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A pursuit game with range-only measurements

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***Abstract*—We address a discrete-time, pursuit-evasion game with alternate moves played between two kinds of players: the pursuer and the evader. The pursuer wishes to capture the evader while the evader's goal is to avoid capture. By capture, we mean that the distance between the players is no greater than 1 unit. We assume simple, first-order motion kinematics for the players. The pursuer can move with a step size of at most 1 unit while the evader can move with a maximum step size of $\beta < 1$ units. The pursuer is able to measure *only* its distance from the evader, before and after the evader's move. We propose a capture strategy and first show that for the game played in $\mathbb{R}^2$, if $\beta < 0.5$, then a single pursuer captures the evader in finite time. Next, we show that if the game is played in $\mathbb{R}^3$ and if $\beta < 0.5$, then with a modified strategy, two identical cooperative pursuers capture the evader in finite time. Finally, we shed light on the performance of the capture strategy in the case of $\beta \in [0.5, 1[$ and the case of sensing errors via simulations. We also present a simulation study of a version of this game with simultaneous moves.**

I. Introduction

The game of pursuit can be posed as to determine a strategy for a pursuer to capture an evader in a given environment. By *capture*, we mean that the evader and the pursuer are within a specified distance after a finite time. The aim of the pursuer is to capture the evader for any evader strategy. The evader wins the game if it can avoid capture indefinitely. Capture strategies are important in surveillance where the goal is to detect and capture intruders that move unpredictably. Another application is search-and-rescue operations where a worst-case capture strategy guarantees a rescue, in spite of any unpredictable motion of the victim.

A. Related Work

There has been tremendous interest in pursuit-evasion games ever since their formal introduction [1]. Various versions of these games have been studied over the past four decades - for instance [2], [3] and [4] to cite a few. Recently, there has been a surge of interest in the algorithmic approach to the game in discrete-time. [5] gives sufficient conditions and a strategy for a single pursuer to capture an evader in a semi-open environment. [6] and [7] analyze pursuer strategies of moving towards the current and towards the last position of the evader respectively. [8] and [9] address visibility-based pursuit evasion. With respect to multiple cooperative pursuers, [10] addresses capturing an equally fast evader in a boundaryless environment while [11] deals with locating and capture in polygonal environments.

With respect to pursuit under sensing constraints, [12] deals with a version of pursuer's visibility limited to a cone. [13] considers a graph environment, with the visibility limited to adjacent nodes, while [14] and [15] propose a framework assuming probabilistic models for sensing. [16] addresses the case in which the pursuer only knows an approximate location of the evader. [17] and [18] present a solution to the game under bounded measurement uncertainty in sensing the evader.

Other areas of research related to the problem we address, are target tracking and localization. Using distance-only measurements, [19] determines optimal motions for multiple mobile sensors to minimize the error in the posterior estimate of the target position. Using the Fisher Information Matrix, [20] characterizes a condition for local system observability of tracking a moving target in a plane with range-only measurements. [21] and [22] present an established estimation method to track targets moving with bounded speeds. We refer to this method as the *Grow-Intersect* algorithm.

B. Contributions

We address a discrete-time alternate-motion pursuit-evasion game played between two kinds of players, the pursuer and the evader. The pursuer wishes to capture the evader while the evader's goal is to avoid capture. By capture, we mean that the distance between the pursuer and the evader is no greater than 1 unit. The game is played in $\mathbb{R}^2$, i.e., the unbounded plane. We assume simple, first-order motion kinematics for both players. The pursuer can move with a step size of at most 1 unit while the evader can move with a maximum step size of $\beta < 1$. The pursuer is able to measure its distance from the evader before as well as after the evader's move, while the evader is assumed to have complete information of the pursuer's location. In continuous time, this is analogous to the pursuer being equipped with a sensor that measures the distance to the evader as well as the rate of change of this distance. [23] presents an example of one such sensor.

In the proposed game, we present a strategy inspired by the *Grow-Intersect* algorithm for the pursuer and show that: (i) if the maximum evader step size $\beta < 0.5$, then the pursuer captures the evader in finite time, (ii) for the game played in $\mathbb{R}^3$: if $\beta < 0.5$, then two identical, cooperative pursuers capture the evader in finite time, and (iii) we provide upper bounds on the time taken to capture the evader in parts (i) and (ii). Finally, we present simulation studies in the planar case to address: (i) the case of $\beta \in [0.5, 1[$, (ii) the effect of additive, zero-mean Gaussian noise with variance proportional to the square of the distance between the evader and the pursuer on the outcome of the game, and (iii) a game with simultaneous moves.

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C. Organization

The problem formulation is described in Section II. The capture strategy and main result is presented in Section III. A cooperative pursuit version of this game is presented in Section IV. The proofs of the main results in Sections III and IV are presented in Section V. Simulations that address the case of evader speed $\beta \in [0.5, 1[$ and sensor noise are presented in Section VI. A version of the present game with simultaneous moves and a simulation study of the application of a modified capture strategy are presented in Section VII.

II. Problem Set-up

We assume a discrete-time model with alternate motion of the evader and the pursuer. The game is played in the unbounded plane. We assume simple, first-order motion kinematics for both players. The pursuer can move with a step size of at most 1 unit while the evader can move with a step size of $\beta < 1$. The pursuer is equipped with a range-only sensor that measures its distance from the evader. The evader is assumed to know exact information of the pursuer's location. Further, we assume that at each time instant, the players take measurements before and after the pursuer's move. Thus a *sequence* of the game consists of the following: (i) the evader moves, (ii) players take measurements, (iii) the pursuer moves, (iv) the players take measurements. This is shown in Figure 1. Capture is defined when the evader is not greater than a *unit* distance from the pursuer.

