

# Equitable Partitioning Policies for Robotic Networks

Marco Pavone, Alessandro Arsie, Emilio Frazzoli, Francesco Bullo

**Abstract**—The most widely applied resource allocation strategy is to balance, or equalize, the total workload assigned to each resource. In mobile multi-agent systems, this principle directly leads to equitable partitioning policies in which (i) the workspace is divided into subregions of equal measure, (ii) there is a bijective correspondence between agents and subregions, and (iii) each agent is responsible for service requests originating within its own subregion. In this paper, we provide the first distributed algorithm that provably allows  $m$  agents to converge to an equitable partition of the workspace, from any initial configuration, i.e., globally. Our approach is related to the classic Lloyd algorithm, and provides novel insights into the properties of Power Diagrams. Simulation results are presented and discussed.

## I. INTRODUCTION

The most widely applied resource allocation strategy is to balance, or equalize, the total workload assigned to each resource. While, in principle, several strategies are able to guarantee workload-balancing in multi-agent systems (where agents can be interpreted as resources to be allocated), *equitable partitioning policies* are predominant [1], [2], [3], [4]. A *partitioning policy* is an algorithm that, as a function of the number  $m$  of agents and, possibly, of their position and other information, partitions a bounded workspace  $A$  into subregions  $A_i$ , for  $i \in \{1, \dots, m\}$ . Then, each agent  $i$  is assigned to subregion  $A_i$ , and each service request in  $A_i$  receives service from the agent assigned to  $A_i$ . Accordingly, if we model the *workload* for subregion  $S \subseteq A$  as  $\lambda_S \doteq \int_S \lambda(x) dx$ , where  $\lambda(x)$  is a measure over  $A$ , then the workload for agent  $i$  is  $\lambda_{A_i}$ . Then, load balancing calls for equalizing the workload  $\lambda_{A_i}$  in the  $m$  subregions or, in equivalent words, requires an *equitable partition* of the workspace  $A$  (i.e., a partition where  $\lambda_{A_i} = \lambda_A/m$ , for all  $i$ ).

Equitable partitioning policies are predominant for three main reasons: (i) efficiency, (ii) ease of design, (iii) ease of analysis. Consider, for instance, the well-known dynamic version of the classic Vehicle Routing Problem: the Dynamic Traveling Repairman Problem (DTRP) [1]. In the DTRP,  $m$  agents operating in a workspace  $A$  must service demands whose time of arrival, location and on-site service are stochastic; the objective is to find a policy to service

demands over an infinite horizon that minimizes the expected system time (wait plus service) of the demands. Equitable partitioning policies are, indeed, optimal for the DTRP when the traffic intensity is close to one (see [1], [5], [6]).

Despite their relevance in robotic network applications, to the best of our knowledge, the only available *distributed* equitable partitioning policy is the one proposed by the authors in [7]. However, the policy presented in [7] only guarantees *local* (i.e., for a subset of initial conditions) convergence to equitable partitions.

Building upon our previous work [7], in this paper we design a distributed control law that allows a team of agents to achieve *globally*, i.e., starting from any initial configuration, a partition of the workspace into subregions of equal measure. Our approach is related to the classic Lloyd algorithm, and exploits the unique features of Power Diagrams. We mention that our algorithms, although motivated in the context of multi-agent systems, are a novel contribution to the field of computational geometry; moreover, our results provide new insights in the geometry of Power Diagrams.

## II. BACKGROUND

In this section, we introduce some notation and briefly review some concepts from geometry and degree theory, on which we will rely extensively later in the paper.

### A. Notation

Let  $\|\cdot\|$  denote the Euclidean norm. Let  $A$  be a compact, convex subset of  $\mathbb{R}^d$ . We denote the boundary of  $A$  as  $\partial A$  and the Lebesgue measure of  $A$  as  $|A|$ . We define the diameter of  $A$  as:  $\text{diameter}(A) \doteq \sup\{\|p - q\| \mid p, q \in A\}$ . The distance from a point  $x$  to a set  $M$  is defined as  $\text{dist}(x, M) \doteq \inf_{p \in M} \|x - p\|$ . We define  $I_m \doteq \{1, 2, \dots, m\}$ . Let  $G = (g_1, \dots, g_m) \subset A^m$  denote the location of  $m$  points in  $A$ . A *partition* (or tessellation) of  $A$  is a collection of  $m$  closed subsets  $\mathcal{A} = \{A_1, \dots, A_m\}$  with disjoint interiors whose union is  $A$ . The partition of  $A$  is *convex*, if each subset  $A_i$ ,  $i \in I_m$ , is convex.

