A cooperative Homicidal Chauffeur game

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Abstract

We address a pursuit-evasion problem involving an unbounded planar environment, a single evader and multiple pursuers moving along curves of bounded curvature. The problem amounts to a multi-agent version of the classic *homicidal chauffeur* problem; we identify parameter ranges in which a single pursuer is not sufficient to guarantee evader capture. We propose a novel multi-phase cooperative strategy in which the pursuers move in specific formations and confine the evader to a bounded region. The proposed strategy is inspired by hunting and foraging behaviors of various fish species. We characterize the required number of pursuers for which our strategy is guaranteed to lead to confinement.

Key words: Pursuit evasion games, Homicidal Chauffeur game, cooperative control.

1 Introduction

The homicidal chauffeur game has been studied in great detail. Proposed originally by Isaacs [1], this problem involves a pursuer who wants to overrun an evader, both moving with fixed speeds. The pursuer has greater speed but has constraints on its turning radius, while the evader can make arbitrarily sharp turns. The evader is said to be *captured* when the distance between the pursuer and evader becomes less than a specified *capture radius*. In this paper, we consider the multi-agent homicidal chauffeur problem in which a single pursuer is not sufficient to guarantee evader capture. We propose a cooperative strategy for multiple pursuers to confine the evader in a bounded region which the evader cannot leave without being captured. Such strategies are important in surveillance, as well as in search-and-rescue operations where a guarantee of rescue is desired, in spite of any unpredictable motion of the victim.

The classic homicidal chauffeur problem was proposed and solved by Isaacs [1]. The pursuer moves at fixed speed along planar paths with bounded curvature. The evader moves with a fixed speed lower than that of the pursuer and governed by a simple first-order-integrator dynamics. Isaacs gives a condition on the game parameters, i.e., the speed ratio between the players and the ratio of the capture radius to the minimum turning radius of the pursuer, such that the evader can evade indefinitely. Numerous variations of this problem have been studied, e.g., capture inside a cone sector [2], effects of stochastic noise [3] and a version without a priori assignment of the role of pursuer or evader [4] to cite a few.

Recent research attention has focused on cooperative control strategies for detection of targets. McLain *et al.* [5] have addressed the problem of cooperative rendezvous in which multiple UAVs are to arrive simultaneously at their targets. Polycarpou *et al.* [6] have presented a cooperative target search using online learning and computing guidance trajectories for the agents. Recently, Tang *et al.* [7] have presented cooperative motion planning methods for first-order mobile sensing agents to detect a moving target that lies in a known initial region. McGee *et al.* [8] have proposed guaranteed strategies to search for mobile evaders in a plane. Recently, Kim *et al.* [9] and Belkhouche *et al.* [10] have proposed schemes for agents with first-order dynamics to capture a target by arriving on a circle with specified radius around it.

Based on Isaacs' analysis of the Homicidal Chauffeur game, we identify regimes for the game parameters, i.e., the evader/pursuer speed ratio and the ratio of the capture radius to the pursuer's minimum turning radius, for which there exists a strategy for the evader to avoid capture. This motivates a multiple pursuer formulation of the game. We seek to confine the evader within a

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bounded region, for which we propose a multiple pursuer formation and a novel multi-phase, cooperative strategy for the pursuers. During all phases, the pursuers move in a specific formation, whereby some pursuer plays the role of "leader" and all other pursuers play the role of "followers." The strategy is partly decentralized, in the sense that it suffices to specify only the motion of the leader in each phase. For the followers, the only information required is the motion of the neighboring pursuer and the evader position. In the initial PRE-ALIGN and ALIGN phases of the strategy, the leader pursuer moves in such a way that the evader lies at a required distance directly ahead of the leader pursuer, while the followers move so as to maintain a straight line formation. In the remaining SWERVE, ENCIRCLE and CLOSE phases, the pursuers get into a "daisy-chain" formation and move to approach, encircle and finally close the chain around the evader. Independent of the evader motion, the final closed daisy-chain formation confines the evader within a bounded region, from which there exists no evader trajectory avoiding capture. Thus, given (i) the evader/pursuer speed ratio which is less than unity and (ii) the ratio of the capture radius to the pursuers' minimum turning radius, we characterize the required number of pursuers for which confinement is guaranteed.

The inspiration for our strategy comes from certain aspects of fish behavior. Gazda *et al.* [11] reported that in Cedar Key, Florida, USA, individual "driver" dolphins herd slower, more agile prey in circles as well as towards the tightly-grouped "barrier" dolphins. Pitman *et al.* [12] reported a herd of killer whales imposing confinement on pantropical spotted dolphins. The whales cut out up to three dolphins from a school and proceeded to take turns chasing a single dolphin and keeping it within a confined area.

This paper is organized as follows. The mathematical model and assumptions are presented in Section 2. The daisy-chain formation, the confinement strategy and the main analysis result are presented in Section 3. Section 4 contains the proof of the main result and some intermediate results. Section 5 contains our concluding remarks.

2 Problem Set-up

Our cooperative homicidal chauffeur game is played in an unbounded, planar environment between a single evader and multiple pursuers. The pursuers have identical motion abilities and possess greater speed than that of the evader. However, the evader can make arbitrarily sharp turns, while the pursuers are *Dubins* vehicles [13], i.e., fixed-speed non-holonomic vehicles constrained to move along paths of bounded curvature. We assume that the instantaneous position and velocity of the evader is available to all pursuers.