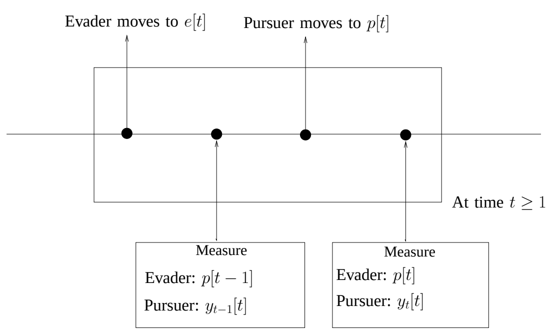


Fig. 1. A *sequence* at each time instant $t \in \{1, 2, \dots\}$ in our alternate motion model. The players take measurements before and after the pursuer's move.

Let $e[t] \in \mathbb{R}^2$ and $p[t] \in \mathbb{R}^2$ denote the positions of the evader and the pursuer respectively, at time $t \in \mathbb{Z}_{\geq 1}$. The discrete-time equations of motion are

$$\begin{aligned} e[t] &= e[t-1] + u^e(e[t-1], \{p[\tau]\}_{\tau=0}^{t-1}), \\ p[t] &= p[t-1] + u^p(p[t-1], y_{t-1}[t], y_t[t]), \end{aligned} \tag{1}$$

where $\{p[\tau]\}_{\tau=1}^{t-1}$ denotes the set $\{p[0], p[1], \dots, p[t-1]\}$. For the pursuer, at the t^{th} time instant , $y_{t-1}[t], y_t[t] \in \mathbb{R}_{\geq 0}$ are the distances of the evader's position from the pursuer before and after the evader's move respectively. Thus, $y_{t-1}[t] = \|e[t] - p[t-1]\|$ and $y_t[t] = \|e[t] - p[t]\|$. The functions $u^e : \mathbb{R}^2 \times \underbrace{\mathbb{R}^2 \times \dots \times \mathbb{R}^2}_{t \text{ times}} \to \mathbb{R}^2$ and $u^p : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ are termed as *strategies* for the evader and pursuer respectively. Notice that in this formulation, we allow the evader the access to the entire history of the pursuer's motion, while we allow the pursuer the access to *only two* of the most recent evader measurements. The lack of symmetry between the number of arguments in strategies of the evader and the pursuer is due to the alternate motion model and due to the assumptions on the measurement models of the players.

Since the step sizes of each player are bounded, we have

$$\|u^e\| \leq \beta, \quad \text{and} \quad \|u^p\| \leq 1, \tag{2}$$

where $\beta < 1$. *Capture* takes place when for some $T_{\text{cap}} \in \mathbb{Z}_{\geq 0}$,

$$\|e[T_{\text{cap}}] - p[T_{\text{cap}} - 1]\| \leq 1 \quad \text{or} \quad \|e[T_{\text{cap}}] - p[T_{\text{cap}}]\| \leq 1. \tag{3}$$

The problem is *to determine a pursuer strategy u^p that guarantees capture for any evader strategy u^e.*

Remark II.1 (Continuous-time analogy) Such a model arises when one discretizes the continuous time pursuit-evasion game in which the pursuer is equipped with a sensor that continuously measures the distance to the evader as well as the rate of change of this distance.

III. The Capture Strategy and Main Result

In this section, we describe our capture strategy and the corresponding main result.

Our capture strategy has two phases: Initialization and Pursuit. These are described as follows.

A. Initialization phase

This phase lasts for only the first sequence. In the first sequence,

(i) The evader moves to $e[1]$.
(ii) The pursuer gets the measurement $y_0[1]$ and it constructs $\partial\mathcal{B}_{y_0[1]}(p[0])$ which is a circle of radius $y_0[1]$ around the point $p[0]$.
(iii) The pursuer randomly selects a direction to move and moves along it with unit step size.
(iv) The pursuer gets the measurement $y_1[1]$ and it constructs $\partial\mathcal{B}_{y_1[1]}(p[1])$ and computes the estimate

$$\hat{E}[1] := \partial\mathcal{B}_{y_1[1]}(p[1]) \cap \partial\mathcal{B}_{y_0[1]}(p[0]). \tag{4}$$

Since $\hat{E}[1]$ is an intersection of two non-concentric circles described in the right hand side of (4), we have the following result.

Proposition III.1 (Initialization) $\hat{E}[1] = (\hat{e}_a[1], \hat{e}_b[1]) \in \mathbb{R}^2 \times \mathbb{R}^2$ *is an estimate of* $e[1]$.

If $\hat{e}_a[1] = \hat{e}_b[1]$, then the pursuer has accurately determined $e[1]$. In general, the pursuer is unable to distinguish between the two estimates.

1) Pursuit phase: We now present our pursuit strategy. Until the evader is *not* captured, at time $t \geq 2$,

(i) the pursuer selects a point $\hat{e}[t-1] \in \hat{E}[t-1]$ at random and moves towards it with full step size. Thus,

$$p[t] = p[t-1] + \frac{\hat{e}[t-1] - p[t-1]}{\|\hat{e}[t-1] - p[t-1]\|}. \qquad (5)$$

(ii) The pursuer updates the estimate of the evader's position using

$$\hat{E}[t] := \partial\mathcal{B}_{y_{t-1}[t]}(p[t-1]) \cap \Big(\hat{E}[t-1] \oplus \mathcal{B}_\beta(0)\Big) \cap \partial\mathcal{B}_{y_t[t]}(p[t]), \qquad (6)$$

where $\mathcal{B}_\beta(0) \subset \mathbb{R}^2$ denotes the closed circular region of radius β around the origin $0 \in \mathbb{R}^2$ and the operation $\oplus$ denotes the Minkowski sum in the plane.

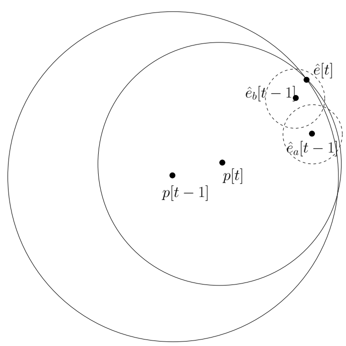


Fig. 2. An instance of the pursuit strategy. The dotted circles have radii equal to β and denote the region where the evader can step at time t. This figure illustrates the case when the pursuer moves towards $\hat{e}_a[t-1]$ while the evader was actually at $\hat{e}_b[t-1]$ and consequently exactly localizes the evader at time t.