### B. Voronoi Diagrams and Power Diagrams

We refer the reader to [8] and [9] for comprehensive treatments, respectively, of Voronoi Diagrams and Power Diagrams. Assume that  $G$  is an ordered set of distinct points. The *Voronoi Diagram*  $\mathcal{V}(G) = (V_1(G), \dots, V_m(G))$  of  $A$  generated by points  $G = (g_1, \dots, g_m)$  is defined by

$$V_i(G) = \{x \in A \mid \|x - g_i\| \leq \|x - g_j\|, \forall j \neq i, j \in I_m\}. \quad (1)$$

We refer to  $G$  as the set of *generators* of  $\mathcal{V}(G)$ , and to  $V_i(G)$  as the Voronoi cell or region of dominance of the  $i$ -th

Marco Pavone and Emilio Frazzoli are with the Laboratory for Information and Decision Systems, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, {pavone, frazzoli}@mit.edu.

Alessandro Arsie is with the Department of Mathematics, Pennsylvania State University, arsie@math.psu.edu.

Francesco Bullo is with the Center for Control Engineering and Computation, University of California at Santa Barbara, bullo@engineering.ucsb.edu.

generator. For  $g_i, g_j \in G$ ,  $i \neq j$ , we define the *bisector* between  $g_i$  and  $g_j$  as  $b(g_i, g_j) = \{x \in A \mid \|x - g_i\| = \|x - g_j\|\}$ . The face  $b(g_i, g_j)$  bisects the line segment joining  $g_i$  and  $g_j$ , and this line segment is orthogonal to the face (*Perpendicular Bisector Property*). It is easy to verify that each Voronoi cell is a *convex* set.

Assume, now, that each generator  $g_i \in G$  has assigned an individual weight  $w_i \in \mathbb{R}$ ,  $i \in I_m$ . We define  $W = (w_1, \dots, w_m)$ . We define the power distance as  $d_P(x, g_i; w_i) \doteq \|x - g_i\|^2 - w_i$ . We refer to the pair  $(g_i, w_i)$  as a *power point*. We define  $G_W = ((g_1, w_1), \dots, (g_m, w_m)) \in (A \times \mathbb{R})^m$ . Two power points  $(g_i, w_i)$  and  $(g_j, w_j)$  are *coincident* if  $g_i = g_j$  and  $w_i = w_j$ . Assume that  $G_W$  is an ordered set of *distinct* power points. Similarly as before, the *Power Diagram*  $\mathcal{V}(G_W) = (V_1(G_W), \dots, V_m(G_W))$  generated by power points  $G_W = ((g_1, w_1), \dots, (g_m, w_m))$  is

$$V_i(G_W) = \{x \in A \mid \|x - g_i\|^2 - w_i \leq \|x - g_j\|^2 - w_j, \forall j \neq i, j \in I_m\}. \quad (2)$$

We refer to  $G_W$  as the set of *power generators* of  $\mathcal{V}(G_W)$ , and to  $V_i(G_W)$  as the power cell or region of dominance of the  $i$ -th power generator; moreover we call  $g_i$  and  $w_i$ , respectively, the position and the weight of power generator  $(g_i, w_i)$ . Notice that, when all weights are the same, the Power Diagram coincides with the Voronoi Diagram. Each power cell is a convex set (as it can be easily verified). Notice that (i) a power cell might be empty, and (ii)  $g_i$  might not be in its power cell (see Fig. 1). Finally, the bisector of  $(g_i, w_i)$  and  $(g_j, w_j)$ ,  $i \neq j$ , is defined as

$$b((g_i, w_i), (g_j, w_j)) = \{x \in A \mid (g_j - g_i)^T x = \frac{1}{2}(\|g_j\|^2 - \|g_i\|^2 + w_i - w_j)\}. \quad (3)$$

Hence,  $b((g_i, w_i), (g_j, w_j))$  is a face orthogonal to the line segment  $\overline{g_i g_j}$ . Notice that the Power Diagram of an ordered set of possibly *coincident* power points is not well-defined. We define

$$\Gamma_{\text{coinc}} = \left\{ G_W \mid g_i = g_j \text{ and } w_i = w_j \text{ for some } i \neq j \in I_m \right\}. \quad (4)$$

For simplicity, we will refer to  $V_i(G)$  ( $V_i(G_W)$ ) as  $V_i$ . When the two Voronoi (power) cells  $V_i$  and  $V_j$  are adjacent (i.e., they share a face),  $g_i$  ( $(g_i, w_i)$ ) is called a *Voronoi (power) neighbor* of  $g_j$  ( $(g_j, w_j)$ ), and vice-versa. The set of indices of the Voronoi (power) neighbors of  $g_i$  ( $(g_i, w_i)$ ) is denoted by  $N_i$ . We also define the  $(i, j)$ -face as  $\Delta_{ij} \doteq V_i \cap V_j$ .

