical systems and explore its relation with behavior-based rules in multiple-vehicle coordination. Consider the following modifications of the gradient dynamical systems (5.2)–(5.3):

$$\dot{p}_i = -\operatorname{Ln}(\partial \operatorname{lg}_{V_i(P)})(P), \tag{5.4}$$

$$\dot{p}_i = \operatorname{Ln}(\partial \operatorname{sm}_{V_i(P)})(P), \tag{5.5}$$

for $i \in \{1, \dots, n\}$. Note that the system (5.4) is Voronoi-distributed, since $\operatorname{Ln}(\partial \operatorname{lg}_{V_i(P)})(P)$ is determined only by the position of p_i and of its Voronoi neighbors $\mathcal{N}(P, i)$. On the other hand, the system (5.5) is nearest-neighbor-distributed, since $\operatorname{Ln}(\partial \operatorname{sm}_{V_i(P)})(P)$ is determined only by the position of p_i and its nearest neighbors.

For future reference, let $\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})(P) = (\operatorname{Ln}(\partial \operatorname{lg}_{V_1(P)})(P), \dots, \operatorname{Ln}(\partial \operatorname{lg}_{V_n(P)})(P))$, $\operatorname{Ln}(\partial \operatorname{sm}_{\mathcal{V}})(P) = (\operatorname{Ln}(\partial \operatorname{sm}_{V_1(P)})(P), \dots, \operatorname{Ln}(\partial \operatorname{sm}_{V_n(P)})(P))$, and write

$$\dot{P} = -\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})(P), \quad \dot{P} = \operatorname{Ln}(\partial \operatorname{sm}_{\mathcal{V}})(P).$$

As for the previous dynamical systems, note that these vector fields are discontinuous, and therefore we understand their solutions in the Filippov sense. One can see that the compact set Q^n is strongly invariant for both vector fields $-\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})$ and $\operatorname{Ln}(\partial \operatorname{sm}_{\mathcal{V}})$. This fact is a consequence of the expressions for the generalized gradients of lg and sm in Proposition 3.3. Note that in the 1-center case, (5.2) (respectively, (5.3)) coincides with (5.4) (respectively, with (5.5)).

PROPOSITION 5.4. *Let $P \in Q^n$. Then the solutions of the dynamical systems* (5.4) *and* (5.5) *starting at P are unique.*

Remark 5.5 (relation with behavior-based robotics: move toward the furthest-away vertex). The distributed gradient control law in the disk-covering setting (5.4) has an interesting interpretation in the context of behavior-based robotics. Consider the ith generator. If the maximum of $\operatorname{lg}_{V_i(P)}$ is attained at a single vertex v of its Voronoi cell V_i, then $\operatorname{lg}_{V_i(P)}$ is differentiable at that configuration and its derivative corresponds to $\operatorname{vrs}(p_i - v)$. Therefore, the control law (5.4) corresponds to the behavior "move toward the furthest vertex in own Voronoi cell." If there are two or more vertexes of V_i where the value $\operatorname{lg}_{V_i(P)}(p_i)$ is attained, then (5.4) provides an average behavior by computing the least-norm element in the convex hull of all $\operatorname{vrs}(p_i - v)$ such that $\|p_i - v\| = \operatorname{lg}_{V_i(P)}(p_i)$.

Remark 5.6 (relation with behavior-based robotics: move away from the nearest neighbor). The distributed gradient control law in the sphere-packing setting (5.5) also has an interesting interpretation. For the ith generator, if the minimum of $\operatorname{sm}_{V_i(P)}$ is attained at a single edge e, then $\operatorname{sm}_{V_i(P)}$ is differentiable at that configuration, and its derivative is n_e. The control law (5.5) corresponds to the behavior "move away from the nearest neighbor" (where a neighbor can also be the boundary of the environment). If there are two or more edges where the value $\operatorname{sm}_{V_i(P)}(p_i)$ is attained, then (5.5) provides an average behavior in an analogous manner as before.