An instance of this strategy is shown in Figure 2. A simple induction argument gives the following result, the proof of which is presented in Section V.

Lemma III.2 (Evader estimate) *At every time instant $t \in \mathbb{Z}_{\geq 1}$,*

(i) *The evader's position $e[t] \in \hat{E}[t]$, where $\hat{E}[t]$ is recursively defined using* (4) *and* (6).

(ii) *The set $\hat{E}[t]$ contains at most two points $(\hat{e}_a[t], \hat{e}_b[t]) \in \mathbb{R}^2 \times \mathbb{R}^2$. Further, $\|\hat{e}_a[t] - p[t]\| = \|\hat{e}_b[t] - p[t]\|$, for every t.*

We now present the main result of this section.

Theorem III.3 (Capture in $\mathbb{R}^2$) *If $\beta < 0.5$, then a single pursuer captures the evader using our capture strategy and in at most $\lceil \frac{\|e[0]-p[0]\|+(1+2\beta)}{2(1-2\beta)} \rceil$ time steps.*

Remark III.4 (Single pursuer in $\mathbb{R}^3$) In $\mathbb{R}^3$, it is not clear whether it is possible to guarantee capture with a single pursuer using the proposed strategy. At each time instant t, the set of evader estimates $\hat{E}[t]$ in general contains more than just two points. This motivates the use of another cooperative pursuer in $\mathbb{R}^3$, which we address in the next section.

Remark III.5 (Other Sensor-based formulations) The Grow-Intersect algorithm can also be adapted to design a pursuit strategy when the evader is a transmitter device and the pursuer is equipped with a sensor that determines *only* the line that contains their positions, without the orientation sense. Our strategy guarantees that the pursuer simultaneously captures the evader as well as ascertains that it is within unit radius if the maximum evader speed $\beta < 0.25$.

IV. Cooperative pursuit in $\mathbb{R}^3$

We now present the pursuit problem in $\mathbb{R}^3$ played with two cooperative pursuers.

A. Problem statement and notation

The problem formulation is almost identical to the planar case except that now we have two identical pursuers which move simultaneously at their turn. The game is played in $\mathbb{R}^3$. Akin to (1), the equations of motion are given by

$$e[t] = e[t-1] + u^e(e[t-1], \{p[\tau]\}_{\tau=0}^{t-1}),$$
$$p_i[t] = p_i[t-1] + u^{p_i}(p_i[t-1], y_{t-1}^i[t], y_t^i[t]),$$

where for the i^{th} pursuer, $p_i[t] \in \mathbb{R}^3$ denotes its position at time t, $y_{t-1}^i[t], y_t^i[t] \in \mathbb{R}_{\geq 0}$ are the distances of the evader from it before and after the evader's move respectively and u^{p_i} is its strategy. The strategies satisfy (2) and capture is defined when for some $i \in \{1,2\}$, (3) is satisfied.

The problem is *to determine pursuer strategies u^{p_i} that guarantee capture for any evader strategy u^e.*

B. Capture strategy and Main result

We present our solution to the cooperative pursuit game played in $\mathbb{R}^3$. Again, our capture strategy has two phases: Initialization and Pursuit. These are described as follows.

1) Initialization phase: This phase lasts for only the first sequence. In the first sequence,

(i) The evader moves to $e[1]$.

(ii) For $i = \{1,2\}$, pursuer p_i gets the measurement $y_0^i[1]$ and it constructs $\partial\mathcal{B}_{y_0^i[1]}(p_i[0])$ which is the surface of a sphere of radius $y_0^i[1]$ around the point $p_i[0]$.

(iii) Pursuer p_i selects a direction to move ensuring that $p_1[1] \neq p_2[1]$ and moves along it with unit step size.

(iv) Each pursuer p_i gets the measurement $y_1^i[1]$ and it constructs $\partial\mathcal{B}_{y_1^i[1]}(p_i[1])$ and computes the estimate

$$\hat{E}[1] := \bigcap_{i\in\{1,2\}} \Big(\partial\mathcal{B}_{y_1^i[1]}(p_i[1]) \cap \partial\mathcal{B}_{y_0^i[1]}(p_i[0])\Big). \qquad (7)$$

For each $i \in \{1,2\}$, the term in the outer parentheses in (7) is an intersection of the surfaces of two spheres in $\mathbb{R}^3$ and hence is a circle. Hence, $\hat{E}[1]$ is an intersection of two non-concentric circles and thus contains at most two points.

2) Pursuit phase: We now present our pursuit strategy.

Until the evader is *not* captured, at time $t \geq 2$,

(i) If $\hat{E}[t-1]$ contains only one point $\hat{e}[t-1]$, then the pursuer closer to it, say p_1 moves towards it with full step size. The other pursuer p_2 moves:

a) towards $\hat{e}[t-1]$ with maximum step size, if the three points $\hat{e}[t-1]$, $p_1[t-1]$ and $p_2[t-1]$ are not collinear.

b) anywhere inside except on the axis of a cone with half-angle equal to $\arcsin(\beta/\|e[t-1]-p_2[t-1]\|)$, vertex at $p_2[t-1]$ and with $e[t-1]-p_2[t-1]$ as the axis, with maximum step size, if the points $\hat{e}[t-1]$, $p_1[t-1]$ and $p_2[t-1]$ are collinear. Refer to Figure 5 for an illustration.

In case both pursuers are equidistant, then pursuer p_1 is the one that moves directly towards the evader.