### C. A Basic Result in Degree Theory

In this section, we state some results in degree theory that will be useful in the remainder of the paper. For a thoroughly

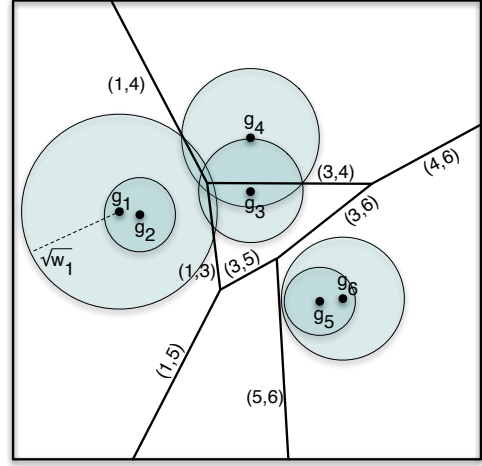


Fig. 1. A Power Diagram. The weights  $w_i$  are all positive in this example. Circles represent the magnitudes of weights. Power generator  $(g_2, w_2)$  has an empty cell. Power generator  $(g_5, w_5)$  is outside its region of dominance.

introduction to the theory of degree we refer the reader to [10].

Let us just recall the simplest definition of degree of a map  $f$ . Let  $f : X \rightarrow Y$  be a smooth map between connected compact manifolds  $X$  and  $Y$  of the same dimension, and let  $p \in Y$  a regular value for  $f$  (regular values abound due to Sard's lemma). Since  $X$  is compact,  $f^{-1}(p) = \{x_1, \dots, x_n\}$  is a finite set of points and since  $p$  is a regular value, it means that  $f_{U_i} : U_i \rightarrow f(U_i)$  is a local diffeomorphism, where  $U_i$  is a suitable open neighborhood of  $x_i$ . Diffeomorphisms can be either orientation preserving or orientation reversing. Let  $d^+$  be the number of points  $x_i$  in  $f^{-1}(p)$  at which  $f$  is orientation preserving (i.e.  $\det(\text{Jac}(f)) > 0$ , where  $\text{Jac}(f)$  is the Jacobian matrix of  $f$ ) and  $d^-$  be the number of points in  $f^{-1}(p)$  at which  $f$  is orientation reversing (i.e.  $\det(\text{Jac}(f)) < 0$ ). Since  $X$  is connected, it can be proved that the number  $d^+ - d^-$  is independent on the choice of  $p \in Y$  and one defines the degree  $\text{deg}(f) \doteq d^+ - d^-$ . The degree can be also defined for a *continuous* map  $f : X \rightarrow Y$  among connected oriented topological manifolds of the same dimensions. For more details see [10].

The following result will be fundamental to prove some existence theorems and it is a direct consequence of the theory of degree of continuous maps among spheres.

**Theorem 2.1:** Let  $f : B^n \rightarrow B^n$  be a continuous map from a closed  $n$ -ball to itself. Call  $S^{n-1}$  the boundary of  $B^n$ , namely the  $(n - 1)$ -sphere and assume that  $f_{S^{n-1}} : S^{n-1} \rightarrow S^{n-1}$  is a map with degree  $\text{deg}(f) \neq 0$ . Then  $f$  is onto  $B^n$ .

In the sequel we will need also the following:

**Lemma 2.2:** Let  $f : S^n \rightarrow S^n$  a continuous bijective map from the  $n$ -dimensional sphere to itself ( $n \geq 1$ ). Then the degree is  $\text{deg}(f) = \pm 1$ .

## III. PROBLEM FORMULATION

A total of  $m$  identical mobile agents provide service in a compact, convex service region  $A \subseteq \mathbb{R}^2$ . Let  $\lambda$  be a measure

whose bounded support is  $A$  (in equivalent words,  $\lambda$  is not zero only on  $A$ ); for any set  $S$ , we define the *workload* for region  $S$  as  $\lambda_S \doteq \int_S \lambda(x) dx$ . The measure  $\lambda$  models service requests, and can represent, for example, the density of customers over  $A$ , or, in a stochastic setting, their arrival rate. Given the measure  $\lambda$ , a partition  $\{A_i\}_i$  of the workspace  $A$  is *equitable* if  $\lambda_{A_i} = \lambda_{A_j}$  for all  $i, j \in I_m$ .

A *partitioning policy* is an algorithm that, as a function of the number  $m$  of agents and, possibly, of their position and other information, partitions a bounded workspace  $A$  into  $m$  subregions  $A_i$ ,  $i \in I_m$ . Then, each agent  $i$  is assigned to subregion  $A_i$ , and each service request in  $A_i$  receives service from the agent assigned to  $A_i$ . We refer to subregion  $A_i$  as the *region of dominance* of agent  $i$ . Given a measure  $\lambda$  and a partitioning policy,  $m$  agents are in a *convex equipartition configuration* with respect to  $\lambda$  if the associated partition is equitable and convex.