Let $e(t)$ and $p_k(t)$, for $k \in \{1, \dots, N\}$, denote the positions of the evader and the k^{th} pursuer in $\mathbb{R}^2$ at time t, as shown in Figure 1. Let $v_{\rm e}$ and $v_{\rm p}$ denote the speeds of the evader and of all pursuers, respectively. Given a *minimum turning radius* $\rho > 0$, the equations of motion are given by

$$\begin{aligned} \dot p_{k,x}(t) &= v_{\rm p}\cos\theta_{{\rm p},k}(t), \quad \dot e_x(t) = v_{\rm e}\cos\theta_{\rm e}(t), \\ \dot p_{k,y}(t) &= v_{\rm p}\sin\theta_{{\rm p},k}(t), \quad \dot e_y(t) = v_{\rm e}\sin\theta_{\rm e}(t), \\ \dot\theta_{{\rm p},k} &= \frac{v_{\rm p}}{\rho} u_{{\rm p},k}, \end{aligned} \tag{1}$$

where $\theta_{\rm e}(t)$ (resp. $\theta_{{\rm p},k}(t)$) is the angle between the velocity vector of the evader (resp. of the k^{th} pursuer) measured counterclockwise from a reference horizontal axis [1]. The control input for the evader is $\theta_{\rm e}(t) : [0, \infty[\to [0, 2\pi]$, which we assume is a measurable function of time. $u_{{\rm p},k} \in [-1, 1]$, is the control applied by the k^{th} pursuer. We define the *evader/pursuer speed ratio* $\gamma := v_{\rm e}/v_{\rm p}$ and assume $\gamma < 1$. Given a *capture radius* $c > 0$, the evader is said to be *captured* if, at some time t and for some k, the evader is at a distance of at most c units from pursuer p_k. In what follows, without loss of generality, we set the capture radius c and the pursuers speed $v_{\rm e}$ to 1. In summary, our cooperative homicidal chauffeur game is described by the number of pursuers $N \in \mathbb{N}$, the minimum turning radius $\rho \in \mathbb{R}_{>0}$, and the evader/pursuers speed ratio $\gamma \in]0, 1[$.

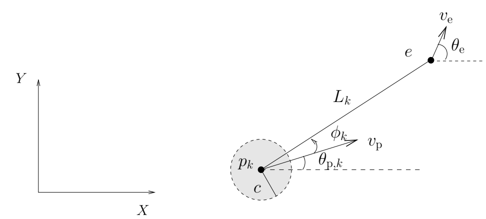

Fig. 1. Intermediate variables in the homicidal chauffeur game: L_k is the distance between the evader and the k^{th} pursuer; $\phi_k \in [0, \pi]$ is the unsigned angle between the k^{th} pursuer's velocity vector and the vector $e - p_k$. The shaded region around the pursuer is its capture disc.

In the case of a single pursuer and single evader, it can be shown in that for $\rho \geq 5/2$, there exists an evasion policy if the evader/pursuer speed ratio satisfies $\gamma \geq \gamma_{\min}(\rho)$, where $\gamma_{\min} : [5/2, +\infty[\to]0, 1[$ is the unique solution to

$$\frac{1}{x} = \sqrt{1 - \gamma_{\min}(x)^2} + \gamma_{\min}(x)\arcsin(\gamma_{\min}(x)) - 1,$$

[1]. This motivates our cooperative version of the homicidal chauffeur game. The use of a game-theoretic approach to determine capture strategies involves solving the Hamilton-Jacobi-Bellman-Isaacs equation, which is difficult to solve in the present context. Therefore, taking motivation from biology, we introduce the notion of evader confinement as follows.

Definition 2.1 (Confinement) *The evader is said to be* confined *to a bounded region* $\mathcal{G} \subset \mathbb{R}^2$ *at time* t^* *if* $e(t^*) \in \mathcal{G}$ *and there exist pursuer trajectories* $p_k : [t^*, +\infty[\to \mathbb{R}^2$ *solutions to equation* (1) *such that the evader cannot leave* $\mathcal{G}$ *without being captured. A set of functions* $\{u_{\mathrm{p},k}\}$*, for* $k \in \{1, \ldots, N\}$*, leading to evader confinement is termed as a* confinement strategy*.*

In our cooperative homicidal chauffeur game with the evader/pursuer speed ratio $\gamma < 1$, we seek deterministic multiple-pursuer strategies that guarantee evader confinement given any value of the pursuer's minimum turning radius $\rho > 0$.

3 A Confinement Strategy

In this section, we design a cooperative strategy for evader confinement and state our main analysis result. We begin by proposing two useful pursuer formations. We denote the velocity vector of the k^{th} pursuer by $\overline{v}_{\mathrm{p},k}$.

Definition 3.1 (Line formation) *The set* $\{p_1, \ldots, p_N, \overline{v}_{\mathrm{p},1}, \ldots, \overline{v}_{\mathrm{p},N}\}$ *is in a* line formation *if*

(i) $p_1, \ldots, p_N$ *are on a straight line with the velocity vectors* $\overline{v}_{\mathrm{p},1}, \ldots, \overline{v}_{\mathrm{p},N}$ *parallel to one-another, and*

(ii) For every $k \in \{1, \ldots, N-2\}$*,* $\|p_k - p_{k+1}\| = \|p_{k+1} - p_{k+2}\| > 0$*.*

Figure 2 shows an example of a line formation. In what follows, we refer to pursuer p_1 as the *leader*, unless specified otherwise. A line formation has the property that,

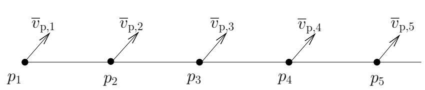

Fig. 2. A pursuer line formation with $N = 5$ pursuers.

if all pursuers start in a line formation and use identical control inputs, then they remain in a line formation.

Definition 3.2 (Daisy-chain formation) *Given* $s_{\mathrm{ip}} > 0$*, the set* $\{p_1, \ldots, p_N, \overline{v}_{\mathrm{p},1}, \ldots, \overline{v}_{\mathrm{p},N}\}$ *is said to be in a* daisy-chain formation *at time* t *if, for every* $k \in \{2, \ldots, N\}$*, pursuer* p_k *can attain at time* $t + s_{\mathrm{ip}}$*, the position and orientation at time* t *of pursuer* p_{k-1}*. Formally, for every* $k \in \{2, \ldots, N\}$*, there exists a solution* $\eta : [t, t + s_{\mathrm{ip}}] \to \mathbb{R}^2$ *to equation* (1) *satisfying*

$$\eta(t) = p_k, \qquad \dot{\eta}(t) = \overline{v}_{\mathrm{p},k},$$
$$\eta(t + s_{\mathrm{ip}}) = p_{k-1}, \qquad \dot{\eta}(t + s_{\mathrm{ip}}) = \overline{v}_{\mathrm{p},k-1}.$$

The quantity s_{ip} in Definition 3.2 is also the *inter-pursuer separation* distance, since the pursuers' speed is normalized to unity. Figure 3 shows an example of a daisy-chain formation. A daisy-chain formation has the property that any time instant, a path taken by the leader pursuer can be exactly traversed by the k^{th} follower pursuer, for every $k \in \{2, \ldots, N\}$, in the daisy-chain after a time delay of $(k-1)s_{\mathrm{ip}}$.