Otherwise, for $i=\{1,2\}$, each pursuer p_i is assigned a unique point $\hat{e}_i[t-1]$ in $\hat{E}[t-1]$ and it moves towards it with full step size. Thus,

$$p_i[t]=p_i[t-1]+\frac{\hat{e}_i[t-1]-p_i[t-1]}{\|\hat{e}_i[t-1]-p_i[t-1]\|}. \tag{8}$$

(ii) The pursuer updates the estimate of the evader's position using

$$\hat{E}[t]:=\Big(\hat{E}[t-1]\oplus\mathcal{B}_\beta(0)\Big)\bigcap_{i\in\{1,2\}}\Big(\partial\mathcal{B}_{y_t^i[t]}(p_i[t])\cap \partial\mathcal{B}_{y_{t-1}^i[t]}(p_i[t-1])\Big). \tag{9}$$

where $\mathcal{B}_\beta(0)\subset\mathbb{R}^2$ denotes the closed sphere of radius β around the origin $0\in\mathbb{R}^2$ and the operation $\oplus$ denotes the Minkowski sum in $\mathbb{R}^3$.

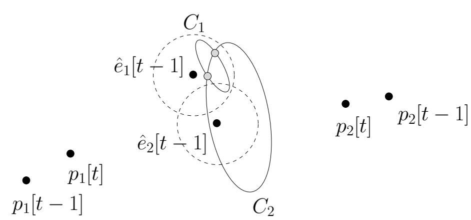


Fig. 3. An instance of the cooperative pursuit in $\mathbb{R}^3$. The dotted circles have radii equal to β and denote the region where the evader can step at time t. Circles C_1 and C_2 (shown as ellipses here) are the intersections of the two spheres (not shown to preserve clarity) associated with each measurement for each pursuer. The lightly shaded dots is the set $\hat{E}[t]$.

An instance of this strategy is shown in Figure 3. Akin to Lemma III.2 in the single pursuer problem, we have the following result.

Lemma IV.1 (Evader estimate) *At every time instant* $t\in\mathbb{Z}_{\geq 1}$,

(i) *Using the proposed cooperative pursuit strategy, the two pursuers are at distinct locations in* $\mathbb{R}^3$.
(ii) *The set* $\hat{E}[t]$ *contains at most two points* $(\hat{e}_1[t],\hat{e}_1[t])\in\mathbb{R}^3\times\mathbb{R}^3$. *Further, for each* $i\in\{1,2\}$, $\|\hat{e}_1[t]-p_i[t]\|=\|\hat{e}_2[t]-p_i[t]\|$, *for every* t.
(iii) *The evader's position* $e[t]\in\hat{E}[t]$, *where* $\hat{E}[t]$ *is recursively defined using* (7) *and* (9).

We now present the main result of this section.

Theorem IV.2 (Capture in $\mathbb{R}^3$) *If* $\beta<0.5$, *then two pursuers capture the evader using the cooperative capture strategy and in at most* $\lceil\frac{\|e[0]-p_1[0]\|+\|e[0]-p_2[0]\|+2(1+2\beta)}{(1-2\beta)}\rceil$ *time steps.*

V. Proofs of the Main Results

In this section, we present the proofs of the main results presented in Sections III and IV.

A. Single pursuer in $\mathbb{R}^2$

We begin by proving Lemma III.2.

Proof of Lemma III.2:

We prove parts (i) using mathematical induction. Proposition III.1 serves as the base of induction. Now assume $e[t-1]\in\hat{E}[t-1]$. Since the evader's step size is upper bounded by β, $e[t]\in\hat{E}[t-1]\oplus\mathcal{B}_\beta(0)$. From the definition of a sequence (ref. Section II), $e[t]$ is contained in both $\partial\mathcal{B}_{y_{t-1}[t]}(p[t-1])$ and $\partial\mathcal{B}_{y_t[t]}(p[t])$. Thus, $e[t]$ is contained in the intersection of these three quantities and part (i) follows via the principle of induction.

By part (i) of this lemma, since both $\partial\mathcal{B}_{y_{t-1}[t]}(p[t-1])$ and $\partial\mathcal{B}_{y_t[t]}(p[t])$ contain $e[t]$, their intersection is non-empty and can contain at most two points due to the fact that they are non-concentric circles. The final statement follows from the fact that the intersection points of two circles are equidistant from their centers. ■

We also have the following useful result.

Lemma V.1 *For every* $t\in\mathbb{Z}_{\geq 2}$, $\|\hat{e}_a[t]-\hat{e}_b[t]\|\leq 2\beta$, *where* $\hat{e}_a[t]$ *and* $\hat{e}_b[t]$ *are elements of the evader estimate set* $\hat{E}[t]$.

Proof: At time t, let the pursuer choose to move towards $\hat{e}_a[t]$ while executing part (i) of the pursuit strategy. From Lemma III.2, $\hat{E}[t+1]$ contains at most two points, $\hat{e}_a[t+1]$ and $\hat{e}_b[t+1]$ and $e[t+1]\in\hat{E}[t]\oplus\mathcal{B}_\beta(0)$, which implies that $e[t+1]\in\mathcal{B}_\beta(\hat{e}_a[t])\cup\mathcal{B}_\beta(\hat{e}_b[t])$. From geometry, the points $\hat{e}_a[t+1]$ and $\hat{e}_b[t+1]$ can be distinct *only if* both are contained inside $\mathcal{B}_\beta(\hat{e}_a[t])$. Thus, the result follows. ■

The last two lines of the proof of Lemma V.1 lead to a useful corollary.

Corollary V.2 *At the end of any sequence at time* $t\in\mathbb{Z}_{\geq 2}$, *if the evader estimates* $\hat{e}_a[t]$ *and* $\hat{e}_b[t]$ *are distinct, then they must be contained inside* $\mathcal{B}_\beta(\hat{e}[t-1])$, *where* $\hat{e}[t-1]$ *is the point the pursuer goes toward at the time step* t.

At each instant $t\in\mathbb{Z}_{\geq 2}$, recall that $y_t[t]:=\|e[t]-p[t]\|=\|\hat{e}_a[t]-p[t]\|=\|\hat{e}_b[t]-p[t]\|$. We have the following useful result.

Lemma V.3 *If* $\beta<0.5$, *then at every instant* $t\in\mathbb{Z}_{\geq 2}$ *for which* $y_t[t]>1$, $y_{t+1}[t+1]<y_t[t]+\beta$.