PROPOSITION 5.7. *For the dynamical system* (5.4)*, the generators' location $P = (p_1, \dots, p_n)$ converges asymptotically to the largest weakly invariant set contained in the closure of $A_{\mathrm{DC}}(Q) = \{P \in Q^n \mid i \in I(P) \implies p_i = \mathrm{CC}(V_i)\}$.*

Proof. Let $a \in \widetilde{\mathcal{L}}_{-\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})}\mathcal{H}_{\mathrm{DC}}(P)$. By definition, $a = -\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})(P) \cdot \zeta$, for all $\zeta \in \partial \mathcal{H}_{\mathrm{DC}}(P)$. Let $v \in \mathrm{Ve}_{\mathrm{DC}}(\mathcal{V}(P))$. From Proposition 4.3 and Lemma 4.5, we know that, independently of the degenerate/nondegenerate character of the Voronoi partition at v, there always exist either an edge e of Q and generators p_i and p_j, or generators p_i, p_j, and p_k, such that $\lambda(e,i,j)$, $\lambda(e,j,i) > 0$ (respectively, $\mu(i,j,k)$, $\mu(j,k,i)$, $\mu(k,i,j) > 0$). If v is a vertex of type (b), then

(5.6)
$$
\begin{aligned}
a &= -\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})(P) \cdot \partial_v G_i \\
&= -\operatorname{Ln}(\partial \operatorname{lg}_{V_i(P)})(P) \cdot \lambda(e,i,j) \operatorname{vrs}(p_i - v) - \operatorname{Ln}(\partial \operatorname{lg}_{V_j(P)})(P) \cdot \lambda(e,j,i) \operatorname{vrs}(p_j - v).
\end{aligned}
$$

From Lemma 3.6(i) we conclude that $a \leq 0$, and the inequality is strict if either $p_i \neq \mathrm{CC}(V_i)$ or $p_j \neq \mathrm{CC}(V_j)$. The same conclusion can be derived if v is a vertex of type (a). Therefore, $\max \widetilde{\mathcal{L}}_{-\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})}\mathcal{H}_{\mathrm{DC}}(P) \leq 0$ or $\widetilde{\mathcal{L}}_{-\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})}\mathcal{H}_{\mathrm{DC}}(P) = \emptyset$. Now, resorting to the LaSalle Invariance Principle (Theorem 2.7), we deduce that the solution $P : [0, +\infty) \to Q^n$ starting from P_0 converges to the largest weakly invariant set contained in $\overline{Z}_{-\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}}), \mathcal{H}_{\mathrm{DC}}} \cap \mathcal{H}_{\mathrm{DC}}^{-1}(\leq \mathcal{H}_{\mathrm{DC}}(P_0), P_0) \cap Q^n$.

Let us see that $Z_{-\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}}), \mathcal{H}_{\mathrm{DC}}} \cap Q^n$ is equal to $A_{\mathrm{DC}}(Q)$. Take a configuration $P \in A_{\mathrm{DC}}(Q)$. Then $\operatorname{Ln}(\partial \operatorname{lg}_{V_i(P)})(P) = 0$ if $i \in I(P)$, and $\pi_i(\zeta) = 0$ if $i \notin I(P)$, for any $\zeta \in \partial \mathcal{H}_{\mathrm{DC}}(P)$ (cf. Proposition 4.6). Consequently, $0 = -\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})(P) \cdot \zeta$, for all $\zeta \in \partial \mathcal{H}_{\mathrm{DC}}(P)$, and so $0 \in \widetilde{\mathcal{L}}_{-\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}})}\mathcal{H}_{\mathrm{DC}}(P)$. Therefore, $A_{\mathrm{DC}}(Q) \subset Z_{-\operatorname{Ln}(\partial \operatorname{lg}_{\mathcal{V}}), \mathcal{H}_{\mathrm{DC}}}$.