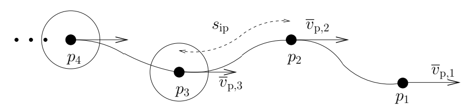

Fig. 3. A daisy-chain formation with inter-pursuer separation s_{ip}. The curve between two consecutive pursuers is an example of a solution η as described in Definition 3.2. The discs around the pursuers represent their capture discs.

Next, we characterize a possible evader motion. For $q \in \mathbb{R}^2$, let $B_r(q) \subset \mathbb{R}^2$ denote the closed ball of radius r centered at q. Given $\{p_{k-1}, p_k, \overline{v}_{\mathrm{p},k-1}, \overline{v}_{\mathrm{p},k}\}$ in a daisy-chain formation at time t with inter-pursuer separation s_{ip}, let $\mathcal{C}^{k-1,k}_{\mathrm{left}}(t)$ and $\mathcal{C}^{k-1,k}_{\mathrm{right}}(t)$ be curves which are tangent to $B_{\mathrm{c}}(\eta(\tau))$ for every $t \in [t, t + s_{\mathrm{ip}}]$. Here, η is a curve described in Definition 3.2. Then, the evader is said to *move between* p_{k-1} *and* p_k if $e(t) \in \mathcal{C}^{k-1,k}_{\mathrm{left}}(t)$ and $e(t + \overline{\tau}) \in \mathcal{C}^{k-1,k}_{\mathrm{right}}(t)$ or if $e(t) \in \mathcal{C}^{k-1,k}_{\mathrm{right}}(t)$ and $e(t + \overline{\tau}) \in \mathcal{C}^{k-1,k}_{\mathrm{left}}(t)$, for some $\overline{\tau} < s_{\mathrm{ip}}$. Figure 4 shows an example of such an evader trajectory. Given the pursuers' min-

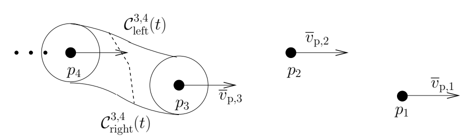

Fig. 4. An example of the evader moving between pursuers p_3 and p_4. The dotted line between curves $\mathcal{C}^{3,4}_{\mathrm{left}}(t)$ and $\mathcal{C}^{3,4}_{\mathrm{right}}(t)$ shows one possible evader trajectory.

imum turning radius ρ, for the evader/pursuers speed ratio γ, we define the *critical inter-pursuer separation* as

$$s^*_{\mathrm{ip}}(\gamma, \rho) := \max\{2, \rho \cdot \Theta(\gamma, \rho)\}, \text{ where} \qquad (2)$$

$$\Theta(\gamma, \rho) := \sqrt{\frac{(1+\rho)^2}{\gamma^2\rho^2} - 1} - \arctan\sqrt{\frac{(1+\rho)^2}{\gamma^2\rho^2} - 1} - \sqrt{\frac{1}{\gamma^2} - 1} + \arctan\sqrt{\frac{1}{\gamma^2} - 1}.$$

The quantity $s^*_{\mathrm{ip}}(\gamma, \rho)$ has the following property.

Lemma 3.3 (Critical inter-pursuer separation) *Given* $\{p_{k-1}, p_k, \overline{v}_{\mathrm{p},k-1}, \overline{v}_{\mathrm{p},k}\}$ *in a daisy-chain formation, an inter-pursuer separation* $s_{\mathrm{ip}} \leq s^*_{\mathrm{ip}}(\gamma, \rho)$ *ensures that the evader cannot not move between* p_{k-1} *and* p_k *without being captured.*

Finally, we define two useful notions. First, a point $q \in \mathbb{R}^2$ is said to be *aligned* with $\{p_k, \overline{v}_{p,k}\}$ if the velocity vector $\overline{v}_{\mathrm{p},k}$ is parallel to $(q - p_k)$. Second, a daisy-chain formation with separation s_{ip} is said to be *closed* if there exists some $k \in \{2, \ldots, N\}$ and a path of length no more than s_{ip} that leads the leader pursuer to the position and orientation of the k^{th} pursuer. More specifically, a daisy-chain formation with separation s_{ip} is closed if for some $k \in \{2, \ldots, N\}$, there exists a $t_k \leq s^*_{\mathrm{ip}}(\gamma, \rho)$ and a solution $\eta : [0, t_k] \to \mathbb{R}^2$ to equation (1) satisfying

$$\eta(0) = p_1, \qquad \dot{\eta}(0) = \overline{v}_{\mathrm{p},1},$$
$$\eta(t_k) = p_k, \qquad \dot{\eta}(t_k) = \overline{v}_{\mathrm{p},k}.$$

We now present our CONFINEMENT strategy. The pursuers begin in a line formation such that the distance between every two consecutive pursuers is $s^*_{\mathrm{ip}}(\gamma, \rho)$. Pursuer p_1 is elected as leader of the line formation. We describe the strategy in the following five phases:

[Phase 1: PRE-ALIGN] The aim of the PRE-ALIGN phase is to ensure that the evader becomes aligned with $\{p_1, \overline{v}_{p,1}\}$ after some finite time, and that all the pursuers are in a line formation with the same initial separation $s^*_{\mathrm{ip}}(\gamma, \rho)$. If the pursuers are already in this configuration, then proceed to Phase 2. Otherwise, pursuer p_1 performs the following maneuver: p_1 moves sufficiently far from the evader and turns on a tightest circle until the evader gets aligned with $\{p_1, \overline{v}_{p,1}\}$. All other pursuers move using identical control inputs to maintain the line formation. We refer the reader to [14] for details of this maneuver.

