On visibility maintenance via controlled invariance for leader-follower Dubins-like vehicles

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Abstract— The paper studies the *visibility maintenance problem* (VMP) for a leader-follower pair of robots modelled as first-order dynamic systems and proposes an original solution based on the notion of controlled invariance. The nonlinear model describing the relative dynamics of the vehicles is interpreted as linear uncertain system, with the leader robot acting as an external disturbance. The VMP can then be reformulated as a *linear constrained regulation problem with additive disturbances* (DLCRP). New positive $\mathcal{D}$-invariance conditions for linear uncertain systems with parametric disturbance matrix are introduced and used to solve the VMP when box bounds on the state, control input and disturbance are considered. The proposed design procedure can be easily adapted to provide the control with UBB disturbances rejection capabilities. As an extension, the paper addresses the VMP on a circle. Simulation experiments show the effectiveness of the proposed designs.

I. Introduction

A. Motivation and related works

The last few years have seen a growing interest in coordination and control of multi-agent systems [?], [?], [?], [1]. The research in this area has been stimulated by the recent technological advances in wireless communications and processing units and by the observation that multiple agents can perform tasks more efficiently and reliably than a single robot. However, multiple robots must respect suitable constraints in order to accomplish a common task. Two moving agents, for instance, can communicate only if the first one keeps always inside a disk region (representing the extent of the electromagnetic field used for data exchange), centered on the second robot. When extended to n robots, this problem is usually referred to as the *connectivity maintenance problem*. It received a special attention in the literature, where agents modelled as first or second-order dynamic systems have been considered [?], [2], [3].

If the robots are equipped with sensors (e.g., panoramic cameras, laser range finders, high resolution radars, etc.) having limited sensing capabilities, then a *visibility maintenance problem* (VMP), instead of connectivity maintenance problem, naturally arises between the robots. Visibility constraints have been introduced in several works dealing with pursuit-evasion [4], deployment [5] and rendezvous [6]

problems. However the VMP is not the primary concern of these works and general conditions for its solution are not provided. In addition these papers do not consider any bound on the control inputs of the robots.

B. Contributions

The setup considered in this paper consists of two agents with first-order dynamics: a leader (or evader) L and a follower (or pursuer) F. The robots can rotate, but similarly to Dubins' vehicles can only move forward. The follower is equipped with a sensing device characterized by a *visibility set $\mathcal{S}$*, a compact and convex polyhedral region embedding both position and angle information. The leader moves along an arbitrary trajectory: the aim of the follower is to keep L always inside its visibility set $\mathcal{S}$, while respecting suitable bounds on the control inputs.

Inspired by [7], where the concept of cone invariance is employed to solve the multi-agent rendezvous problem and by the results in [8], [9], this paper addresses the VMP using the notion of controlled invariance. The key idea of the work is to interpret the nonlinear model describing the relative dynamics of the leader and the follower, as a linear system with model parameter uncertainty, with the leader acting as an external disturbance. The VMP can then be easily reformulated as *linear constrained regulation problem with additive disturbances* (DLCRP) [9]. New positive $\mathcal{D}$-invariance conditions for general linear uncertain systems with parametric disturbance matrix are introduced and used to solve the VMP when box bounds on the visibility set, control inputs and disturbances are considered. Analytical conditions for the solution of the VMP are obtained by symbolically solving the set of linear inequalities defining the polytope of all the feasible state feedback matrices, using the Fourier-Motzkin elimination method. The proposed design procedure can be readily adapted to provide the control with UBB disturbances rejection capabilities. As a final contribution, the paper presents conditions for the solution of the VMP on a circle.

C. Organization

In Sect. II the linear constrained regulation problem is reviewed and new positive $\mathcal{D}$-invariance conditions for linear systems with parameter uncertainty are presented. In Sect. III we introduce the VMP and prove the main results of the paper. In Sect. IV simulation experiments illustrate the theory and show the effectiveness of the proposed designs. In Sect. V the main contributions of the paper are summarized and future research lines are highlighted.

