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Dynamic Vehicle Routing with Moving Demands – Part II: High speed demands or low arrival rates

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***Abstract*— In the companion paper we introduced a vehicle routing problem in which demands arrive via a temporal Poisson process, and uniformly distributed along a line segment. Upon arrival, the demands move perpendicular to the line with a fixed speed. A service vehicle, with speed greater than that of the demands, seeks to provide service by reaching the location of each mobile demand. In this paper we study a first-come-first-served (FCFS) policy in which the service vehicle serves the demands in the order in which they arrive. When the demand arrival rate is very low, we show that the FCFS policy can be used to minimize the expected time, or the worst-case time, spent by a demand before being served. We determine necessary and sufficient conditions on the arrival rate of the demands (as a function of the problem parameters) for the stability of the FCFS policy. When the demands are much slower than the service vehicle the necessary and sufficient conditions become equal. We also show that in the limiting case when the demands move nearly as fast as the service vehicle; (i) the demand arrival rate must tend to zero; (ii) every stabilizing policy must service the demands in the order in which they arrive, and; (iii) the FCFS policy is the optimal policy.**

I. INTRODUCTION

In companion paper [1] we introduced a vehicle routing problem in which demands arrive via a temporal Poisson arrival process with rate λ at a uniformly random location on a line segment of length W. The demands move in a fixed direction perpendicular to the line with fixed speed $v < 1$. A service vehicle, modeled as a unit speed first-order integrator, seeks to serve these mobile demands by reaching each demands location. The goal is to determine conditions on the arrival rate λ, which ensure stability of the system (i.e., ensure a finite expected time spent by a demand in the environment). We refer the reader to [1] for related work and motivation. In [1] we showed that to ensure the existence of a stabilizing policy, we must have $\lambda \leq 4/vW$. We proposed a service policy which relied on the computation of the translational traveling salesperson path (t-TSP) through unserviced demands, and showed that for small v the policy ensures stability for all λ's up to a constant factor of the necessary condition.

We now focus on the case when the arrival rate is low (if v is close to, but strictly smaller than one, we will see that this is a necessary condition for stability). For this case we propose a first-come-first-served (FCFS) policy; such policies are common in classical queuing theory [2], [3]. In the regime where v is fixed and λ tends to zero, the problem becomes one of providing optimal coverage. Related works include geometric location problems such as [4], and [5], where given a set of static demand points in the plane, the goal is to find supply points that minimize a cost function of the distance from each demand to its nearest supply point. The authors in [6] study the problem of deploying robots into a region so as to provide optimal coverage of the region.

The contributions of this paper can be summarized as follows. We study a first-come-first-served (FCFS) policy in which demands are served in the order in which they arrive, and when the environment contains no outstanding demands, the vehicle moves to a location which minimizes the expected (or worst-case) travel time to a demand. We show that for fixed v, as the demand arrival rate λ tends to zero, the FCFS policy is the optimal policy in terms of minimizing the expected (or worst-case) delay between a demands arrival and its service completion. Next, we determine necessary and sufficient conditions on λ for the stability of the FCFS policy. As $v \to 0^+$, the necessary and sufficient conditions become equal. When v approaches one, we show that: (i) for existence of a stabilizing policy, λ must converge to zero as $1/\sqrt{-\log(1-v)}$, (ii) every stabilizing policy must service the demands in the order in which they arrive, and (iii) the FCFS policy is the optimal policy. When compared to the TSP-based policy introduced in companion paper [1], the FCFS policy has a larger stability region when v is large, but a smaller stability region when v is small. This is summarized in Fig. 1.

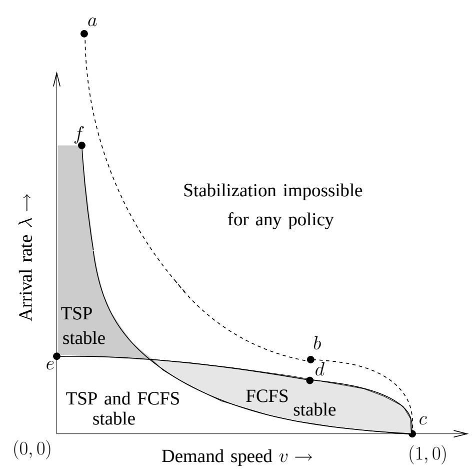

Fig. 1. A summary of stability regions for the TSP-based policy and the FCFS policy. Stable service policies can exist only for the region under the dotted line. Curve b-c is due to Theorem IV.2, curves c-d and d-e are due to Theorem V.1. For curves a-b and c-f, refer to [1].

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This paper is organized as follows: the problem is formalized in Section II. The FCFS policy is introduced in Section II-B. We determine the optimal placement to minimize the expected time in Section III-A, and the worst-case time in Section III-B. In Section IV we determine a necessary condition for stability as v tends to one, and in Section V we determine a sufficient condition for the stability of the FCFS policy. In Section VI, we present simulation results for the FCFS and its comparison with the TSP-based policy.

II. Problem Formulation and Service Policy

We consider a single service vehicle that seeks to service mobile demands that arrive via a spatio-temporal process on a line segment with length W along the x-axis, termed the *generator*. The vehicle is modeled as a first-order integrator with speed upper bounded by one. The demands arrive uniformly distributed on the generator via a temporal Poisson process with intensity $\lambda > 0$, and move with constant speed $v < 1$ along the positive y-axis. We assume that once the vehicle reaches a demand, the demand is served instantaneously. The vehicle is assumed to have unlimited fuel and demand servicing capacity.