Now, consider $P \in Z_{-\operatorname{Ln}(\partial \lg_{\mathcal{V}}),\mathcal{H}_{\mathrm{DC}}}$. Then $0 \in \widetilde{\mathcal{L}}_{-\operatorname{Ln}(\partial \lg_{\mathcal{V}})}\mathcal{H}_{\mathrm{DC}}(P)$, that is, $0 = -\operatorname{Ln}(\partial \lg_{\mathcal{V}})(P) \cdot \zeta$, for all $\zeta \in \partial\mathcal{H}_{\mathrm{DC}}(P)$. If P is nondegenerate, then we deduce from (5.6) and Lemma 3.6 that all the active generators are centered, i.e., $P \in A_{\mathrm{DC}}(Q)$. If P is degenerate, then consider a degenerate vertex v where the value of $\mathcal{H}_{\mathrm{DC}}(P)$ is attained. For simplicity, we deal with the case where v is contained in an edge e of Q (the case $v \in \operatorname{int}(Q)$ is treated analogously). From Lemma 4.5 we know that there exist generators p_i, p_j determining v on opposite sides of l, the orthogonal line to the edge e passing through v. From (5.6) and Lemma 3.6 we deduce that both p_i and p_j are centered. Now, for each generator p_k with $v \in V_k$ in the same side of l as p_i (respectively, p_j), we consider the triplet (e, j, k) (respectively, (e, i, k)). Again resorting to (5.6) and Lemma 3.6, we conclude that p_k is also centered. Finally, if a generator p_k with $v \in V_k$ is such that $p_k \in l$, any of the triplets (e, j, k) or (e, i, k) can be invoked in a similar argument to ensure that p_k is centered. Therefore, $P \in A_{\mathrm{DC}}(Q)$, and hence $(Z_{-\operatorname{Ln}(\partial \lg_{\mathcal{V}}),\mathcal{H}_{\mathrm{DC}}} \cap Q^n) \subset A_{\mathrm{DC}}(Q)$. □

PROPOSITION 5.8. *For the dynamical system* (5.5)*, the generators' location* $P = (p_1, \dots, p_n)$ *converges asymptotically to the largest weakly invariant set contained in the closure of* $A_{\mathrm{SP}}(Q) = \{P \in Q^n \mid i \in I(P) \implies p_i \in \operatorname{IC}(V_i)\}$.

Remark 5.9. The sets $A_{\mathrm{DC}}(Q)$ and $A_{\mathrm{SP}}(Q)$ are not closed in general. If $\dim Q = 1$, then it can be seen that they indeed are. In higher dimensions one can find sequences $\{P_k \in Q^n \mid k \in \mathbb{N}\}$ in these sets which converge to configurations P where not all active generators are centered.

5.3. Distributed dynamical systems based on geometric centering. Here, we propose alternative distributed dynamical systems for the multicenter functions. Our design is directly inspired by the results in Theorems 4.7 and 4.8 on the critical points of the multicenter functions $\mathcal{H}_{\mathrm{DC}}$ and $\mathcal{H}_{\mathrm{SP}}$. For $i \in \{1, \dots, n\}$, consider the dynamical systems

$$\dot{p}_i = \operatorname{CC}(V_i) - p_i, \tag{5.7}$$

$$\dot{p}_i \in \operatorname{IC}(V_i) - p_i. \tag{5.8}$$

Alternatively, we may write $\dot{P} = \operatorname{CC}(\mathcal{V}(P)) - P$ and $\dot{P} \in \operatorname{IC}(\mathcal{V}(P)) - P$. Note that both systems are Voronoi-distributed. Also, note that the vector field (5.7) is continuous, since the circumcenter of a polygon depends continuously on the location of its vertexes, and the location of the vertexes of the Voronoi partition depends continuously on the location of the generators; see [34]. However, (5.8) is a differential inclusion, since the incenter sets may not be singletons. By Lemma 2.5, the existence of solutions to (5.8) is guaranteed by the following result.

PROPOSITION 5.10. *Consider the set-valued map* $\operatorname{IC}(\mathcal{V}) - \operatorname{Id} : Q^n \to 2^{(\mathbb{R}^2)^n}$ *given by* $P \mapsto \operatorname{IC}(\mathcal{V}(P)) - P$. *Then* $\operatorname{IC}(\mathcal{V}) - \operatorname{Id}$ *is upper semicontinuous with nonempty, compact, and convex values.*

Having established the existence of solutions, one can also see that the compact set Q^n is strongly invariant for the vector field $\operatorname{CC}(\mathcal{V}) - \operatorname{Id}$ and for the differential inclusion $\operatorname{IC}(\mathcal{V}) - \operatorname{Id}$. Next, we characterize the asymptotic convergence of the dynamical systems under study.