II. The Linear Constrained Regulation Problem

This section presents a set of results that are instrumental to address the VMP in Sect. III. Theorem 10, Corollary 11, Theorem 12 and Corollary 13 extend the results in [8], [9] to linear uncertain systems with parametric disturbance matrix and are original contributions of this paper. Consider the following system,

$$\dot{s}(t) = A(q(t))\, s(t) + B(q(t))\, u(t)\,, \qquad (1)$$

where $s(t) \in \mathcal{X} \subset \mathbb{R}^n$ and $u(t) \in \mathcal{U} \subset \mathbb{R}^m$ are respectively the state and input vectors, $q(t) \in \mathcal{Q} \subset \mathbb{R}^p$ is the model parameter uncertainty vector, while $\mathcal{U}, \mathcal{X}, \mathcal{Q}$ are assigned sets containing the origin, with $\mathcal{U}$ and $\mathcal{Q}$ compact. We assume that $A(q)$ and $B(q)$ are matrices of suitable dimensions whose entries are continuous functions of q. We shall assume $q(t)$ to be a piecewise continuous function of time.

Definition 1 (Positive invariance): The set $\mathcal{S} \subset \mathbb{R}^n$ is positively invariant for system (1), if and only if, for every initial condition $s(0) \in \mathcal{S}$ and every admissible $q(t) \in \mathcal{Q}$, the solution obtained for $u(t) \equiv 0$, satisfies the condition $s(t) \in \mathcal{S}$ for $t > 0$.

Definition 2 (Admissible region): A region $\mathcal{S} \subset \mathbb{R}^n$ is said to be *admissible* for the feedback control law $u = Ks$, if and only if, for every $s \in \mathcal{S}$, the condition $u \in \mathcal{U}$ holds.

If $\mathcal{U}$ and $\mathcal{S}$ are convex polyhedral sets containing the origin, the admissibility of $\mathcal{S}$ is simply equivalent to,

$$K\, \mathrm{v}_i \in \mathcal{U}, \; \mathrm{v}_i \in \mathrm{vert}(\mathcal{S}), \; i \in \{1, \ldots, \mu\}, \qquad (2)$$

where $\mathrm{vert}(\mathcal{S})$ denotes the set of vertices of $\mathcal{S}$.

We can now introduce the *linear constrained regulation problem* (LCRP) [9].

Problem 3 (LCRP): Given a system in the form (1), find a linear feedback control law $u(t) = Ks(t)$ and a set $\mathcal{S} \subset \mathcal{X}$ such that, for every initial condition $s(0) \in \mathcal{S}$ and every admissible function $q(t) \in \mathcal{Q}$, the conditions $s(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$ are fulfilled for $t > 0$.

Theorem 4: The LCRP has a solution if and only if there exists a feedback matrix K and a set $\mathcal{S} \subset \mathcal{X}$ that is positive invariant and admissible for the closed loop system,

$$\dot{s}(t) = F(q(t))\, s(t)\,, \qquad (3)$$

where $F(q(t)) = A(q(t)) + B(q(t))K$.

Theorem 5 (Sub-tangentiality condition): Let $\mathcal{S} \subset \mathbb{R}^n$ be a compact and convex set with nonempty interior. The positive invariance of $\mathcal{S}$ for (1) is equivalent to the following condition: for every $s_0 \in \partial \mathcal{S}$ and $q \in \mathcal{Q}$,

$$A(q)\, s_0 \in T_{\mathcal{S}}(s_0)\,, \qquad (4)$$

where $T_{\mathcal{S}}(s_0)$ is the tangent cone to $\mathcal{S}$ at s_0 [10].

The main difficulty in exploiting condition (4) to study the positive invariance of an assigned region $\mathcal{S}$ is that it has to be checked on the boundary of $\mathcal{S}$. However, if convex polyhedral sets are considered, only their vertices must be taken into account and easy algebraic conditions can be derived. In this respect, let us consider a system of the form (1), with $A(q(t)) = A_0 + \sum_{l=1}^{p} A_l\, q_l(t)$ and $B(q(t)) =$

$B_0 + \sum_{l=1}^{p} B_l\, q_l(t)$ where A_l and B_l, $l \in \{1, \ldots, p\}$, are constant matrices of appropriate dimension and $q(t)$ takes values in a compact and convex polyhedron $\mathcal{Q} \subset \mathbb{R}^p$. Let the set $\mathcal{U}$ be compact, convex and polyhedral as well. We consider a candidate convex and compact polyhedral set $\mathcal{S}$ containing the origin in its interior and we search for a feedback matrix K that assures the positive invariance of $\mathcal{S}$ for the closed loop system (3). Since $\mathcal{S}$ is polyhedral, then condition (4) is fulfilled on $\partial \mathcal{S}$ if and only if is fulfilled on every vertex of $\mathcal{S}$.