We define the environment as $\mathcal{E} := [0, W] \times \mathbb{R}_{\geq 0} \subset \mathbb{R}^2$, and let $\mathbf{p}(t) = [X(t), Y(t)]^T \in \mathcal{E}$ denote the position of the service vehicle at time t. Let $\mathcal{Q}(t) \subset \mathcal{E}$ denote the set of all demand locations at time t, and $n(t)$ the cardinality of $\mathcal{Q}(t)$. Servicing of a demand $\mathbf{q}_i \in \mathcal{Q}$ and removing it from the set $\mathcal{Q}$ occurs when the service vehicle reaches the location of the demand. A static feedback control policy for the system is a map $\mathcal{P} : \mathcal{E} \times 2^{\mathcal{E}} \to \mathbb{R}^2$, assigning a commanded velocity to the service vehicle as a function of the current state of the system: $\dot{\mathbf{p}}(t) = \mathcal{P}(\mathbf{p}(t), \mathcal{Q}(t))$. Let D_i denote the time that the ith demand spends within the set $\mathcal{Q}$, i.e., the delay between the generation of the ith demand and the time it is serviced. The policy $\mathcal{P}$ is *stable* if under its action, $\lim_{i \to +\infty} \mathbb{E}[D_i] < +\infty$, i.e., the steady state expected delay is finite. Equivalently, the policy $\mathcal{P}$ is stable if under its action,

$$\lim_{t \to +\infty} \mathbb{E}[n(t)] < +\infty,$$

that is, if the vehicle is able to service demands at a rate that is—on average—at least as fast as the rate at which new demands arrive. In what follows, our goal is to *design stable control policies* for the system.

A. Constant Bearing Control

In this paper we will use the following result on catching a demand in minimum time.

Definition II.1 (Constant bearing control) *Given the locations $\mathbf{p} := (X, Y) \in \mathcal{E}$ and $\mathbf{q} := (x, y) \in \mathcal{E}$ at time t of the service vehicle and a demand, respectively, with the demand moving in the positive y-direction with constant speed v, the motion of the vehicle towards the point $(x, y + vT)$, where*

$$T(\mathbf{p}, \mathbf{q}) := \frac{\sqrt{(1 - v^2)(X - x)^2 + (Y - y)^2}}{1 - v^2} - \frac{v(Y - y)}{1 - v^2}, \tag{1}$$

with unit speed is defined as the constant bearing control.

Constant bearing control is illustrated in Fig. 2.

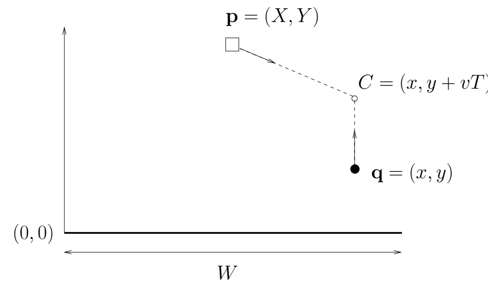

Fig. 2. Constant bearing control. The vehicle moves towards the point $C := (x, y + vT)$, where x, y, v and T are as per Definition II.1, to reach the demand.

The following result on constant bearing control is established in [7].

Proposition II.2 (Minimum time control, [7]) *The constant bearing control is the minimum time control for the vehicle to reach the demand.*

B. The First-Come-First-Served (FCFS) Policy

We are now ready introduce the FCFS policy, which will be the focus of this paper. In this policy the service vehicle uses constant bearing control and services the demands in the order in which they arrive. If the environment contains no demands, the vehicle moves to the location (X^*, Y^*) which minimizes the expected, or worst-case, time to catch the next demand to arrive. We can state this policy as follows.

The FCFS policy

Assumes: Given the optimal location $(X^*, Y^*) \in \mathcal{E}$.
if no unserviced demands in $\mathcal{E}$ **then**
  Move toward (X^*, Y^*) until the next demand arrives.
else
  Move using the constant bearing control to service the furthest demand from the generator.
Repeat.

Fig. 3 illustrates an instance of the FCFS policy. The

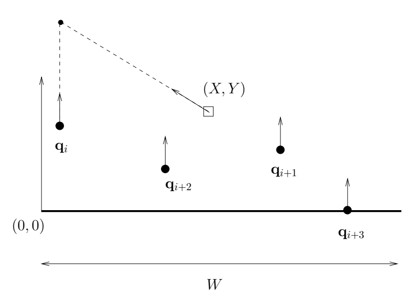

Fig. 3. The FCFS policy. The vehicle services the demands in the order of their arrival in the environment, using the constant bearing control.

first question is, how do we compute the optimal position (X^*, Y^*)? This will be answered in the following section.

III. Optimal Vehicle Placement

In this section we study the FCFS policy when $v < 1$ is fixed and $\lambda \to 0^+$. In this regime stability is not an issue, as demands arrive very rarely, and the problem becomes one of optimally placing the service vehicle (ie., determining (X^*, Y^*) in the statement of the FCFS policy). We determine placements that minimize the expected time and the worst-case time.

A. Minimizing the Expected Time

We seek to place the vehicle at location that minimizes the expected time to service a demand once it appears on the generator. Demands appear at uniformly random positions on the generator and the vehicle uses the constant bearing control to reach the demand. Thus, the expected time to reach a demand generated at position $\mathbf{q} = (x, 0)$ from vehicle position $\mathbf{p} = (X, Y)$ is given by

$$\mathbb{E}\left[T(\mathbf{p},\mathbf{q})\right] = \frac{1}{W(1-v^2)} \int_0^W \left(\sqrt{(1-v^2)(X-x)^2 + Y^2} - vY\right) dx. \tag{2}$$

The following lemma characterizes the way in which this expectation varies with the position $\mathbf{p}$.