Proof: There are two main possibilities: either $\hat{E}[t]$ contains only one point, i.e., $e(t)$ or $\hat{E}[t]=(\hat{e}_a[t],\hat{e}_b[t])$. In the first case, the pursuer moves towards $e[t]$ and on applying the triangle inequality, we have $y_{t+1}[t+1]\leq y_t[t]-(1-\beta)<y_t[t]+\beta$, and the proposition is verified. In the second case, let us assume that the pursuer moves towards $\hat{e}_a[t]$. There are two possibilities now. If the evader was at $\hat{e}_a[t]$ at time t, then the result is verified to be true since this possibility is exactly similar to the first case. But if the evader was at $\hat{e}_b[t]$ at time t,

then observe that since $y_t[t] > 1$, $p[t+1]$ will lie somewhere between $p[t]$ and $\hat{e}_a[t]$. This is shown in Figure 4. By triangle inequality,

$$\begin{aligned} y_{t+1}[t+1] &= \|e[t+1] - p[t+1]\|, \\ &\leq \|\hat{e}_b[t] - p[t+1]\| + \|e[t+1] - e[t]\|. \end{aligned}$$

Since $\beta < 0.5$ and $y_t[t] > 1$, $\|\hat{e}_b[t] - p[t+1]\| < \|\hat{e}_b[t] - p[t]\| =: y_t[t]$. Thus, the result follows since $\|e[t+1] - e[t]\| \leq \beta$. ■

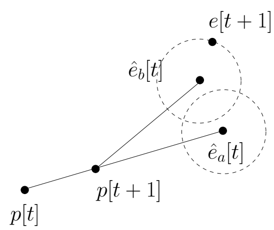


Fig. 4. Illustration of a case in Lemma V.3. The evader is at $\hat{e}_b[t]$ and the pursuer moves towards $\hat{e}_a[t]$.

We present another important result.

Lemma V.4 *For every time step $t \in \mathbb{Z}_{\geq 2}$, if $\beta < 0.5$ and as long as the evader is not captured,*

$$y_{t+2}[t+2] < y_t[t] - (1 - 2\beta).$$

Proof: At any time $t \in \mathbb{Z}_{\geq 2}$, there are two main possibilities:

(i) $\hat{e}_a[t] = \hat{e}_b[t] = e[t]$: In this case, the pursuer moves towards $e[t]$ at time $t+1$. Thus, by the triangle inequality at this step,

$$y_{t+1}[t+1] \leq y_t[t] - (1 - \beta). \tag{10}$$

At time $t+1$, there are two further cases,

1) If $\hat{e}_a[t+1] \neq \hat{e}_b[t+1]$, then by Lemma V.3, we have, $y_{t+2}[t+2] < y_{t+1}[t+1] + \beta$. This combined with (10) gives,

$$y_{t+2}[t+2] < y_{t+1}[t+1] + \beta < y_t[t] - (1 - 2\beta).$$

Thus, the lemma holds for this case.

2) If $\hat{e}_a[t+1] = \hat{e}_b[t+1]$, then akin to (10), we have,

$$\begin{aligned} y_{t+2}[t+2] &\leq y_{t+1}[t+1] - (1 - \beta), \\ &< y_t[t] - 2(1 - \beta) < y_t[t] - (1 - 2\beta). \end{aligned}$$

Thus, the lemma holds for this case.

(ii) $\hat{e}_a[t] \neq \hat{e}_b[t]$: Let the pursuer choose to move towards $\hat{e}_a[t]$ at time $t+1$. Then, there are two further possibilities.

1) $\hat{e}_a[t+1] \neq \hat{e}_b[t+1]$: From Corollary V.2, we know that $\hat{e}_a[t+1]$ and $\hat{e}_b[t+1]$ are contained in $\mathcal{B}_\beta(\hat{e}_a[t])$. So by triangle inequality,

$$y_{t+1}[t+1] \leq y_t[t] - (1 - \beta). \tag{11}$$

At time step $t+2$, independent of which point in $\hat{E}[t+1]$ the pursuer decides to move toward, by Lemma V.3, $y_{t+2}[t+2] < y_{t+1}[t+1] + \beta$. Combining this with (11), we get,

$$\begin{aligned} y_{t+2}[t+2] &\leq y_{t+1}[t+1] - (1 - \beta) + \beta, \\ &< y_t[t] - (1 - 2\beta). \end{aligned}$$

Thus, the lemma holds for this case.

2) $\hat{e}_a[t+1] = \hat{e}_b[t+1] = e[t+1]$: Applying Lemma V.3 at time step $t+1$, we get $y_{t+1}[t+1] < y_t[t] + \beta$. Before its move at time $t+2$, the pursuer knows the exact location $e[t+1]$. So at the end of time step $t+2$, by applying triangle inequality, akin to (10), we have,

$$y_{t+2}[t+2] \leq y_{t+1}[t+1] - (1 - \beta) < y_t[t] - (1 - 2\beta).$$

Thus, the lemma holds for this case.

We have verified that this lemma holds for all the possibilities. ■

The proof of Theorem III.3 is almost immediate due to Lemma V.4.

Proof of Theorem III.3: If $\beta < 0.5$, then Lemma V.4 states that for every time step $t \geq 2$ and as long as $y_t[t] > 1$, the distance $y_t[t]$ strictly decreases by a positive quantity $1 - 2\beta$ after every two time steps. Thus, after a time of at most $\frac{y_2[2]-1}{2(1-2\beta)}$, we obtain $y_t[t] \leq 1$, i.e., the evader is captured.

For the expression of the upper bound on the capture time, we seek an upper bound on $y_2[2]$. In the initialization phase, it is possible that the pursuer and evader both move in a direction away from each other. Thus, $y_1[1] \leq \|e[0] - p[0]\| + (1 + \beta)$. This can also take place at time step $t = 2$, since Lemma V.3 does not hold at time step $t = 1$. Thus, $y_2[2] \leq y_1[1] + (1 + \beta)$. Thus, a conservative upper bound on $y_2[2]$ is $\|e[0] - p[0]\| + 2(1 + \beta)$. The result now follows. ■

B. Cooperative pursuit in $\mathbb{R}^3$

We begin by proving Lemma IV.1.