In this paper we are interested in the following problem: find a *spatially-distributed* equitable partitioning policy that allows  $m$  mobile agents to *globally* (i.e., from any initial configuration) reach a convex equipartition configuration (with respect to  $\lambda$ ).

#### IV. ON THE EXISTENCE OF EQUITABLE POWER DIAGRAMS

The key advantage of Power Diagrams is that an equitable Power Diagram always exists for any  $\lambda$ . Indeed, as shown in the next theorem, an equitable Power Diagram (with respect to any  $\lambda$ ) exists for *any* vector of *distinct* points  $G = (g_1, \dots, g_m)$  in  $A$ .

*Theorem 4.1:* Let  $A$  be a bounded, connected domain in  $\mathbb{R}^2$ , and  $\lambda$  be a measure on  $A$ . Let  $G = (g_1, \dots, g_m)$  be the positions of  $m \geq 1$  distinct points in  $A$ . Then, there exist weights  $w_i$ ,  $i \in I_m$ , such that the power points  $\left((g_1, w_1), \dots, (g_m, w_m)\right)$  generate a Power Diagram that is equitable with respect to  $\lambda$ . Moreover, given a vector of weights  $W^*$  that yields an equitable partition, the set of all vectors of weights yielding an equitable partition is  $\mathcal{W} \doteq \{W^* + t[1, \dots, 1]\}$ , with  $t \in \mathbb{R}$ .

*Proof:* It is not restrictive to assume that  $\lambda_A = 1$  (i.e., we normalize the measure of  $A$ ), since  $A$  is bounded. The strategy of the proof is to use a topological argument to force existence.

First, we construct a weight space. Let  $D = \text{diameter}(A)$ , and consider the cube  $\mathcal{C} := [-D, D]^m$ . This is the weight space and we consider weight vectors  $W$  taking value in  $\mathcal{C}$ . Second, consider the standard  $m$ -simplex of measures  $\lambda_{A_1}, \dots, \lambda_{A_m}$  (where  $A_1, \dots, A_m$  are, as usual, the regions of dominance). This can be realized in  $\mathbb{R}^m$  as the subset of defined by  $\sum_{i=1}^m \lambda_{A_i} = 1$  with the condition  $\lambda_{A_i} \geq 0$ . Let us call this set “the measure simplex  $\mathcal{A}$ ” (notice that it is  $(m-1)$ -dimensional).

There is a map  $f : \mathcal{C} \rightarrow \mathcal{A}$  associating, according to the power distance, a weight vector  $W$  with the corresponding vector of measures  $(\lambda_{A_1}, \dots, \lambda_{A_m})$ . Since the points in  $G$  are assumed to be distinct, this map is continuous.

We prove the case for  $m = 3$  (the statement for  $m = 1$  and  $m = 2$  is trivially checked) while the complete proof, that uses induction on  $m$ , can be found in [11]. When  $m = 3$ , the weight space  $\mathcal{C}$  is a three dimensional cube with vertices  $v_0 = [-D, -D, -D]$ ,  $v_1 = [D, -D, -D]$ ,  $v_2 = [-D, D, -D]$ ,  $v_3 = [-D, -D, D]$ ,  $v_4 = [D, -D, D]$ ,  $v_5 = [-D, D, D]$ ,  $v_6 = [D, D, -D]$  and  $v_7 = [D, D, D]$ . The measure simplex  $\mathcal{A}$  is, instead, a triangle with vertices  $u_1, u_2, u_3$  that correspond to the cases  $\lambda_{A_1} = 1, \lambda_{A_2} = 0, \lambda_{A_3} = 0$ ,  $\lambda_{A_1} = 0, \lambda_{A_2} = 1, \lambda_{A_3} = 0$ , and  $\lambda_{A_1} = 0, \lambda_{A_2} = 0, \lambda_{A_3} = 1$ , respectively. Moreover, call  $e_1, e_2$  and  $e_3$  the edges opposite to the vertices  $u_1, u_2, u_3$  respectively. The edges  $e_i$  are, therefore, given by the condition  $\{\lambda_{A_i} = 0\}$  (see Fig. 2).