Lemma III.1 (Properties of the expected time) *(i) The expected time $\mathbb{E}[T(\mathbf{p},\mathbf{q})]$ is convex in $\mathbf{p}$, for all $\mathbf{p} \in [0, W] \times \mathbb{R}_{>0}$. (ii) There exists a unique point $\mathbf{p}^* := (W/2, Y^*) \in \mathbb{R}^2$ that minimizes $\mathbb{E}[T(\mathbf{p},\mathbf{q})]$.*

Proof: For part (i), it suffices to show that the integrand in Eq. (2), $T(\mathbf{p},\mathbf{q})$ is convex. To do this we compute the Hessian of $T((X,Y),(0,x))$ with respect to X and Y. Thus, for $Y > 0$,

$$\begin{bmatrix} \frac{\partial^2 T}{\partial X^2} & \frac{\partial^2 T}{\partial X \partial Y} \\ \frac{\partial^2 T}{\partial Y \partial X} & \frac{\partial^2 T}{\partial Y^2} \end{bmatrix} = \frac{\begin{bmatrix} Y^2 & Y(X-x) \\ Y(X-x) & (X-x)^2 \end{bmatrix}}{\left((1-v^2)(X-x)^2 + Y^2\right)^{3/2}} \geq 0.$$

The Hessian is positive semi-definite, which implies that $T(\mathbf{p},\mathbf{q})$ is convex in $\mathbf{p}$ for each $\mathbf{q} = (0, x)$.

For part (ii), observe that since demands are uniformly randomly generated on the interval $[0, W]$, X^* has a unique minimum of $W/2$. If we can show that $\mathbb{E}[T(\mathbf{p},\mathbf{q})]$ is strictly convex in Y when $X = W/2$, then we have proved part (ii). From the $\partial^2 T/\partial Y^2$ term of the Hessian we see that $T(\mathbf{p},\mathbf{q})$ is strictly convex for all $x \neq W/2$. But, letting $\mathbf{p} = (W/2, Y)$ and $\mathbf{q} = (0, x)$ we can write

$$\mathbb{E}\left[T(\mathbf{p},\mathbf{q})\right] = \frac{1}{W(1-v^2)} \int_{x \in [0,W] \setminus \{W/2\}} T(\mathbf{p},\mathbf{q}) dx.$$

The integrand is strictly convex for all $x \in [0, W] \setminus \{W/2\}$, implying $\mathbb{E}[T(\mathbf{p},\mathbf{q})]$ is strictly convex on the line $X = W/2$, and the existence of a unique minimizer $(W/2, Y^*)$. ■

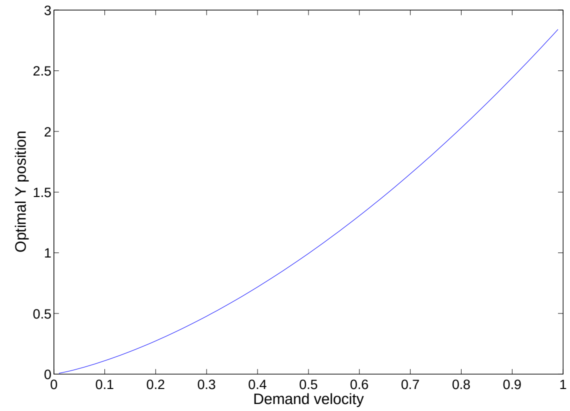

Fig. 4. The Y position of the service vehicle which minimizes the expected distance to a demand, as a function of v. In this plot the generator has length $W = 10$.

Lemma III.1 tells us that there exists a unique point $\mathbf{p}^* := (X^*, Y^*)$ which minimizes the expected travel time. In addition, we know that $X^* = W/2$. Obtaining a closed form expression for Y^* does not appear to be possible. Computing the integral in Eq. (2), with $X = W/2$, one can obtain

$$\mathbb{E}\left[T(\mathbf{p},\mathbf{q})\right] = \frac{Y}{a}\left(\frac{1}{2}\sqrt{1 + \frac{aW^2}{4Y^2}} - \frac{Y}{\sqrt{a}W} \log\left(\sqrt{1 + \frac{aW^2}{4Y^2}} - \sqrt{\frac{aW^2}{4Y^2}}\right) - v\right),$$

where $a = 1 - v^2$. For each value of v and W, this convex expression can be easily numerically minimized over Y, to obtain Y^*. A plot of Y^* as a function of v for $W = 10$ is shown in Fig. 4.

For the optimal position $\mathbf{p}^*$, the expected delay between a demand's arrival and its service completion is

$$D^* := \mathbb{E}\left[T(\mathbf{p}^*, (0, x))\right].$$

Thus, a lower bound on the steady-state expected delay of any policy is D^*. We now characterize the steady-state expected delay of the FCFS policy D_{FCFS}, as λ tends to zero.

Theorem III.2 (FCFS optimality) *Fix any $v < 1$. Then as $\lambda \to 0^+$, $D_{\text{FCFS}} \to D^*$, and the FCFS policy minimizes the expected time to service a demand.*

Proof: We have shown how to compute the position $\mathbf{p}^* := (X^*, Y^*)$ which minimizes Eq. (2). Thus, if the vehicle is located at $\mathbf{p}^*$, then the expected time to service the demand is minimized. But, as $\lambda \to 0^+$, the probability that demand $i+1$ arrives before the vehicle completes service of demand i and returns to $\mathbf{p}^*$ tends to zero. Thus, the FCFS policy is optimal as $\lambda \to 0^+$. ■

B. Minimizing the Worst-Case Time

In the previous section we looked at the expected time to service a demand. This was the metric studied in the companion paper, and will be the metric of interest in

Section V when we study the FCFS policy for $\lambda > 0$. However, another metric that can be used to determine (X^*, Y^*) is the worst-case time to service a demand.