Proof of Lemma IV.1: Observe that for each i, the set $\partial\mathcal{B}_{y^i_t[t]}(p_i[t]) \cap \partial\mathcal{B}_{y^i_{t-1}[t]}(p_i[t-1])$ is a circle with $p_i[t]$ located on its *axis*, i.e., the line passing through its center and perpendicular to the plane containing the circle. Thus, for each time instant t, the points in $\hat{E}[t]$ are equidistant from both pursuers.

We prove parts (i) and (ii) by mathematical induction. The lemma holds at time $t = 1$, as a consequence of the Initialization phase. Now assume that at some time t, the pursuers are at distinct locations and there are at most two points in $\hat{E}[t]$. Then there are two possibilities:

1) There are two distinct points $\hat{e}_1[t]$ and $\hat{e}_2[t]$ in $\hat{E}[t]$: If the four points $p_1[t], \hat{e}_1[t], p_2[t], \hat{e}_2[t]$ are co-planar, then $\hat{e}_1[t]$ and $\hat{e}_2[t]$ lie on opposite sides of the line joining $p_1[t]$ and $p_2[t]$. By the pursuit strategy, since each pursuer moves towards its respective $\hat{e}[t]$, the points $p_1[t+1]$ and $p_2[t+1]$ also lie on opposite sides of the line joining $p_1[t]$ and $p_2[t]$ and thus are distinct. If $p_1[t], \hat{e}_1[t], p_2[t], \hat{e}_2[t]$ are not co-planar, then the line joining $p_1[t]$ and $\hat{e}_1[t]$ and the line joining $p_2[t]$ and $\hat{e}_2[t]$ are skew in $\mathbb{R}^3$. Thus, any point on the first line is distinct from any on the second.

2) $\hat{e}_1[t] = \hat{e}_2[t] = e[t]$: If $e[t]$, $p_1[t]$ and $p_2[t]$ are not collinear, then by part (a) of item (i) in the pursuit strategy, the points $p_1[t+1]$ and $p_2[t+1]$ are distinct. If they are collinear, then the axis of the cone described in part (b) of item (i) of the pursuit strategy is the line l passing through $e[t]$, $p_1[t]$ and $p_2[t]$. The pursuer closer to the evader, say p_1 moves towards

$e[t]$ and hence is still on the line l, while p_2 moves to a point not contained in l and thus $p_1[t+1]$ and $p_2[t+1]$ are distinct.

Thus, the pursuers are at distinct locations at time $t+1$. For part (ii), observe that at time instant t, pursuer p_2 does not move towards $p_1[t-1]$. This means that the axes (defined in the first line of this proof) of the two circles $\partial\mathcal{B}_{y^i_{t+1}[t+1]}(p_i[t+1]) \cap \partial\mathcal{B}_{y^i_t[t+1]}(p_i[t])$ are never parallel to each other. Thus, their intersection and hence $\hat{E}[t+1]$ contains at most two points. Thus, the result holds by mathematical induction.

Proof of item (iii) is on similar lines as that of item (i) of Lemma III.2. ■

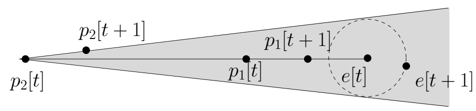


Fig. 5. Illustration of case 2 in Lemma IV.1. The shaded region is the cone described in part (b) of item (i) of the Pursuit phase.

Lemma V.5 *For every $t \in \mathbb{Z}_{\geq 2}$, $\|\hat{e}_1[t] - \hat{e}_2[t]\| \leq 2\beta$, where $\hat{e}_1[t]$ and $\hat{e}_1[t]$ are elements of the evader estimate set $\hat{E}[t]$.*

Proof: Let the evader be located at $\hat{e}_1[t-1]$ at time $t-1$. p_1 moves towards $\hat{e}_1[t-1]$ and hence $e[t]$ must be contained in $\left(\partial\mathcal{B}_{y^1_t[t]}(p_1[t]) \cap \partial\mathcal{B}_{y^1_{t-1}[t]}(p_1[t-1])\right) \bigcap \mathcal{B}_\beta(\hat{e}_1[t-1]) \subset \mathcal{B}_\beta(\hat{e}_1[t-1])$, which is a circle. The intersection points of this circle with the other circle due to p_2, must be contained inside $\mathcal{B}_\beta(\hat{e}_1[t-1])$ and thus the result follows. ■

Next, we observe that Lemma V.3 holds for the cooperative pursuit strategy as well. This follows from Lemma V.5 and the fact that only the triangle inequality was being used in the proof of Lemma V.3. The only extra technicality is the possibility of occurrence of case 2 as in the proof of Lemma IV.1 (refer Figure 5). However, a simple calculation reveals that Lemma V.3 still holds due to the motion as per part (b) of item (i) in the Pursuit strategy.

We now present a useful result.

Lemma V.6 *For every time step $t \in \mathbb{Z}_{\geq 2}$, if $\beta < 0.5$ and as long as the evader is not captured,*

$$y^1_{t+1}[t+1] + y^2_{t+1}[t+1] < y^1_t[t] + y^2_t[t] - (1-2\beta).$$

Proof: At any instant t, by Lemma IV.1, it is clear that the evader is in $\hat{E}[t]$, which contains at most two points, $\hat{e}_1[t], \hat{e}_2[t]$. Let the evader be located at $\hat{e}_1[t]$. Then, by triangle inequality

$$\begin{aligned} y^1_{t+1}[t+1] &= \|e[t+1] - p_1[t+1]\| \leq \|e[t] - p_1[t]\| - (1-\beta), \\ &= y^1_t[t] - (1-\beta). \end{aligned}$$

From Lemma V.3, we have $y^2_{t+1}[t+1] < y^2_t[t] + \beta$. Thus, adding the two inequalities, we get the required result. ■

Proof of Theorem IV.2: If $\beta < 0.5$ and if both $y^1_t[t]$ and $y^2_t[t]$ are greater than 1 for $t \in \mathbb{Z}_{\geq 2}$, then by Lemma V.6, their sum $y^1_t[t] + y^2_t[t]$ strictly decreases by a positive quantity $(1-2\beta)$ at every instant of time. Thus, after at most $\lceil \frac{y^1_2[2]+y^2_2[2]-2}{(1-2\beta)} \rceil$ time steps, $y^1_t[t] + y^2_t[t] \leq 2$, which means either $y^1_t[t] \leq 1$ or $y^2_t[t] \leq 1$, i.e., the evader is captured.