Let us return to the map  $f : \mathcal{C} \rightarrow \mathcal{A}$ . Observe that the map  $f$  sends  $v_0$  to the unique point  $p_0$  of  $\mathcal{A}$  corresponding to the measures of usual Voronoi cells (since the weights are all equal). Observe that only the *differences* among the weights change the vector  $(\lambda_{A_1}, \lambda_{A_2}, \lambda_{A_3})$ , i.e., if all weights are increased by the same quantity, the vector  $(\lambda_{A_1}, \lambda_{A_2}, \lambda_{A_3})$  does not change. In particular, the image of the diagonal  $v_0 v_7$  is exactly the point for which the measures are those of a Voronoi partition. Now let us understand what are the “fibers” of  $f$ , that is to say, the loci where  $f$  is constant. Since the measures of the regions of dominance do not change if the differences among the weights are kept constant, then the fibers of  $f$  in the weight space  $\mathcal{C}$  are given by the equations  $w_1 - w_2 = c_1$  and  $w_2 - w_3 = c_2$ . Rearranging these equations, it is immediate to see that  $w_1 = w_3 + c_1 + c_2$ ,  $w_2 = w_3 + c_2$  and  $w_3 = w_3$ , therefore taking  $w_3$  as parameter we see that the fibers of  $f$  are straight lines *parallel* to the main diagonal  $v_0 v_7$ . Therefore we can conclude that if a particular weight vector  $W^*$  yields a specific area vector  $\lambda^*$ , then all the weight vectors of the form  $W^* + t[1, \dots, 1]$ ,  $t \in \mathbb{R}$  will give rise to the same area vector  $\lambda^*$ . On the weight space  $\mathcal{C}$  let us define the following equivalence relation:  $w \equiv w'$  if and only if they are on a line parallel to the main diagonal  $v_0 v_7$ . Map  $f : \mathcal{C} \rightarrow \mathcal{A}$  induces a continuous map (still called  $f$  by abuse of notation) from  $\mathcal{C}/\equiv \rightarrow \mathcal{A}$  having the same image. Let us identify  $\mathcal{C}/\equiv$ . It is easy to see that any line in the cube parallel to the main diagonal  $v_0 v_7$  is entirely determined by its intersection with the three faces  $F_3 = \{w_3 = -D\} \cap \mathcal{C}$ ,  $F_2 = \{w_2 = -D\} \cap \mathcal{C}$  and  $F_1 = \{w_1 = -D\} \cap \mathcal{C}$ . Call  $\mathcal{F}$  the union of these faces. We therefore have a continuous map  $f : \mathcal{F} \rightarrow \mathcal{A}$  that has the same image of the original  $f$ ; besides, the induced map  $f : \mathcal{F} \rightarrow \mathcal{A}$  is *injective* by construction, since each fiber intersects  $\mathcal{F}$  in only one point.

Observe that  $\mathcal{F}$  is homeomorphic to  $B^2$ , the 2-dimensional ball, like  $\mathcal{A}$  itself. Up to homeomorphisms, therefore, the map  $f : \mathcal{F} \rightarrow \mathcal{A}$  can be viewed as a map (again called  $f$  by abuse of notation),  $f : B^2 \rightarrow B^2$ . Consider the closed loop  $\alpha$  given by  $v_2 v_5, v_5 v_3, v_3 v_4, v_4 v_1, v_1 v_6, v_6 v_2$  with this orientation (see Fig. 2). This loop is the boundary of  $\mathcal{F}$  and we think of it also as the boundary of  $B^2$ . Taking into account the continuity of  $f$ , it is easy to see that  $f$  maps  $\alpha$  onto the boundary of  $\mathcal{A}$ . Indeed, as a point moves along the path  $\alpha$  one of the weights is always equal to  $-D$  and

another one is always equal to  $+D$  and therefore the area  $\lambda_{A_i}$  corresponding to weight  $w_i = -D$  is zero, thus describing one of the edges of the area simplex. For instance, along the edge  $v_3v_4$  on the cube we have  $w_3 = +D$  identically and  $w_2 = -D$ , while on the edge  $v_5v_3$  we have  $w_1 = -D$  and  $w_3 = +D$ , and analogously for all the other edges in the path  $\alpha$  as it is easy to verify. More precisely, while we move on the edges  $v_2v_5$  and  $v_5v_3$  that are characterized by having  $w_1 = -D$  the corresponding point on the area simplex moves on the edge  $e_1$ .

Moreover since  $f$  is injective by construction, the inverse image of any point in the boundary of  $A$  is just one element of  $\alpha$ . Identifying the boundary of  $\mathcal{A}$  with  $S^1$  (up to homeomorphisms) and the loop  $\alpha$  with  $S^1$  (up to homeomorphisms) we have a bijective continuous map  $f_{S^1} : S^1 \rightarrow S^1$ . By Lemma (2.2)  $\deg(f) = \pm 1$  and therefore  $f$  is onto  $\mathcal{A}$ , using Theorem (2.1).

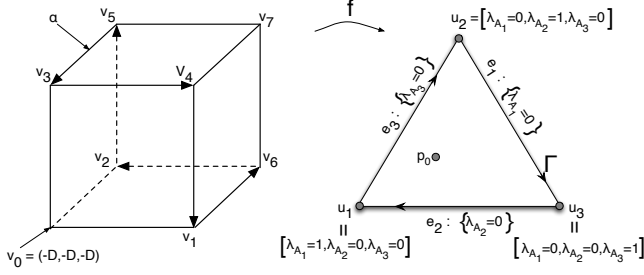


Fig. 2. Construction used for the proof of existence of equitable Power Diagrams.