Lemma III.3 (Optimal placement for worst-case) *The location (X^*, Y^*) that minimizes the worst-case time to service the demand is $(W/2, vW/2)$. At this location, in the worst-case, the vehicle moves a distance of $W/2$ horizontally.*

Proof: Assume that $X^* > W/2$ (or $X^* < W/2$). Then the worst-case time is achieved when a target appears at the location $(0, 0)$ (resp. $(W, 0)$). One can then move the vehicle parallel to the x-axis such that $X = W/2$ and thus decrease the time taken to service the same target as compared to its previous position. This is a contradiction. Hence, $X^* = W/2$.

Applying the constant bearing control for the vehicle, we obtain the worst case system time as

$$T_w(Y) = \frac{1}{1-v^2}\Big(\sqrt{(1-v^2)W^2/4 + Y^2} - vY\Big).$$

Since T_w is solely a function of Y, to minimize T_w, we must have

$$\left.\frac{dT_w(Y)}{dY}\right|_{Y^*} = 0 \Rightarrow \frac{Y^*}{\sqrt{(1-v^2)W^2/4 + (Y^*)^2}} - v = 0.$$

Simplifying, we get $Y^* = vW/2$. ■

Using an argument identical to that in the proof of Theorem III.2 we have the following: For fixed $v < 1$, and as $\lambda \to 0^+$, the FCFS policy, with $(X^*, Y^*) = (W/2, vW/2)$, minimizes the worst-case time to service a demand.

IV. Necessary Conditions for Stability

In the previous section, we studied the case of fixed v and low $\lambda \to 0^+$. In this section we look at the problem when $\lambda > 0$, and determine necessary conditions on the magnitude of λ that ensure the FCFS policy remains stable. We also determine a necessary condition on λ for the stability of any policy as $v \to 1^-$, and establish the optimality of the FCFS policy. We begin with the following.

Lemma IV.1 (Special case of equal speeds) *For $v = 1$ there does not exist a stabilizing policy.*

Proof: When $v = 1$, given a vehicle location $\mathbf{p} := (X, Y)$ and a demand location with initial location $\mathbf{q} := (x, y)$, the minimum time T in which the vehicle can reach the demand is given by

$$T(\mathbf{p}, \mathbf{q}) = \frac{(X-x)^2 + (Y-y)^2}{2(Y-y)} \quad \text{if } Y > y, \qquad (3)$$

and is undefined if $Y \leq y$. Thus, a demand can only be reached if the vehicle is above the demand. From Eq. (3) we see that a necessary stability condition is that a demand's y-coordinate never exceeds that of the service vehicle. The only policy that can ensure this is the FCFS policy. Thus, we prove the result by computing the expected time to travel between demands using the FCFS policy, and show that for every $\lambda > 0$, more than one demand arrives during this travel time. To do this, assume there are many outstanding demands below the service vehicle, and none above. Suppose the service vehicle completed the service of demand i at time t_i and position $(x_i(t_i), y_i(t_i))$. Let us compute the expected time to reach demand $i+1$, with location $(x_{i+1}(t_i), y_{i+1}(t_i))$. Since arrivals are Poisson it follows that $y_i(t_i) > y_{i+1}(t_i)$. To simplify notation we define $\Delta x = |x_i(t_i) - x_{i+1}(t_i)|$ and $\Delta y = y_i(t_i) - y_{i+1}(t_i)$. Then, from Eq. (3)

$$T(\mathbf{q}_i, \mathbf{q}_{i+1}) = \frac{\Delta x^2 + \Delta y^2}{2\Delta y} = \frac{1}{2}\left(\frac{\Delta x^2}{\Delta y} + \Delta y\right).$$

Taking expectation and noting that Δx and Δy are independent

$$\mathbb{E}\left[T(\mathbf{q}_i, \mathbf{q}_{i+1})\right] = \frac{1}{2}\Big(\mathbb{E}\left[\Delta x^2\right]\mathbb{E}\left[\tfrac{1}{\Delta y}\right] + \mathbb{E}\left[\Delta y\right]\Big).$$

Now, $\mathbb{E}\left[\Delta y\right] = 1/\lambda$, $\mathbb{E}\left[\Delta x^2\right]$ is a positive constant independent of λ and

$$\mathbb{E}\left[\tfrac{1}{\Delta y}\right] = \int_{y=0}^{+\infty} \frac{1}{y}\lambda e^{-\lambda y} dy = +\infty.$$

Thus $\mathbb{E}\left[T(\mathbf{q}_i, \mathbf{q}_{i+1})\right] = +\infty$, and for every $\lambda > 0$,

$$\lambda \mathbb{E}\left[T(\mathbf{q}_i, \mathbf{q}_{i+1})\right] = +\infty,$$

implying that an infinite number of demands arrive in the time required to service one. ■

Next we look at the FCFS policy and give a necessary condition for its stability.

Theorem IV.2 (Necessary stability condition for FCFS) *A necessary condition for the stability of the FCFS policy is*

$$\lambda \leq \begin{cases} \dfrac{3}{W}, & \text{for } v \leq \frac{4}{5}, \\[2ex] \dfrac{3\sqrt{2v}}{W\sqrt{(1+v)\left(A - \log\left(\frac{\sqrt{1-v^2}}{v}\right)\right)}}, & \text{o.w.}, \end{cases}$$

where $A \approx 0.62$.