[Phase 2: ALIGN] The aim of the ALIGN phase is to bring pursuer p_1 within distance $\gamma\rho$ of the evader, that is, to achieve $\|e - p_1\| \leq \gamma\rho$, while maintaining the evader aligned with $\{p_1, \overline{v}_{p,1}\}$ (this property was achieved by the PRE-ALIGN phase). During the ALIGN phase each pursuer p_k, $k \in \{1, \ldots, N\}$, moves according to

$$u_{\mathrm{p},k}(\theta_{\mathrm{e}}, e, \theta_{\mathrm{p},1}, p_1) = \frac{\rho\gamma}{\|p_1 - e\|} \sin(\theta_{\mathrm{e}} - \theta_{\mathrm{p},1}). \qquad (3)$$

We will show later that at the end of this phase, $\|p_1 - e\| \leq \gamma\rho$, e is aligned with $\{p_1, \overline{v}_{p,1}\}$, and all pursuers are in a line formation, see Figure 5.

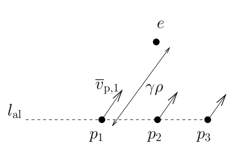

Fig. 5. End of the ALIGN phase (beginning of the SWERVE phase); all pursuers are on a line formation. l_{al} denotes the line defining the line formation at the end of the ALIGN phase.

[Phase 3: SWERVE] This phase has two aims. First, the pursuers move into a straight-line daisy-chain formation with separation $s^*_{\mathrm{ip}}(\gamma, \rho)$. Second, once the daisy-chain is formed, a new pursuer is elected as leader based on the relative position of the evader. These two steps are described as follows:

(i) **Form daisy-chain:** Each pursuer p_k, $k \in \{1, \ldots, N\}$, moves with maximal angular velocity $|u_{\mathrm{p},k}| = 1$ until all the pursuers are in a straight-line daisy-chain formation, as shown in Figure 6. This straight line through the pursuer positions is denoted by l_{sw}. The pursuers turn counterclockwise (resp. clockwise) if all other pursuers are located to the right (resp. left) side of pursuer p_1 in the line formation.

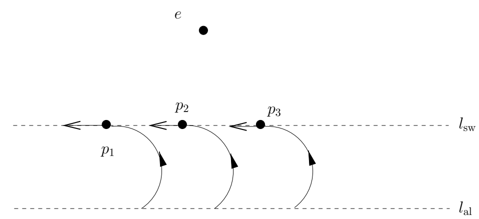

Fig. 6. Forming a straight-line daisy-chain in the SWERVE phase. Starting from the configuration in Figure 5, the pursuers have turned counterclockwise and are now on the line l_{sw} with headings along l_{sw}.

(ii) **Re-elect leader:** Compute the angle $\phi_k = \phi_k(p_k, \overline{v}_{p,k}, e)$, for $k \in \{1, \ldots, N\}$, according to the definition in Figure 1. If there exists k for which $|\phi_k| \geq \frac{\pi}{2}$ (see Figure 7), then set $l := \max\{k \in \{1, \ldots, N\} | \ |\phi_k| \geq \frac{\pi}{2}\}$, discard from consideration the motion of the pursuers $p_1, \ldots, p_{l-1}$, and select pursuer p_l as the leader for the remaining daisy-chain formation. Otherwise, if $|\phi_k| < \frac{\pi}{2}$ for all $k \in \{1, \ldots, N\}$, then set $l = 1$, retain p_1 as the leader, and move p_l straight until $\phi_l = \pi/2$.

We shall show later that with a sufficiently large number of pursuers, at the end of this step, there are more than one pursuer in the remaining daisy-chain formation.

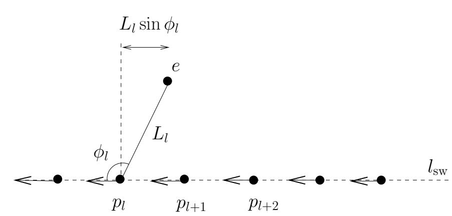

Fig. 7. Election of the leader and the end of the SWERVE phase (beginning of the ENCIRCLE phase). All pursuers in front of p_l do not play any role in the subsequent phases.

[Phase 4: ENCIRCLE] The aim of the ENCIRCLE phase is to move to pursuers towards a closed shape and enclose the evader inside it. This is achieved via a alternating sequence of *turn* and *move straight* maneuvers. The strategy for the leader p_l is as follows:

(i) **Turn:** The pursuer p_l moves on a circular arc of appropriate radius and angle if the evader is "sufficiently behind" it. Specifically, if $|L_l \cos\phi_l| \geq \rho$ and

$\phi_l \geq \pi/2$, then p_l moves on the circle with radius $R := \max\{L_l \sin\phi_l, \rho\}$ and with center in the half-plane that (i) is formed by the line along $\overline{v}_{\mathrm{p},l}$, and (ii) contains the evader. This maneuver lasts for a time interval $\Delta t := R \arctan(\sqrt{1-\gamma^2}/\gamma)$.

(ii) **Move straight:** If the evader is not "sufficiently behind" pursuer p_l, then p_l moves on a straight line to ensure that the evader gets "sufficiently behind" it, i.e., moves with $u_{\mathrm{p},l} = 0$, until $|L_l \cos\phi_l| \geq \rho$ and $\phi_l \geq \pi/2$.

The remaining pursuers follow the path of p_l, as shown in Figure 8. The ENCIRCLE phase ends when the velocity vector $\overline{v}_{p,l}$ has rotated by at least $3\pi/2$ with respect to its orientation at the start of the ENCIRCLE phase.

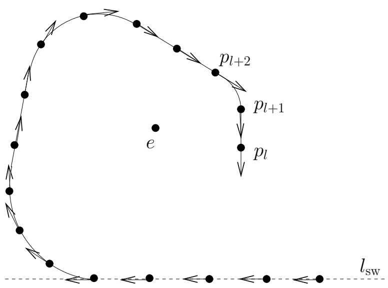

Fig. 8. End of the ENCIRCLE phase (beginning of the CLOSE phase). The leader p_l keeps the evader on the same side of its velocity vector with the alternating turn-move straight maneuvers, until its velocity vector rotates by at least $3\pi/2$.

[Phase 5: CLOSE] The aim of the CLOSE phase is to close the daisy-chain around the evader in two steps:

(i) Pursuer p_l moves straight until it lies on the path between two pursuers in the daisy-chain, (cf. Figure 9).