PROPOSITION 5.11. *For the dynamical system* (5.7) *(respectively,* (5.8)*), the generators' location* $P = (p_1, \dots, p_n)$ *converges asymptotically to the largest weakly invariant set contained in the closure of* $A_{\mathrm{DC}}(Q)$ *(respectively, in the closure of* $A_{\mathrm{SP}}(Q)$*).*

Proof. The proof of this result is parallel to the proof of Proposition 5.7. The sequence of steps is the same as before, though now one resorts to Lemma 3.6(ii) and Lemma 3.6(iv). The only additional observation is that when computing the set-valued Lie derivative for (5.8), one has that $a \in \widetilde{\mathcal{L}}_{\mathrm{IC}(\mathcal{V})-\mathrm{Id}}\mathcal{H}_{\mathrm{SP}}(P)$ if and only if there exists $x \in \mathrm{IC}(\mathcal{V}(P))$ such that $a = (x - P) \cdot \zeta$, for any $\zeta \in \partial\mathcal{H}_{\mathrm{SP}}(P)$. The application of Lemma 3.6 guarantees that $a \geq 0$ and that the inequality is strict if any of the active generators is not in its corresponding incenter set. $\square$

5.4. Simulations. To illustrate the performance of the distributed coordination algorithms, we include some simulation results. The algorithms are implemented in `Mathematica` as a single centralized program. We compute the bounded Voronoi diagram of a collection of points using the `Mathematica` package `ComputationalGeometry`. We compute the circumcenter of a polygon via the algorithm in [43] and the incenter set via the `LinearProgramming` solver in `Mathematica`. Measuring displacements in meters, we consider the domain determined by the vertexes

$$\{(0,0),(2.5,0),(3.45,1.5),(3.5,1.6),(3.45,1.7),(2.7,2.1),(1.,2.4),(.2,1.2)\}.$$

In Figures 5.2 and 5.3, we illustrate the performance of the dynamical systems (5.4) and (5.7), respectively, minimizing the multi-circumcenter function $\mathcal{H}_{\mathrm{DC}}$. In Figures 5.4 and 5.5, we illustrate the performance of the dynamical systems (5.5) and (5.8), respectively, maximizing the multi-incenter function $\mathcal{H}_{\mathrm{SP}}$. Observing the final configurations in the four figures, one can verify, visually and numerically, that the active generators are asymptotically centered as forecast by our analysis.

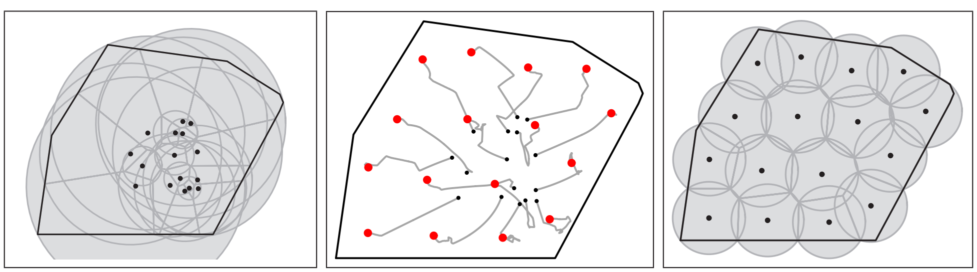


FIG. 5.2. *"Toward the furthest" algorithm for* 16 *generators in a convex polygonal environment. The left (respectively, right) figure illustrates the initial (respectively, final) locations and Voronoi partition. The central figure illustrates the network evolution. After* 2 *seconds, the multicenter function is approximately* .39504 *meters.*

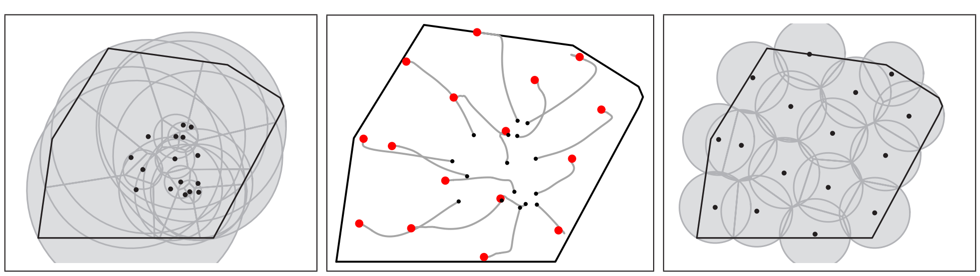


FIG. 5.3. *"Move-toward-the-circumcenter" algorithm for* 16 *generators in a convex polygonal environment. The left (respectively, right) figure illustrates the initial (respectively, final) locations and Voronoi partition. The central figure illustrates the network evolution. After* 20 *seconds, the multicenter function is approximately* 0.43273 *meters.*