**Remark 4.2:** Since all vectors of weights in  $\mathcal{W}$  yield exactly the *same* Power Diagram, we conclude that the positions of the generators *uniquely* induces an equitable Power Diagram.

## V. DISTRIBUTED GRADIENT DESCENT LAW FOR EQUITABLE PARTITIONING

In this section, we design distributed policies that allow a team of agents to achieve a convex equipartition configuration.

### A. Virtual Generators and Locational Optimization

The first step is to associate to each agent  $i$  a *virtual power generator* (virtual generator for short)  $(g_i, w_i)$ . We define the region of dominance for agent  $i$  as the power cell  $V_i = V_i(G_W)$ , where  $G_W = ((g_1, w_1), \dots, (g_m, w_m))$  (see Fig. 3). We refer to the partition into regions of dominance induced by the set of virtual generators  $G_W$  as  $\mathcal{V}(G_W)$ . A virtual generator  $(g_i, w_i)$  is simply an artificial variable locally controlled by the  $i$ -th agent; in particular,  $g_i$  is a virtual point and  $w_i$  is its weight.

Virtual generators allow us to decouple the problem of achieving an equitable partition into regions of dominance

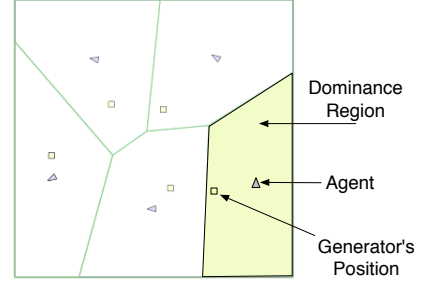


Fig. 3. Agents, virtual generators and regions of dominance.

from that of positioning an agent inside its own region of dominance.

In light of Theorem 4.1, the key idea is to enable the weights of the virtual generators to follow a (distributed) gradient descent law (while maintaining the positions of the generators *fixed*) such that an equitable partition is reached.

Assume, henceforth, that the positions of the virtual generators are *distinct*, i.e.  $g_i \neq g_j$  for  $i \neq j$ . Define the set

$$S \doteq \left\{ (w_1, \dots, w_m) \in \mathbb{R}^m \mid \lambda_{V_i} > 0 \ \forall i \in I_m \right\}. \quad (5)$$

Set  $S$  contains all possible vectors of weights such that no region of dominance has measure *zero*.

We introduce the locational optimization function  $H_V : S \mapsto \mathbb{R}_{>0}$ :

$$H_V(W) \doteq \sum_{i=1}^m \left( \int_{V_i(W)} \lambda(x) dx \right)^{-1} = \sum_{i=1}^m \lambda_{V_i}^{-1}(W). \quad (6)$$

### B. Smoothness and Gradient of $H_V$

We now analyze the smoothness properties of the locational optimization function  $H_V$ . In the following, let  $\gamma_{ij} \doteq \|g_j - g_i\|$ .

**Theorem 5.1:** Assume that the positions of the virtual generators are *distinct*, i.e.,  $g_i \neq g_j$  for  $i \neq j$ . Given a measure  $\lambda$ , the locational optimization function  $H_V$  is continuously differentiable on  $S$ , where for each  $i \in \{1, \dots, m\}$

$$\frac{\partial H_V}{\partial w_i} = \sum_{j \in N_i} \frac{1}{2\gamma_{ij}} \left( \frac{1}{\lambda_{V_j}^2} - \frac{1}{\lambda_{V_i}^2} \right) \int_{\Delta_{ij}} \lambda(x) dx. \quad (7)$$

Furthermore, the critical points of  $H_V$  are vectors of weights that yield an equitable Power Diagram.

**Proof:** The proof is almost identical to that of Theorem 5.1 in [7] and, thus, it is omitted.  $\blacksquare$

**Remark 5.2:** The computation of the gradient in Theorem 5.1 is *spatially-distributed* over the dual graph of the Power Diagram, since it depends only on information from the agents with contiguous power cells.

**Example 5.3 (Gradient of  $H_V$  for uniform measure):** The gradient of  $H_V$  simplifies considerably when  $\lambda$  is constant. In such case, it is straightforward to verify that (assuming that  $\lambda$  is normalized)

$$\frac{\partial H_V}{\partial w_i} = \frac{1}{2|A|} \sum_{j \in N_i} \frac{\delta_{ij}}{\gamma_{ij}} \left( \frac{1}{|V_j|^2} - \frac{1}{|V_i|^2} \right), \quad (8)$$

where  $\delta_{ij}$  is the length of the border  $\Delta_{ij}$ .