(ii) Next, pursuer p_l moves on a circle C_1 of radius ρ centered at O_1, where O_1 is on the same side of the line along $\overline{v}_{\mathrm{p},l}$ as the evader. Then, it determines the location of center O_2 of circle C_2 of radius ρ which is tangent to C_1 and either l_{sw} or the path followed by p_l. Of the two possible locations for O_2, it selects the one which is further away from location of p_l at the end of part (i). Pursuer p_l moves along C_2 after reaching the tangency point until it closes the daisy-chain. This path is illustrated in Figure 9.

This five-phase strategy gives us our main result.

Theorem 3.4 (Confinement) *Consider a cooperative homicidal chauffeur game with parameters $N \in \mathbb{N}$, $\rho > 0$, and $\gamma < 1$. The proposed five-phase strategy guarantees evader confinement if the number of pursuers satisfies*

$$N \geq N_{\min}(\gamma,\rho) := \lceil \rho(3+\gamma\pi)/s^*_{\mathrm{ip}}(\gamma,\rho) \rceil + \Big\lceil \frac{2(1+\gamma)\rho}{s^*_{\mathrm{ip}}(\gamma,\rho)} \Big(K^{i_{\max}} \Big(\frac{4+\gamma\pi}{1-\gamma}\Big) + \frac{i_{\max}}{1-\gamma} + 2\pi \Big) \Big\rceil,$$

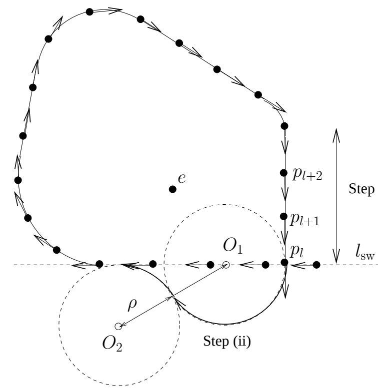

Fig. 9. Maneuvers in the CLOSE phase for pursuer p_l. Step (i): move straight to intersect the daisy-chain. Step (ii): moves on the shortest path to close the daisy-chain.

where $K := 1 + (1/\gamma)\sqrt{(1+\gamma)/(1-\gamma)}$, $s^*_{\mathrm{ip}}(\gamma,\rho)$ is as per equation (2) and the maximum number of turns in the ENCIRCLE *phase is $i_{\max} := \lceil 3\pi/(2\arctan(\sqrt{1-\gamma^2}/\gamma)) \rceil$.*

Remark 3.5 (Asymptotic properties) In the limit as $\gamma \to 1^-$, $N_{\min}(\gamma,\rho) \to +\infty$, as is expected. Moreover, for γ very close to 1 and $\rho \to +\infty$, there exist constants $c > 0$ and $\rho_0 > 0$ such that $N_{\min}(\rho) \leq c\rho$, $\forall \rho \geq \rho_0$.

4 Proofs of the Main Results

In this section, we prove the main result from Section 3 along with certain intermediate results.

Proof sketch of Lemma 3.3: The ENCIRCLE phase involves motion either on circular arcs or straight lines. Hence, we consider any two consecutive pursuers in a daisy-chain formation placed on a circle of radius R. The goal is to determine the angular spacing between these pursuers $\Theta(\gamma, R)$, which is sufficient to prevent the evader from moving between these two pursuers without getting captured. The main steps in this proof are: 1) Write equations of motion for the distance r and angular displacement θ of the evader in the reference frame attached to the center of rotation of the pursuers, in which the pursuers are stationary; 2) Determine using a result in [8] the evader motion that maximizes θ at for each r; 3) Integrate the differential equation for θ to get Θ.

The separation $R \cdot \Theta$ ensures that the evader cannot move between consecutive pursuers without getting captured. Also, for a given value of $\gamma < 1$, the quantity $R \cdot \Theta$ decreases monotonically with increasing R. Thus, it suffices to have the inter-pursuer separation equal to $\rho \cdot \Theta$, where ρ is the minimum turning radius. In other words, for a larger minimum turning radius, the pursuers need to be placed closer to each other. ■

Lemma 4.1 (Align phase) *The* ALIGN *phase terminates after a finite time with the evader aligned with $\{p_1, \overline{v}_{p,1}\}$ and $\|p_1 - e\| \leq \gamma\rho$.*

Proof sketch: The central idea is that once the angle $\phi_1 = 0$ at the end of PRE-ALIGN phase, we need to ensure that $\dot{\phi}_1 = 0$. This is achieved using (3). Since $\|u_{\mathrm{p},1}\| \leq 1$, the evader can be kept aligned with $\{p_1, \overline{v}_{p,1}\}$ as long as $\|e - p_1\| \geq \gamma\rho$. Further, the time derivative of $\|e - p_1\|$ is upper bounded by $-(1-\gamma)$, implying that $\|e - p_1\|$ reduces to $\gamma\rho$ in finite time. ■

Lemma 4.2 (Swerve phase) *(i) A sufficient number of pursuers which ensures that after the leader re-election step, there are at least two pursuers in the remaining daisy-chain formation is* $\lceil \rho(3+\gamma\pi)/s^*_{\mathrm{ip}}(\gamma,\rho) \rceil$ *and,*

(ii) Let d_{sw} *denote the distance of the evader from the line* l_{sw} *joining the pursuer positions at the end of the* SWERVE *phase, (cf. Figure 6). Then,* $d_{\mathrm{sw}} \leq \rho(3+\gamma\pi)/(1-\gamma)$.

Proof: In the SWERVE phase, let pursuer p_1 move on the circle of radius ρ centered at O as shown in Figure 10. The time taken for this phase is $\rho\beta$, where $\beta \in [0,\pi]$ is the angle between line l_{al} and the vector $\overline{v}_{\mathrm{p},1}$ as shown in Figure 10. Let d_{x} (resp. d_{y}) denote the magnitude of the component of the vector $p_l - e$ along (resp. perpendicular to) l_{sw} after re-election of the leader. To maximize d_{x}, the evader must move parallel to l_{sw}. From trigonometry,

$$d_{\mathrm{x}} = \gamma\rho\beta + \|\gamma\rho\cos\beta - \rho\sin\beta\| \leq \rho(3+\gamma\pi),$$

where the first term is the radius of the evader's reachability set in time $\rho\beta$ and the second term in the right hand side equality is the x-component of the distance between e and p'_1. To ensure that at least two pursuers exist in the remaining daisy-chain, it suffices to have the length of the original straight-line daisy-chain equal to the upper bound on d_{x}. This proves part (i).