Proof: Suppose the service vehicle completed the service of demand i at time t_i at position $(x_i(t_i), y_i(t_i))$ and demand $i+1$ is located at $(x_{i+1}(t_i), y_{i+1}(t_i))$. Also define $\Delta x = |x_i(t_i) - x_{i+1}(t_i)|$ and $\Delta y = y_i(t_i) - y_{i+1}(t_i)$. For $v < 1$, the travel time between demands is given by

$$T = \frac{1}{1-v^2}\left(\sqrt{(1-v^2)\Delta x^2 + \Delta y^2} - v\Delta y\right).$$

Observe that the function T is convex in Δx and Δy. So by applying Jensen's inequality, we obtain

$$\mathbb{E}\left[T\right] \geq \frac{1}{1-v^2}\Big(\sqrt{(1-v^2)(\mathbb{E}\left[\Delta x\right])^2 + (\mathbb{E}\left[\Delta y\right])^2} - v\mathbb{E}\left[\Delta y\right]\Big).$$

Substituting the expressions for the expected values,

$$\mathbb{E}\left[T\right] \geq \frac{1}{1-v^2}\Big(\sqrt{(1-v^2)\frac{W^2}{9} + \frac{v^2}{\lambda^2}} - \frac{v^2}{\lambda}\Big).$$

From the necessary condition for stability, (cf. [8]), we must have

$$\lambda \mathbb{E}[T] \leq 1 \Leftrightarrow \lambda \frac{1}{1-v^2}\Big(\sqrt{\frac{(1-v^2)W^2}{9}+\frac{v^2}{\lambda^2}}-\frac{v^2}{\lambda}\Big) \leq 1.$$

On simplifying, we obtain

$$\lambda \leq \frac{3}{W}. \tag{4}$$

This provides a good necessary condition for low v, but we will be able to obtain a much better necessary condition for large v.

Since T is convex in Δx, we can apply Jensen's inequality to write

$$\mathbb{E}[T|\Delta y] \geq \frac{1}{1-v^2}\left(\sqrt{(1-v^2)W^2/9+\Delta y^2}-v\Delta y\right), \tag{5}$$

where $\mathbb{E}[\Delta x] = W/3$. Now, the random variable Δy is distributed exponentially with parameter v/λ and probability density function

$$f(y) = \frac{v}{\lambda}\mathrm{e}^{-\lambda y/v}.$$

Un-conditioning Eq. (5) on Δy we obtain

$$\mathbb{E}[T] = \int_0^{+\infty}\mathbb{E}[T|y]f(y)dy \geq \frac{v}{\lambda(1-v^2)}\int_0^{+\infty}\left(\sqrt{\frac{(1-v^2)W^2}{9}+y^2}-vy\right)\mathrm{e}^{-\lambda y/v}dy. \tag{6}$$

The right hand side can be evaluated using the software Maple® and equals

$$\frac{\pi W}{2\cdot 3\sqrt{1-v^2}}\left[\mathbf{H}_1\left(\frac{\lambda W\sqrt{1-v^2}}{3v}\right) - \mathbf{Y}_1\left(\frac{\lambda W\sqrt{1-v^2}}{3v}\right)\right] - \frac{v^2}{\lambda(1-v^2)},$$

where $\mathbf{H}_1(\cdot)$ is the 1st order Struve function and $\mathbf{Y}_1(\cdot)$ is 1st order Bessel function of the 2nd kind. We can perform a Taylor series expansion of the function $\mathbf{H}_1(z)-\mathbf{Y}_1(z)$ about $z=0$ to obtain

$$\mathbf{H}_1(z)-\mathbf{Y}_1(z) \geq \frac{1}{\pi}\left(\frac{2}{z}+Az-z\log(z)\right),$$

where $A = 1/2+\log(2)-\gamma \approx 0.62$. Using the above expression, Eq. (6) can be written as

$$\mathbb{E}[T] \geq \frac{v}{\lambda(1+v)}+\frac{\lambda W}{18v}\left(A-\log\left(\frac{\lambda W\sqrt{1-v^2}}{3v}\right)\right),$$

where we have used the fact that

$$\frac{v}{\lambda(1-v)^2}-\frac{v^2}{\lambda(1-v^2)} = \frac{v}{\lambda(1+v)}.$$

To obtain a stability condition on λ we wish to remove λ from the $\log$ term. To do this, note that from Eq. (4) we have $\lambda W/3 < 1$, and thus

$$\begin{aligned}
&\mathbb{E}[T] \\
&\geq \frac{v}{\lambda(1+v)}+\frac{\lambda W}{18v}\left(A-\log\frac{W\lambda}{3}-\log\frac{W\sqrt{1-v^2}}{3v}\right), \\
&\geq \frac{v}{\lambda(1+v)}+\frac{\lambda W}{18v}\left(A-\log\left(\frac{\sqrt{1-v^2}}{v}\right)\right)
\end{aligned}$$

For stability we require that $\lambda\mathbb{E}[T] \leq 1$, from which we see that a necessary condition for stability is

$$\begin{aligned}
\frac{\lambda^2 W}{18v}\left(A-\log\left(\frac{\sqrt{1-v^2}}{v}\right)\right) &\leq 1-\frac{v}{1+v} \\
&= \frac{1}{1+v}.
\end{aligned}$$

Solving for λ when $A > \log(\sqrt{1-v^2}/v)$ we obtain that

$$\lambda \leq \frac{3\sqrt{2v}}{W\sqrt{(1+v)\left(A-\log\left(\frac{\sqrt{1-v^2}}{v}\right)\right)}}, \tag{7}$$

The condition $A > \log(\sqrt{1-v^2}/v)$, implies that the above bound holds for all $v < 1/\sqrt{1+\mathrm{e}^{2A}} \approx 1/2$. We now have two bounds; Eq. (4) which holds for all $v < 1$, and Eq. (7) which holds for $v > 1/2$. The final step is to determine the values of v for which each bound is active. To do this, we set the right-hand side of Eq. (4) equal to the RHS of Eq. (7) and solve for v^* to obtain $v^* \approx 0.8$. Thus, the necessary condition for stability is given by Eq. (4) when $v \leq 0.8$, and by Eq. (7) when $v > 0.8$. ■

The previous theorem shows that although λ must go to zero as $v \to 1^-$, it can go very slowly to 0. In fact, the necessary condition states that λ goes to zero as

$$\frac{1}{\sqrt{-\log(1-v)}}.$$

This goes to zero more slowly than any polynomial in $(1-v)$.