For the expression of the upper bound on the capture time, we seek an upper bound on $y^1_2[2] + y^2_2[2]$. On similar lines to the proof of Theorem III.3, we have $y^i_2[2] \leq \|e[0] - p_i[0]\| + 2(1+\beta)$. Thus, the result follows. ■

VI. Simulation Studies

We now present simulation studies that address the case of evader speed $\beta \in [0.5, 1[$ and the case of the pursuer measurements being corrupted with additive, zero-mean Gaussian noise, with variance proportional to the square of the distance to the evader. All simulations were run using MATLAB®.

A. The case of $\beta \in [0.5, 1[$

We ran simulations for $\|e[0] - p[0]\| = 20$, 30 and 40 units. An upper limit of 1000 time steps was set to decide whether the capture strategy terminated into capture or evasion.

It is unclear as to what is the optimal evader strategy in this problem. This is because if the evader decides to always move directly away from the pursuer with full step (i.e., *greedy* move), then it would reduce the uncertainty in its position for the pursuer. If it does not make a greedy move, then the distance from the pursuer may reduce. So we adopt the following reasonable evader strategy for simulations - with full step, move to a point selected uniformly randomly in a sector with angle 0.2 radians. This sector is placed symmetrically along the line $e[t]p[t]$ and away from the pursuer.

The plots of probability of success of the strategy versus the evader speed β are presented in Figure 6.

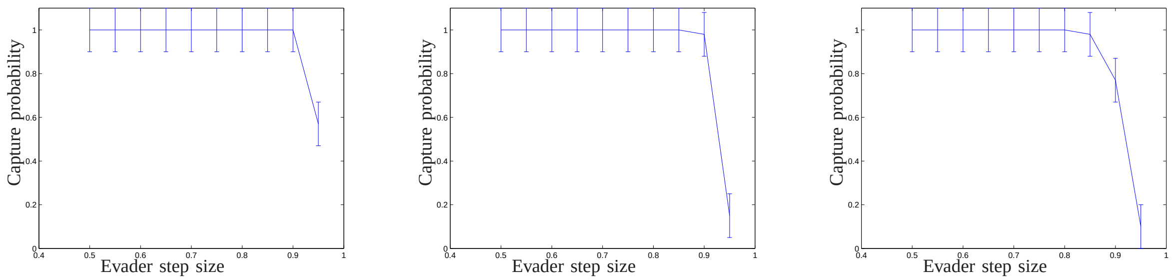
B. Noisy measurements

We now assume that the pursuer measurements are corrupted with zero-mean, additive Gaussian noise whose variance proportional to the square of the distance to the evader. This implies $\sigma_{t1}[t] = \nu\|e[t]p[t1]\|$ and $\sigma_t[t] = \nu\|e[t]p[t]\|$, where $\nu > 0$ is the noise parameter. Thus, in the notation of Section II, $y_{t1}[t] \sim \mathcal{N}(\|e[t]p[t1]\|, \sigma_{t1}[t])$ and $y_t[t] \sim \mathcal{N}(\|e[t]p[t]\|, \sigma_t[t])$, where given $a, b \geq 0$, $\mathcal{N}(a, b)$ denotes the Gaussian distribution with mean a and standard deviation b.

Since it is unclear as to what is the optimal evader strategy in this problem, we adopted the evader strategy in Section VI-A. We ran simulations for $\beta = 0.2$, 0.3 and 0.4 units. The initial distance was set to 20 units. An upper limit of 2000 time steps was set to decide whether the capture strategy terminated into capture or evasion. The plots of probability of success of the strategy versus the noise parameter ν are presented in Figure 7.

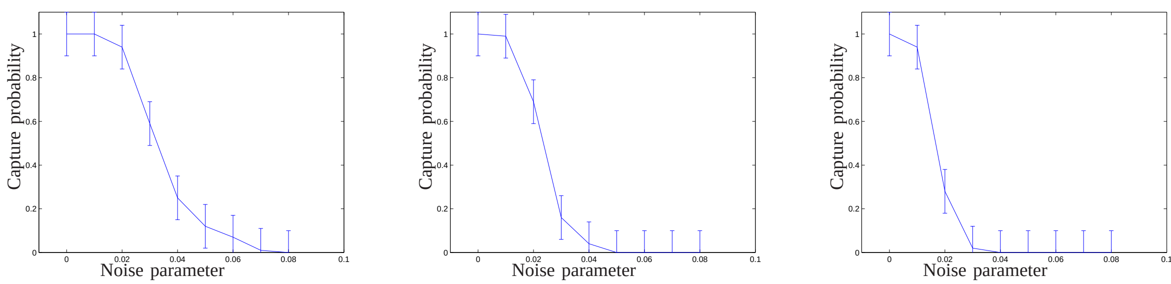
VII. A Game with Simultaneous Moves: Simulation Study

We now consider a discrete-time version of the game in the plane in which the pursuer and the evader move simultaneously. In this version, at each instant of time, each player gets *only one* measurement of its opponent. This is equivalent to