On similar lines, to maximize d_{y}, the evader must move along the line perpendicular to l_{sw}. Thus, we obtain

$$d_{\mathrm{y}} = \gamma\rho\beta + \|\rho(1-\cos\beta) - \gamma\rho\sin\beta\| \leq \rho(3+\gamma\pi).$$

If there exists k for which $|\phi_k| \geq \frac{\pi}{2}$ (see Figure 7), then $d_{\mathrm{sw}} = d_{\mathrm{y}}$ and part (ii) follows. Otherwise, if $|\phi_k| < \frac{\pi}{2}$ for all $k \in \{1,\ldots,N\}$, then pursuer p_1 (who is retained as the leader) moves straight for a time interval of at most $d_{\mathrm{x}}/(1-\gamma)$, which then gives, $d_{\mathrm{sw}} \leq d_{\mathrm{y}} + d_{\mathrm{x}}\gamma/(1-\gamma)$. The result follows from the upper bounds on d_{x} and d_{y}. ■

Lemma 4.3 (Encircle and Close phases) *If the pursuers begin the* ENCIRCLE *phase at time* t^**, then in the* ENCIRCLE *and part (i) of the* CLOSE *phases, there exists no evader trajectory such that the evader is aligned with* $\{p_l, \overline{v}_{p,l}\}$ *at any time* $t \geq t^*$.

To prove Lemma 4.3, we first introduce the following notation: let $\Sigma(t^*)$ denote the local coordinate system

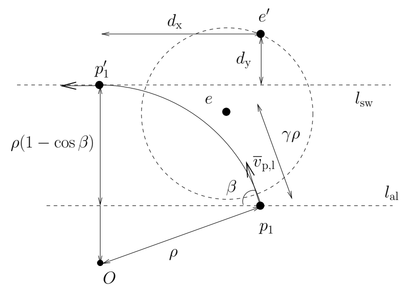

Fig. 10. Illustrating the proof of Lemma 4.2. The primed notation refers to the positions of the players after the pursuers have formed a straight line daisy-chain. The dotted circle shows the reachability set of the evader in time interval $\rho\beta$.

with origin at $p_l(t^*)$ and with the positive Y axis along its heading $\overline{v}_{p,l}$ at time t^*, as shown in Figure 11. Define

$$\mathcal{V}(p_l(t^*), \overline{v}_{p,l}(t^*)) := \Big\{(x^\Sigma, y^\Sigma) \in \Sigma(t^*) |\; x^\Sigma \geq 0, y^\Sigma \leq x^\Sigma \sqrt{1-\gamma^2}/\gamma\Big\}.$$

The set $\mathcal{V}$ possesses the following useful property.

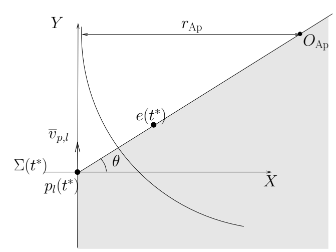

Fig. 11. Proof of Lemma 4.4. For $\theta = \arctan(\sqrt{1-\gamma^2}/\gamma)$, the shaded region denotes the set $\mathcal{V}(p_l(t^*), \overline{v}_{p,l}(t^*))$.

Lemma 4.4 (Property of $\mathcal{V}$) *Given a time instant* t^**, let pursuer* p_l *move with* $u_{p,l} = 0$ *for all subsequent time instants. If* $e(t^*) \in \mathcal{V}(p_l(t^*), \overline{v}_{p,l}(t^*))$*, then there exists no evader trajectory such that the evader is aligned with* $\{p_l, \overline{v}_{p,l}\}$ *at any time* $t \geq t^*$.

Proof: In the coordinate system $\Sigma(t^*)$, denote the point $e(t^*)$ by (x^Σ, y^Σ). Construct the Apollonius circle [1] of the points $p_l(t^*)$ and $e(t^*)$, as shown in Figure 11. This is the set of points that the evader can reach before pursuer p_l does, assuming that the pursuer does not possess turning constraints. The center O_{Ap} and radius r_{Ap} of the Apollonius circle are $O_{\mathrm{Ap}} = \frac{1}{1-\gamma^2}(x^\Sigma, x^\Sigma \tan\theta)$ and $r_{\mathrm{Ap}} = \frac{\gamma x^\Sigma \sec\theta}{1-\gamma^2}$, respectively. Now, let the pursuer p_l move with $u_{p,l} = 0$ for all $t \geq t^*$. In the reference frame $\Sigma(t^*)$, if r_{Ap} does not exceed the X coordinate of O_{Ap}, then the pursuer reaches any point z on the Y axis before the evader can reach z. In other words, the evader cannot align itself with $\{p_l, \overline{v}_{p,l}\}$ at any subsequent time. Thus, $r_{\mathrm{Ap}} \leq x^\Sigma/(1-\gamma^2)$ implies $\tan\theta \leq \sqrt{1-\gamma^2}/\gamma$. ■

Proof of Lemma 4.3: In the ENCIRCLE phase, let t^* be a time instant at which pursuer p_l is about to begin a move straight maneuver. It suffices to show that the evader is at a point $e(t^*)$ contained in the set $\mathcal{V}(p_l(t^*), \overline{v}_{p,l}(t^*))$. Two cases need to be considered:

Case 1: $R := L_l \sin\phi_l$. Note that the angle through which the pursuer turns in a turn maneuver satisfies $\arctan(\sqrt{1-\gamma^2}/\gamma) < \pi/2$. Figure 12 shows the positions of pursuer p_l and the evader just before a turn maneuver (at time instant t_{turn}) and just before the following move straight maneuver (at time instant t^*) in the ENCIRCLE phase. As per the strategy, we have $t^* = t_{\text{turn}} + \Delta t = t_{\text{turn}} + R\arctan(\sqrt{1-\gamma^2}/\gamma)$. Thus, in the time interval Δt, the evader's reachability set is the dotted circle, having radius upper bounded by $R\sqrt{1-\gamma^2}$ as shown in Figure 12. By geometry, the pursuer's center of rotation O in the time interval Δt is precisely at a distance of $R\sqrt{1-\gamma^2}$ from the boundary L defined in Figure 12, of the set $\mathcal{V}(p_l(t^*), \overline{v}_{p,l}(t^*))$. Since $\arctan(\sqrt{1-\gamma^2}/\gamma) < \pi/2$, it follows that the evader's reachability set in time Δt and hence $e(t^*)$ is contained in $\mathcal{V}(p_l(t^*), \overline{v}_{p,l}(t^*))$. Lemma 4.4 completes the proof.