Finally, we prove that the FCFS is the best policy as $v \to 1^-$.

Theorem IV.3 (Optimality of FCFS) *For the limiting case as $v \to 1^-$;*

(i) *every stabilizing policy must serve the demands in the order in which they arrive;*
(ii) *the stability condition in Theorem IV.2 is necessary for all policies; and*
(iii) *no policy can provide a lower expected delay than the FCFS policy.*

Proof: We begin by proving Part (i). Parts (ii) and (iii) are direct consequences. Suppose there is a policy P that is not does not serve demands FCFS, but can stabilize the system with

$$\lambda = B(1-v)^p,$$

for some $p > 0$, and $B > 0$. Let t_i be the first instant at which policy P deviates from FCFS. Then, the demand served immediately after i is demand $i+k$ for some $k > 1$. When the vehicle reaches demand $i+k$ at time t_{i+1}, demand $i+1$ has moved above the vehicle. To ensure stability, demand $i+1$ must eventually be served. The time to travel to demand $i+1$ from any demand $i+j$, where $j > 1$ is

$$\begin{aligned}T(\mathbf{q}_{i+j}, \mathbf{q}_{i+1}) &= \sqrt{\left(\frac{\Delta x}{\sqrt{1-v^2}}\right)^2 + \left(\frac{\Delta y}{1-v^2}\right)^2} + \frac{v\Delta y}{1-v^2} \\ &\geq \frac{\Delta y}{1-v^2} + \frac{v\Delta y}{1-v^2} = \frac{\Delta y}{1-v},\end{aligned}$$

where Δx and Δy are now the minimum of the x and y distances from $\mathbf{q}_{i+j}$ to the $\mathbf{q}_{i+1}$. The random variable Δy is Erlang distributed with shape $j-1 \geq 1$ and rate λ, implying

$$\mathbb{P}[\Delta y \leq c] \leq 1 - \mathrm{e}^{-\lambda c/v}, \quad \text{for each } c > 0.$$

Now, since $\lambda = B(1-v)^p$ as $v \to 1^-$, almost surely $\Delta y > (1-v)^{1/2-p}$. Thus

$$T(\mathbf{q}_{i+j}, \mathbf{q}_{i+1}) \geq (1-v)^{-(p+1/2)},$$

almost surely as $v \to 1^-$. Thus, the expected number of demands that arrive during $T(\mathbf{q}_{i+j}, \mathbf{q}_{i+1})$ is

$$\begin{aligned}\lambda T(\mathbf{q}_{i+j}, \mathbf{q}_{i+1}) &\geq B(1-v)^p(1-v)^{-(p+1/2)} \\ &\geq B(1-v)^{-1/2} \to +\infty,\end{aligned}$$

as $v \to 1^-$. This implies that almost surely the policy P becomes unstable when it deviates from FCFS. Thus, any deviation must occur with probability zero as $v \to 1^-$. Thus, a necessary condition for a policy to stabilize with $\lambda = B(1-v)^p$, is that as $v \to 1^-$, the policy must serve demands in the order in which they arrive. But this holds for every p, and by letting p go to infinity, $B(1-v)^p$ converges to zero for all $v \in (0,1]$. Thus, a non-FCFS policy cannot stabilize the system no matter how quickly $\lambda \to 0^+$ as $v \to 1^-$. Hence, as $v \to 1^-$, every stabilizing policy must serve the demands in the order in which they arrive.

To see parts (ii) and (iii), notice that the definition of the FCFS policy is that it uses the minimum time control (i.e., constant bearing control) to move between demands, thus the necessary stability condition in Theorem IV.2 holds for all policies as $v \to 1^-$. In addition, the FCFS spends the minimum amount of time to travel between demands and thus minimizes the expected delay. ■

V. A SUFFICIENT CONDITION FOR FCFS STABILITY

In Section IV we determined a necessary condition for stability of the FCFS policy. In this section, we will derive the following sufficient condition on the arrival rate that ensures stability for the FCFS policy.

Theorem V.1 (Sufficient stability condition for FCFS) *The FCFS policy is stable if*

$$\lambda < \begin{cases} \frac{3}{W}\sqrt{\frac{1-v}{1+v}}, & \textit{for } v \leq \frac{2}{3}, \\ \frac{\sqrt{12v}}{W\sqrt{(1+v)\left(C - \log\left(\frac{1-v}{v}\right)\right)}}, & \textit{o.w.,}\end{cases}$$

where $C \approx 2.06$.