### C. Spatially-Distributed Algorithm for Equitable Partitioning

Consider the set

$$U \doteq \left\{ (w_1, \dots, w_m) \in \mathbb{R}^m \mid \sum_{i=1}^m w_i = 0 \right\}.$$

Indeed, since adding an identical value to every weight leaves all power cells unchanged, there is *no loss of generality* in restricting the weights to  $U$ ; let  $\Omega \doteq S \cap U$ . Assume the generators' weights obey a first order dynamical behavior described by  $\dot{w}_i = u_i$ . Consider  $H_V$  an objective function to be minimized and impose that the weight  $w_i$  follows the gradient descent given by (7). In more precise terms, we set up the following control law defined over the set  $\Omega$

$$u_i = -\frac{\partial H_V}{\partial w_i}(W), \quad (9)$$

where we assume that the Power Diagram  $\mathcal{V}(W) = \{V_1, \dots, V_m\}$  is continuously updated. One can prove the following result.

*Theorem 5.4:* Assume that the positions of the virtual generators are *distinct*, i.e.,  $g_i \neq g_j$  for  $i \neq j$ . Consider the gradient vector field on  $\Omega$  defined by equation (9). Then generators' weights starting at  $t = 0$  at  $W(0) \in \Omega$ , and evolving under (9) remain in  $\Omega$  and converge asymptotically to a critical point of  $H_V$ , i.e., to a vector of weights that yields an equitable Power Diagram.

*Proof:* We first prove that generators' weights evolving under (9) remain in  $\Omega$  and converge asymptotically to the *set* of critical points of  $H_V$ . By assumption,  $g_i \neq g_j$  for  $i \neq j$ , thus the Power Diagram is well defined. First, we prove that set  $\Omega$  is positively invariant with respect to (9). Recall that  $\Omega = S \cap U$ . Noticing that control law (9) is a gradient descent law, we have

$$\lambda_{V_i}^{-1}(W(t)) \leq H_V(W(t)) \leq H_V(W(0)), \quad i \in I_m, t \geq 0.$$

Since the measures of the power cells depend continuously on the weights, we conclude that the measures of all power cells will be bounded away from zero; thus, the weights will belong to  $S$  for all  $t \geq 0$ , i.e.,  $W(t) \in S \forall t \geq 0$ . Moreover, the sum of the weights is invariant under control law (9). Indeed,

$$\begin{aligned} \frac{\partial \sum_{i=1}^m w_i}{\partial t} &= -\sum_{i=1}^m \frac{\partial H_V}{\partial w_i} = \\ &= -\sum_{i=1}^m \sum_{j \in N_i} \frac{1}{2\gamma_{ij}} \left( \frac{1}{\lambda_{V_j}^2} - \frac{1}{\lambda_{V_i}^2} \right) \int_{\Delta_{ij}} \lambda(x) dx = 0, \end{aligned}$$

since  $\gamma_{ij} = \gamma_{ji}$ ,  $\Delta_{ij} = \Delta_{ji}$ , and  $j \in N_i \Leftrightarrow i \in N_j$ . Thus, we have  $W(t) \in U \forall t \geq 0$ . Since  $W(t) \in S \forall t \geq 0$  and  $W(t) \in U \forall t \geq 0$ , we conclude that  $W(t) \in S \cap U = \Omega$ ,  $\forall t \geq 0$ , i.e., set  $\Omega$  is positively invariant.

Second,  $H_V : \Omega \mapsto \mathbb{R}_{\geq 0}$  is clearly non-increasing along system trajectories, i.e.,  $\dot{H}_V(W) \leq 0$  in  $\Omega$ .

Third, all trajectories with initial conditions in  $\Omega$  are bounded. Indeed, we have already shown that  $\sum_{i=1}^m w_i = 0$

along system trajectories. This implies that weights remain within a bounded set: If, by contradiction, a weight could become arbitrarily positive large, another weight would become arbitrarily negative large (since the sum of weights is constant), and the measure of at least one power cell would vanish, which contradicts the fact that  $S$  is positively invariant.

Finally, by Theorem 5.1,  $H_V$  is continuously differentiable in  $\Omega$ . Hence, by invoking the LaSalle invariance principle (see, for instance, [12]), under the descent flow (9), weights will converge asymptotically to the *set* of critical points of  $H_V$ , that is not empty by Theorem 4.1.

Indeed, by Theorem 4.1, we know that all vectors of weights yielding an equitable Power Diagram differ by a common translation. Thus, the largest invariant set of  $H_V$  in  $\Omega$  contains only one point. This implies that  $\lim_{t \rightarrow \infty} W(t)$  exists and it is equal to a vector of weights that yields an equitable Power Diagram. ■

Some remarks are in order.