Case 2: $R = \rho$. The proof of this case is on similar lines as that of case 1, with the additional property that one need not consider that part of the evader's reachability set which lies on the opposite side of the daisy-chain. ■

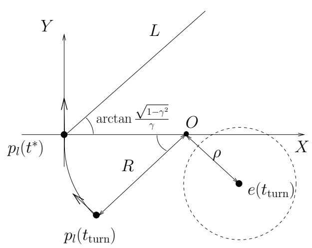

Fig. 12. Case 1 in the proof of Lemma 4.3. The dotted circle is the evader's reachability set in time $R\arctan\sqrt{1-\gamma^2}/\gamma$. Pursuer p_l begins the turn and move straight maneuvers of the ENCIRCLE phase at times t_{turn} and t^*, respectively.

Proof of Theorem 3.4: It suffices to show that all five phases terminate in finite time. This partly follows from Lemmas 4.1 and 4.2. It remains to show that (a) the ENCIRCLE phase terminates in finite time and, (b) the evader is confined at the end of the CLOSE phase.

To show (a), we determine an upper bound T_{enc} on the time taken by the ENCIRCLE phase. From Lemma 4.3, we deduce that in the ENCIRCLE phase, the evader is always the same side of the line along $\overline{v}_{\text{p},l}$. Also, in each turn maneuver, pursuer p_l turns through an angle of at least $\arctan(\sqrt{1-\gamma^2}/\gamma))$. Thus, the turn maneuver is made at most $i_{\max} := \lceil 3\pi/(2\arctan(\sqrt{1-\gamma^2}/\gamma))\rceil$ times. This justifies the expression for $i_{\max}$ in the statement of this theorem.

Let t_0 be the time instant at the end of the SWERVE phase and $d_0 := d_{\text{sw}}$, i.e., the distance of the evader from the line l_{sw} at the end of the SWERVE phase. Let t_i denote the time instant when the pursuer begins the turn maneuver of the ENCIRCLE phase for the i^{th} time and let d_i denote the distance of the evader from the line along $\overline{v}_{\text{p},l}$ at the time instant t_i. We first determine an upper bound for d_i. Let p_l begin the turn maneuver at t_{i-1}, as shown in Figure 13. An upper bound for $t_i - t_{i-1}$

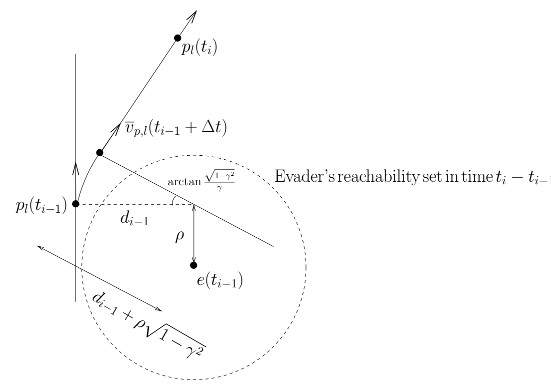

Fig. 13. Determining an upper bound on the interval between two successive times in the ENCIRCLE phase, when the pursuer uses the turn maneuver.

is obtained when the evader decides to move parallel to the line along $\overline{v}_{\text{p},l}(t_{i-1}+\Delta t)$ in the interval $[t_{i-1}, t_i]$. Thus, $t_i - t_{i-1}$ is upper-bounded by

$$d_{i-1}\arctan\frac{\sqrt{1-\gamma^2}}{\gamma} + \frac{\rho + \gamma d_{i-1}\arctan\frac{\sqrt{1-\gamma^2}}{\gamma}}{1-\gamma}$$
$$\leq \frac{d_{i-1}}{\gamma}\sqrt{\frac{1+\gamma}{1-\gamma}} + \frac{\rho}{1-\gamma},$$

where the first term in the first expression is the time for which p_l moves on a circular path and the second is an upper bound on the time taken for the following move straight maneuver, assuming that the evader moves parallel to $\overline{v}_{p,l}(t_{i-1}+\Delta t)$, (cf. Fig. 13). The next inequality follows by using the fact that $\arctan(x) \leq x$, and upon simplification. An upper bound for d_i results when the evader moves normal to the line along $\overline{v}_{\text{p},l}(t_{i-1}+\Delta t)$ in the time interval $[t_{i-1}, t_i]$. Thus,

$$\begin{aligned} d_i &\leq d_{i-1} + \rho\sqrt{1-\gamma^2} + \gamma(t_i - t_{i-1}) \\ &\leq d_{i-1}\Big(1 + \frac{1}{\gamma}\sqrt{\frac{1+\gamma}{1-\gamma}}\Big) + \frac{\rho}{1-\gamma} \\ &\leq K d_{i-1} + \frac{\rho}{1-\gamma} \leq K^i\Big(d_0 + \frac{\rho}{1-\gamma}\Big), \end{aligned}$$

where the second step follows from the upper bound on $t_i - t_{i-1}$ and the fact that $\sin x \leq 1$, and $K := 1 + (1/\gamma)\sqrt{(1+\gamma)/(1-\gamma)}$. The last inequality follows from $K > 2$. Now, for $i \in \{1, \ldots, i_{\max}\}$ where $i_{\max} :=$

$\lceil 3\pi/2(\arctan(\sqrt{1-\gamma^2}/\gamma))\rceil$, the time $T_{\rm enc}$ satisfies

$$T_{\rm enc} \leq \sum_{i=1}^{i_{\max}} t_i - t_{i-1} \leq \sum_{i=1}^{i_{\max}} \frac{d_{i-1}}{\gamma}\sqrt{\frac{1+\gamma}{1-\gamma}} + i_{\max}\frac{\rho}{1-\gamma}.$$

Using the upper bounds for d_{i-1}, and for d_0 (cf. part (ii) of Lemma 4.2),

$$T_{\rm enc} \leq K^{i_{\max}}\rho\Big(\frac{4+\gamma\pi}{1-\gamma}\Big) + \frac{i_{\max}\rho}{1-\gamma}.$$

Note that $T_{\rm enc}$ is also the distance covered by pursuer p_l in the ENCIRCLE phase. So in part (i) of the CLOSE phase, p_l covers a distance of at most $T_{\rm enc}$. Thus, we have shown that the ENCIRCLE phase and part (i) of the CLOSE phase terminate in finite time.