Proof: We begin with the expression for the time taken for the vehicle from the position $\mathbf{p}$, coinciding with a demand, to reach the next demand at $\mathbf{q}$ using the constant bearing control (cf. Definition II.1). Thus,

$$\begin{aligned}T(\mathbf{p}, \mathbf{q}) &= \frac{\sqrt{(1-v^2)(X-x)^2 + (Y-y)^2}}{1-v^2} - \frac{v(Y-y)}{1-v^2} \\ &\leq \frac{|X-x|}{\sqrt{1-v^2}} + \frac{(Y-y)}{1-v^2},\end{aligned} \tag{8}$$

where we used the inequality $\sqrt{a^2+b^2} \leq |a| + |b|$. Taking expectation,

$$\mathbb{E}[T] \leq \frac{W}{3\sqrt{1-v^2}} + \frac{v}{\lambda(1-v^2)},$$

since the demands are distributed uniformly in the x-direction and Poisson in the y-direction. A sufficient condition for stability is (cf. [8])

$$\lambda\mathbb{E}[T] < 1 \Leftrightarrow \lambda < \frac{3}{W}\sqrt{\frac{1-v}{1+v}}. \tag{9}$$

The upper bound on T given by Eq. (8) is very conservative except for the case when v is very small. Alternatively, taking expected value of T conditioned on Δy, and applying Jensen's inequality to the square-root part, we obtain

$$\mathbb{E}[T|\Delta y] \leq \frac{1}{1-v^2}\left(\sqrt{(1-v^2)W^2/6 + \Delta y^2} - v\Delta y\right),$$

since $\mathbb{E}\left[\Delta x^2\right] = W^2/6$. Following steps which are similar to those between Eq. (5) and Eq. (6), we obtain

$$\begin{aligned}\mathbb{E}[T] \leq \frac{\pi W}{2\cdot\sqrt{6}\sqrt{1-v^2}}&\left[\mathbf{H}_1\left(\frac{\lambda W\sqrt{1-v^2}}{\sqrt{6}v}\right)\right. \\ &\left.-\mathbf{Y}_1\left(\frac{\lambda W\sqrt{1-v^2}}{\sqrt{6}v}\right)\right] - \frac{v^2}{\lambda(1-v^2)},\end{aligned} \tag{10}$$

where $\mathbf{H}_1(\cdot)$ is the 1st order Struve function and $\mathbf{Y}_1(\cdot)$ is 1st order Bessel function of the 2nd kind.

In [9], polynomial approximations have been provided for the Struve and Bessel functions in the intervals $[0,3]$ and $[3,+\infty)$. We seek an upper bound for the right-hand side of (10) when v is sufficiently large, i.e., when the argument of $\mathbf{H}_1(\cdot)$ and $\mathbf{Y}_1(\cdot)$ is small. Thus, from [9]

$$\begin{aligned}&\mathbf{H}_1(z) \leq \frac{z}{2}, \\ &\mathbf{Y}_1(z) \geq \frac{2}{\pi}\left(\frac{z}{2}\log\frac{z}{2} - \frac{1}{z}\right), \text{ for } 0 \leq z \leq 3,\end{aligned}$$

where $z := \lambda W\sqrt{1-v^2}/(\sqrt{6}v)$. Substituting into Eq. (10), we obtain

$$\begin{aligned}\mathbb{E}[T] \leq \frac{\pi W}{2\cdot\sqrt{6}\sqrt{1-v^2}}&\left[\frac{\lambda W\sqrt{1-v^2}}{2\sqrt{6}v} + \frac{2}{\pi}\left(\frac{\sqrt{6}v}{\lambda W\sqrt{1-v^2}}\right.\right. \\ &\left.\left.-\frac{\lambda W\sqrt{1-v^2}}{2\sqrt{6}v}\log\frac{\lambda W\sqrt{1-v^2}}{2\sqrt{6}v}\right)\right] - \frac{v^2}{\lambda(1-v^2)},\end{aligned}$$

which yields

$$\mathbb{E}[T] \le \frac{\lambda W^2}{12v}\left(\frac{\pi}{2} - \log\frac{\lambda W}{3} - \log\frac{\sqrt{3}\sqrt{1-v^2}}{2\sqrt{2}v}\right) - \frac{1}{\lambda(1+v)}. \quad (11)$$

Now, let λ^* be the least upper bound on λ for which the FCFS policy is unstable, i.e., for every $\lambda < \lambda^*$, the FCFS policy is stable. To obtain λ^*, we need to solve $\lambda^* \mathbb{E}[T] = 1$. Using Eq. (11), we can obtain a lower bound on λ^* by simplifying

$$\frac{\lambda^{*2}W^2}{12v}\left(\frac{\pi}{2} - \log\frac{\lambda^* W}{3} - \log\frac{\sqrt{3}(1-v^2)}{2\sqrt{2}v(1+v)}\right) - \frac{1}{1+v} \ge 1.$$

From the condition given by Eq. (9), the second term in the parentheses satisfies

$$\frac{\lambda^* W}{3} > \sqrt{\frac{1-v}{1+v}}.$$

Thus, we obtain,

$$\lambda^* \ge \frac{\sqrt{12v}}{W\sqrt{(1+v)\left(C - \log\left(\frac{1-v}{v}\right)\right)}},$$

where the constant $C = \pi/2 - \log(0.5 \cdot \sqrt{3}/\sqrt{2}) \approx 2.06$. Since $\lambda < \lambda^*$ implies stability, a sufficient condition for stability is

$$\lambda < \frac{\sqrt{12v}}{W\sqrt{(1+v)\left(C - \log\left(\frac{1-v}{v}\right)\right)}}. \quad (12)$$

To determine the value of the speed v^* beyond which this is a less conservative condition than Eq. (9), we solve

$$\frac{\sqrt{12v^*}}{W\sqrt{(1+v^*)\left(C - \log\left(\frac{1-v^*}{v^*}\right)\right)}} = \frac{3}{W}\sqrt{\frac{1-v^*}{1+v^*}},$$

which gives $v^* \approx 2/3$. For $v > v^*$, one can verify that the numerical value of the argument of the Struve and Bessel functions is less than 3, and so the approximation used in this analysis is valid. Thus, a sufficient condition for stability is given by Eq. (9) for $v \le 2/3$, and by Eq. (12) for $v > 2/3$. ■

Remark V.2 (Limiting regimes) As $v \to 0^+$, the sufficient condition for FCFS stability becomes $\lambda < 3/W$, which is exactly equal to the necessary condition given by part (ii) of Theorem IV.2. Thus, the condition for stability is asymptotically tight in this limiting regime.