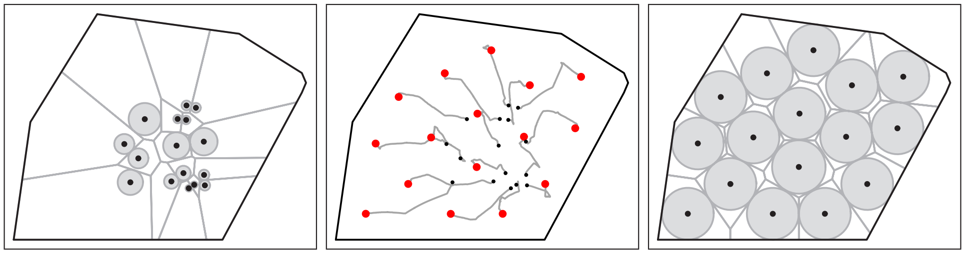


FIG. 5.4. *"Away-from-closest" algorithm for 16 generators in a convex polygonal environment. The left (respectively, right) figure illustrates the initial (respectively, final) locations and Voronoi partition. The central figure illustrates the network evolution. After 2 seconds, the multicenter function is approximately .26347 meters.*

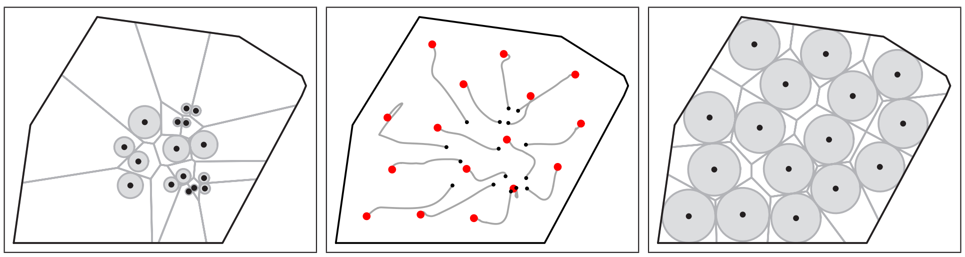


FIG. 5.5. *"Move-toward-the-incenter" algorithm for 16 generators in a convex polygonal environment. The left (respectively, right) figure illustrates the initial (respectively, final) locations and Voronoi partition. The central figure illustrates the network evolution. After 20 seconds, the multicenter function is approximately .2498 meters.*

6. Conclusions. We have introduced two multicenter functions that provide quality-of-service measures for mobile networks. We have shown that both functions are globally Lipschitz, and we have computed their generalized gradients. Furthermore, under certain technical conditions, we have characterized via nonsmooth analysis their critical points as center Voronoi configurations and as solutions of disk-covering and sphere-packing problems. We have also considered various algorithms that extremize the multicenter functions. First, we considered the nonsmooth gradient flows induced by their respective generalized gradients. Second, we devised a novel strategy based on the generalized gradients of the 1-center functions of each generator. Third, we introduced and characterized a geometric centering strategy with resemblances to the classical Lloyd algorithm. We have unveiled the remarkable geometric interpretations of these algorithms, discussed their distributed character, and analyzed their asymptotic behavior using nonsmooth stability analysis.

In summary, this paper has shown the relevance of tools from geometric optimization, nonsmooth analysis, and nonsmooth stability in motion coordination problems. As discussed in the introduction, the concepts adopted in this paper are being developed in a number of directions. Future directions of research include the following specific problems: (i) how to sharpen the asymptotic convergence results for the proposed dynamical systems (e.g., proving that all generators will asymptotically be centered), (ii) how to extend the analysis to the setting of convex polytopes in $\mathbb{R}^N$, for $N > 2$, (iii) in what sense the proposed multi-circumcenter and the multi-incenter problems can be shown to be dual. From a broader viewpoint, we envision that the formal analysis of interaction laws in multi-agent systems will continue to prove fertile ground for research in motion coordination and cooperative control.

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