Pursuer p_l travels a distance of at most $4\pi\rho$ in part (ii) of the CLOSE phase before the daisy-chain gets closed. Thus, the total distance traveled by p_l in the ENCIRCLE and CLOSE phases is at most $2T_{\rm enc} + 4\pi\rho$. In the worst-case, to ensure closure of the daisy-chain, consider the distance between $p_l(t_0)$ and the point at which pursuer p_l intersects the daisy-chain. This distance can be at most $\gamma(2T_{\rm enc} + 4\pi\rho)$, which is the distance covered by the evader if it moves with a fixed heading parallel to the line $l_{\rm sw}$ at the end of the SWERVE phase and in the direction opposite to the pursuers' velocity vectors at time t_0. Thus, a sufficient number of pursuers that ensures confinement in the ENCIRCLE and CLOSE phases is given by

$$\Big\lceil \frac{2(1+\gamma)}{s^*_{\rm ip}(\gamma,\rho)}\Big(K^{i_{\max}}\rho\Big(\frac{4+\gamma\pi}{1-\gamma}\Big) + \frac{\rho i_{\max}}{1-\gamma} + 2\pi\rho\Big)\Big\rceil.$$

The result now follows since an additional $\lceil \rho(3+\gamma\pi)/s^*_{\rm ip}(\gamma,\rho)\rceil$ pursuers are sufficient for the leader reelection step in the SWERVE phase (cf. Lemma 4.2). The evader is confined because the pursuers have now formed a closed daisy-chain around it. ■

5 Conclusions and Future Directions

We addressed a cooperative homicidal chauffeur game in which a single pursuer is unable to capture an evader, given an arbitrary initial condition. We proposed a multi-phase partly-decentralized pursuer strategy that involved role specialization in the form of leader and followers, that guarantees confinement of an evader to a bounded region. We characterized the required number of pursuers for which our pursuer strategy is guaranteed to lead to confinement.

In future, it would be of interest to investigate even more distributed encircling and pursuit strategies in which all pursuers play identical roles. Also of interest would be the optimal number of pursuers for such strategies.

Acknowledgments

This material is based upon work supported in part by ARO MURI Award W911NF-05-1-0219 and ONR Award N00014-07-1-0721 and by the Institute for Collaborative Biotechnologies through the grant DAAD19-03-D-0004 from the U.S. Army Research Office. The authors would like to thank Prof. David Skelly for insightful conversations about animal behavior.

References

[1] R. Isaacs, *Differential games*. John Wiley and Sons, 1965.

[2] V. Y. Glizer, "Homicidal chauffeur game with target set in the shape of a circular angular sector: Conditions for existence of a closed barrier," *J. Optimization Theory & Apps*, vol. 101, no. 3, pp. 581–598, 1999.

[3] M. Pachter and Y. Yavin, "A stochastic homicidal chauffeur pursuit-evasion differential game," *J. Optimization Theory & Apps*, vol. 34, no. 3, pp. 405–424, 1981.

[4] W. M. Getz and M. Pachter, "Two-target pursuit-evasion differential games in the plane," *J. Optimization Theory & Apps*, vol. 34, no. 3, pp. 383–403, 1981.

[5] T. W. McLain, P. R. Chandler, S. Rasmussen, and M. Pachter, "Cooperative control of UAV rendezvous," in *Proc ACC*, Arlington, VA, June 2001, pp. 2309–2314.

[6] M. M. Polycarpou, Y. Yang, and K. M. Passino, "A cooperative search framework for distributed agents," in *IEEE International Symposium on Intelligent Control*, Mexico City, Mexico, Sept. 2001, pp. 1–6.

[7] Z. Tang and Ü. Özgüner, "On non-escape search for a moving target by multiple mobile sensor agents," in *Proc ACC*, Minneapolis, MN, June 2006, pp. 3525–3530.

[8] T. G. McGee and J. K. Hedrick, "Guaranteed strategies to search for mobile evaders in the plane," in *Proc ACC*, Minneapolis, MN, June 2006, pp. 2819–2824.

[9] T. Kim and T. Sugie, "Cooperative control for target-capturing task based on cyclic pursuit strategy," *Automatica*, vol. 43, no. 8, pp. 1426–1431, 2007.

[10] F. Belkhouche, B. Belkhouche, and P. Rastgoufard, "Multi-robot hunting behavior," in *IEEE International Conference on Systems, Man and Cybernetics*, Waikoloa, HI, Oct. 2005, pp. 2299–2304.

[11] S. K. Gazda, R. C. Connor, R. K. Edgar, and F. Cox, "A division of labour with role specialization in group-hunting bottlenose dolphins (Tursiops truncatus) off Cedar Key, Florida," *Proceedings of the Royal Society B: Biological Sciences*, vol. 272, no. 1559, pp. 135–140, 2005.

[12] R. L. Pitman, S. O'Sullivan, and B. Mase, "Killer whales (Ornicus orca) attack a school of pantropical spotted dolphins (Stenella attenuata) in Gulf of Mexico," *Aquatic Mammals*, vol. 29, no. 3, pp. 321–324, 2003.

[13] L. E. Dubins, "On curves of minimal length with a constraint on average curvature and with prescribed initial and terminal positions and tangents," *American Journal of Mathematics*, vol. 79, pp. 497–516, 1957.

[14] S. D. Bopardikar, F. Bullo, and J. P. Hespanha, "A cooperative Homicidal Chauffeur game," in *Proc CDC*, New Orleans, LA, Dec. 2007, pp. 4857–4862.