*Remark 5.5:* By Theorem 5.4, for *any* set of generators' distinct positions, convergence to an equitable power diagram is *global* with respect to  $\Omega$ . Indeed, there is a very natural choice for the initial values of the weights. Assuming that at  $t = 0$  agents are in  $A$  and in distinct positions, each agent initializes its weight to zero. Then, the initial partition is a Voronoi tessellation; since  $\lambda$  is positive on  $A$ , each initial cell has nonzero measure, and therefore  $W(0) \in \Omega$  (the sum of the initial weights is clearly zero).

*Remark 5.6:* As noted in Remark 5.2, the computation of the partial derivative in Eq. (9) only requires information from the agents with neighboring power cells. Therefore, the gradient descent law (9) is indeed *spatially-distributed* over the dual graph of the Power Diagram. We mention that, in a Power Diagram, each power generator has an average number of neighbors less than or equal to six [13]; therefore, the computation of gradient (9) is scalable with the number of agents.

*Remark 5.7:* The previous gradient descent law, although effective in providing a convex and equitable partition, can yield long and "skinny" subregions. Notice that, to obtain an equitable Power Diagram, changing the weights, while maintaining the generators *fixed*, is sufficient. Then, we can use the degrees of freedom given by the positions of the generators to optimize secondary objectives, e.g., to obtain Power Diagrams *similar* to Voronoi Diagrams, or to obtain cells whose shapes show circular symmetry. These extensions are explored in-depth in [11].

## VI. SIMULATION

We simulate ten agents operating in the unit square  $A$ . Agents' initial positions are independently and uniformly distributed over  $A$ ; the initial position of each virtual generator coincides with the initial position of the corresponding agent, and all weights are initialized to zero. Time is discretized with a step  $dt = 0.01$ , and each simulation run consists of 800 iterations (thus, the final time is  $T = 8$ ). Define the area

TABLE I  
PERFORMANCE OF CONTROL LAW (9).

$\lambda$	$\mathbb{E}[\epsilon]$	$\max \epsilon$
unif	$3.8 \cdot 10^{-4}$	0.016
gauss	$3 \cdot 10^{-3}$	$5.3 \cdot 10^{-3}$

error  $\epsilon$  as the difference, at  $T = 8$ , between the measure of the region of dominance with maximum measure and the measure of the region of dominance with minimum measure.

First, we consider a measure  $\lambda$  *uniform* over  $A$ , in particular  $\lambda \equiv 1$ . Therefore, we have  $\lambda_A = 1$  and agents should reach a partition in which each region of dominance has measure equal to 0.1. For this case, we run 50 simulations.

Then, we consider a measure  $\lambda$  that follows a gaussian distribution, namely  $\lambda(x, y) = e^{-5((x-0.8)^2+(y-0.8)^2)}$ ,  $(x, y) \in A$ , whose peak is at the north-east corner of the unit square. Therefore, we have  $\lambda_A \approx 0.336$ , and agents should reach a partition in which each region of dominance has measure equal to 0.0336. For this case, we run 20 simulations.

Table I summarizes simulation results for the uniform  $\lambda$  ( $\lambda$ =unif) and the gaussian  $\lambda$  ( $\lambda$ =gaussian). Expectation and worst case value of the area error  $\epsilon$  are with respect to 50 runs for uniform  $\lambda$ , and 20 runs for gaussian  $\lambda$ . Notice that for both measures, after 800 iterations, the worst case area error is within 16% from the desired measure of dominance regions. Figure 4 shows the typical equitable partitions that are achieved with control law (9).

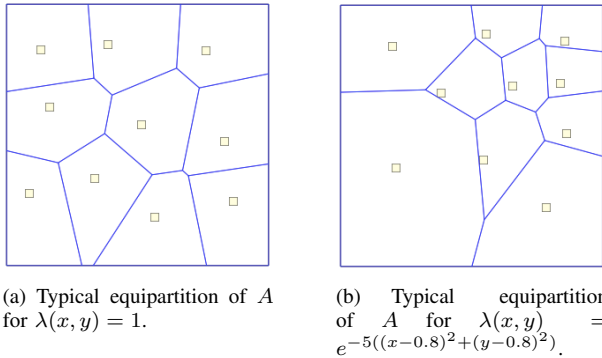


Fig. 4. Typical equipartitions achieved by using control law (9). The yellow squares represent the virtual generators.

## VII. CONCLUSION

We have presented provably correct, spatially-distributed control policies that allow a team of agents to achieve a convex and equitable partition of a convex workspace. Our algorithms could find applications in many problems, including dynamic vehicle routing, and wireless networks. This paper leaves numerous important extensions open for further research. First, it is of interest to study the speed of convergence to an equitable partition. Second, the algorithm that we proposed is synchronous: we plan to devise algorithms that are amenable to asynchronous implementation. Third, we envision considering the setting of structured environments (ranging from simple nonconvex polygons to

more realistic ground environments). Finally, to assess the closed-loop robustness and the feasibility of our algorithms, we plan to implement them on a network of unmanned aerial vehicles.

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