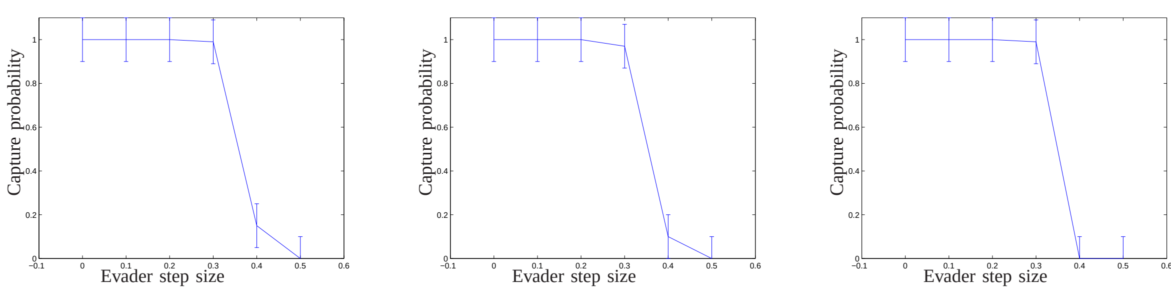
(a) Initial distance: 20 units (b) Initial distance: 30 units (c) Initial distance: 40 units

Fig. 6. Estimate of capture probability versus evader step size β. The vertical bars give a 95% confidence interval about the probability estimate $P(\beta)$ which is given by $\left[P(\beta) - 2\sqrt{\frac{0.25}{n}}, P(\beta) + 2\sqrt{\frac{0.25}{n}}\right]$, where $n = 100$ is the number of trials [24]. For a particular evader strategy, we study how the capture strategy performs for evader step size $\beta \in [0.5, 1[$.



(a) Evader step size: $\beta = 0.2$ (b) Evader step size: $\beta = 0.3$ (c) Evader step size: $\beta = 0.4$

Fig. 7. Estimate of capture probability versus noise parameter ν. The vertical bars give a 95% confidence interval about the probability estimate $P(\nu)$ computed as described in Figure 6. For a particular evader strategy, we study how the capture strategy performs under noisy measurements.



(a) Initial distance: 20 units (b) Initial distance: 30 units (c) Initial distance: 40 units

Fig. 8. Estimate of capture probability versus evader step size β, in the game with simultaneous moves. The vertical bars give a 95% confidence interval about the probability estimate $P(\beta)$, computed as described in Figure 6. For a particular evader strategy, we study the performance of a modified capture strategy presented in Section VII.

a game in which the pursuer receives *only* the distance to the evader at each instant in continuous time. Thus, (1) becomes

$$e[t] = e[t-1] + u^e(e[t-1], \{p[\tau]\}_{\tau=0}^{t-1}),$$
$$p[t] = p[t-1] + u^p(p[t-1], y[t-1]),$$

We modify the capture strategy in Section III as follows.

Initialization phase: The following happens simultaneously for only the first time step:

(i) The evader moves to $e[1]$.

(ii) The pursuer randomly selects a direction to move and moves along it with unit step size.

(iii) The pursuer gets the measurement $y[1]$ and the evader estimate is given by

$$\hat{E}[1] := \partial\mathcal{B}_{y[1]}(p[1]).$$

Pursuit Phase: Until the evader is *not* captured, at time $t \geq 2$,

(i) If $\hat{E}[t-1]$ is a circle, then denote any point in it as $\hat{e}[t-1]$. Otherwise, denote as $\hat{e}[t-1]$ a point chosen uniformly randomly from one of the end points of the arcs in $\hat{E}[t-1]$. The pursuer moves towards $\hat{e}[t-1]$ with full step size.

(ii) The pursuer updates the estimate of the evader's position using

$$\hat{E}[t] := \left(\hat{E}[t-1] \oplus \mathcal{B}_\beta(0)\right) \cap \partial\mathcal{B}_{y[t]}(p[t]).$$

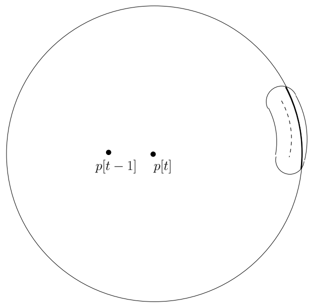


Fig. 9. Illustration of the pursuit strategy in the game with simultaneous moves. The dotted line is the estimate $\hat{E}[t-1]$. The bean-shaped region around it is its Minkowski sum with $\mathcal{B}_\beta(0)$ and the darkly shaded arc is the estimate $\hat{E}[t]$.

The strategy is illustrated in Figure 9. Since it is unclear as to what is the optimal evader strategy in this problem, we adopted the same evader strategy as in Section VI-A. We ran simulations for $\|e[0] - p[0]\| = 20$, 30 and 40 units. An upper limit of 5000 time steps was set to decide whether the capture strategy terminated into capture or evasion. The plots of probability of success of the strategy versus the evader step size β are presented in Figure 8.

VIII. Conclusion and Future Directions

We addressed a discrete-time alternate-motion pursuit-evasion game in the plane in which the pursuer is equipped with a range-only sensor that measures its distance from the evader. We propose a capture strategy based on the established Grow-Intersect algorithm, and show that if the evader's maximum step size $\beta < 0.5$, then the evader is captured. We then posed a variant of this game in $\mathbb{R}^3$ and showed that two cooperative pursuers capture the evader with a modified capture strategy if $\beta < 0.5$. We gave upper bounds on the capture times in both games.

We then presented simulation studies that addressed the case of $\beta \in [0.5, 1[$ and the case of noisy measurements in the planar game. We deduce that the proposed capture strategy performs fairly well against a reasonable evader strategy in the former case while in the latter case, we observe some amount of robustness to small values of the noise parameter. Finally, we presented a simulation study of a variant of this game with simultaneous moves. Based on simulation results, we deduce that the appropriately modified capture strategy succeeds with probability of more than 97% in the regime of $\beta \in [0, 0.3]$.

We have also been able to adapt the Grow-Intersect algorithm to design a pursuit strategy when the evader is a transmitter device and the pursuer is equipped with a sensor that determines *only* the line that contains their positions, without the orientation sense. Details of this case-study will be a part of future work. Also, it would be interesting to design a provably-correct capture strategy for the game with simultaneous moves. Additional directions would be to consider noisy situations in the same.

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