As $v \to 1^-$, the sufficient condition for FCFS stability becomes

$$\lambda < \frac{\sqrt{6}}{W\sqrt{-\log(1-v)}},$$

In comparison the necessary condition scales as

$$\lambda \le \frac{3\sqrt{2}}{W\sqrt{-\log(1-v)}}.$$

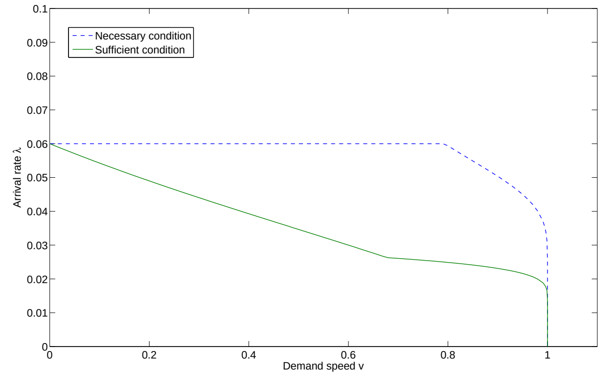

Fig. 5. The necessary and sufficient conditions for the stability of the FCFS policy.

Thus, the necessary and sufficient conditions for the stability of the FCFS policy (and by Theorem IV.3, for any policy) differ by a factor $\sqrt{3}$.

Fig. 5 shows a comparison of the necessary and sufficient stability conditions for the FCFS policy. It should be noted that λ can converge to zero extremely slowly as v tends to one, and still satisfy the sufficient stability condition in Theorem V.1. For example, with $v = 0.999999$, the FCFS policy can stabilize the system for an arrival rate of $3/(5W)$. □

VI. SIMULATIONS

In this section we present results of numerical experiments of the FCFS policy. We use these experiments to study how the steady-state expected delay varies with the generation rate of the demands for different speeds. The policy was simulated for a fixed number of demands, large enough to ensure steady state was reached. We obtained the steady state delay by computing the mean of the delay in the last 200 iterations. This was repeated 10 times to obtain an estimate of the steady state expected delay for one value of the generation rate and for a given speed. We then repeat the experiment for four different values of the speed. The variation of the expected steady-state delay with the generation rate is presented in Fig. 6 for different values of the demand speed. The system is observed to be stable for generation rates up to the sufficient condition for stability, which was theoretically established in Section V.

We also compared the steady-state expected delay of the FCFS policy to the TSP-based policy proposed in companion paper [1], in the region where both policies are stable (c.f., Fig. 1). The comparison was performed for two values of speed, $v = 0.05$ and $v = 0.25$, varying λ from zero up to values close to the stability limit of FCFS. The results are shown in Fig. 7. We observe that in this region, the FCFS performs significantly better in terms of the steady-state expected delay than the TSP-based policy.

VII. CONCLUSIONS

This two part paper has introduced a dynamic vehicle routing problem with moving demands. In this paper we

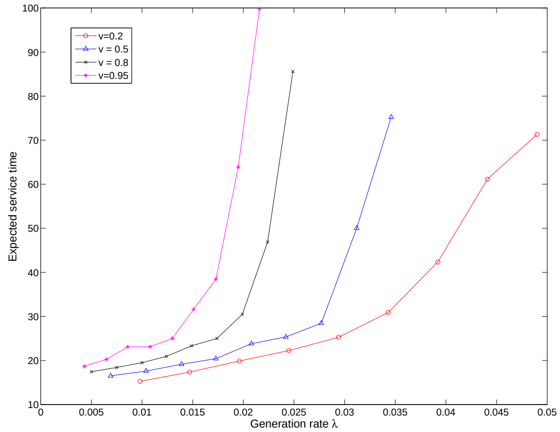

Fig. 6. Simulation results for FCFS policy: variation of the steady state expected delay with the arrival rate for four different values of the demand speeds. The width of the environment is $W = 50$.

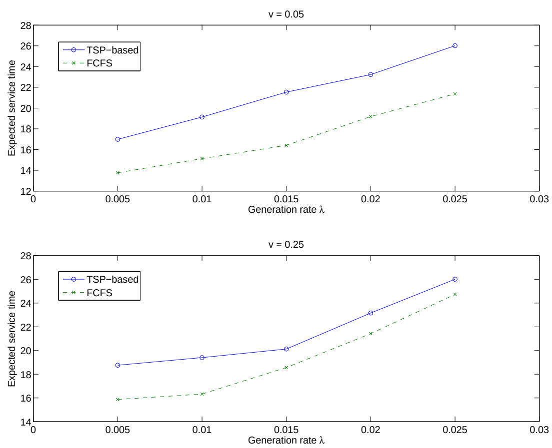

Fig. 7. Comparison of the steady state expected delay using the FCFS and the TSP-based policy, for low values of arrival rate λ. Each data point is obtained after running the policy 30 times. The width of the environment is $W = 50$.

studied the cases where the demands have high speed and where the arrival rate of demands is low. We introduced a first-come-first-served policy and gave necessary and sufficient conditions on the arrival rate for its stability. We also determined the optimal placement of the vehicle so as to minimize the worst-case, and the expected delay in servicing a demand. We then showed that for fixed v, as the arrival rate tends to zero, no policy can perform better than FCFS in terms of minimizing the worst-case service delay, and the expected service delay. Finally we showed that as v tends to one, FCFS is the optimal policy, as every stabilizing policy must service demands in the order in which arrive.

For future work we will extend our results to the case when demands are generated according to a nonuniform distribution on the generator, and the case of multiple vehicles. This extension has been completed for the placement problem.

Acknowledgments

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