Theorem 6: The set $\mathcal{S}$ is positive invariant for system (3) with feedback $u = Ks$, if and only if, for all $\mathrm{v}_i \in \mathrm{vert}(\mathcal{S})$ and $w_j \in \mathrm{vert}(\mathcal{Q})$:

$$F(w_j)\, \mathrm{v}_i \in T_{\mathcal{S}}(\mathrm{v}_i), \; i \in \{1, \ldots, \mu\}, j \in \{1, \ldots, \nu\}.$$

The LCRP as formulated in Problem 3 does not require the stability. However, a desirable property is the global uniform stability of the closed loop system. The relationship between the stability property and the existence of positively invariant regions is established by Theorem 5.2 in [9]. Let us now turn our attention to systems in the form,

$$\dot{s}(t) = A(q(t))\, s(t) + B(q(t))\, u(t) + E(q(t))\, \delta(t)\,, \quad (5)$$

where the unknown external disturbance $\delta(t)$ is constrained in a compact and convex polyhedral set $\mathcal{D} \subset \mathbb{R}^l$ containing the origin. Note that with respect to the systems considered in [9], the structure of (5) is more general since matrix E also depends on the uncertain parameter q. As an immediate extension of the positive invariance property introduced in Definition 1, we may require that the state s remains in $\mathcal{S}$ despite the presence of the disturbance $\delta(t)$.

Definition 7 (Positive $\mathcal{D}$-invariance): The set $\mathcal{S} \subset \mathbb{R}^n$ is positively $\mathcal{D}$-invariant (PDI) for system (5), if for every initial condition $s(0) \in \mathcal{S}$ and all admissible $q(t) \in \mathcal{Q}$ and $\delta(t) \in \mathcal{D}$, the solution obtained for $u(t) \equiv 0$, satisfies the condition $s(t) \in \mathcal{S}$ for $t > 0$.

We can now introduce the *linear constrained regulation problem with additive disturbances* (DLCRP).

Problem 8 (DLCRP): Given a system in the form (5), find a linear feedback control law $u(t) = Ks(t)$ and a set $\mathcal{S} \subset \mathcal{X}$ such that, for every initial condition $s(0) \in \mathcal{S}$ and every admissible $q(t) \in \mathcal{Q}$ and $\delta(t) \in \mathcal{D}$, the conditions $s(t) \in \mathcal{X}$ and $u(t) \in \mathcal{U}$ are fulfilled for $t > 0$.

Theorem 9: The DLCRP has a solution if and only if there exists a feedback matrix K and a set $\mathcal{S} \subset \mathcal{X}$ that is PDI and admissible for the closed loop system $\dot{s}(t) = F(q(t))\, s(t) + E(q(t))\, \delta(t)$.

Similarly to $A(q(t))$ and $B(q(t))$, hereafter we will suppose that $E(q(t)) = E_0 + \sum_{l=1}^{p} E_l\, q_l(t)$. For the sake of brevity, we do not report the proof of the next theorem: it is based on the same ideas as those of Theorem 4.1 in [9] and Theorem 2.1 in [8].

Theorem 10: The set $\mathcal{S}$ is positively $\mathcal{D}$-invariant for system (5) with feedback $u = Ks$, if and only if, for all $\mathrm{v}_i \in \mathrm{vert}(\mathcal{S})$, $\omega_j \in \mathrm{vert}(\mathcal{Q})$ and $r_k \in \mathrm{vert}(\mathcal{D})$,

$$F(w_j)\, \mathrm{v}_i + E(w_j)\, r_k \in T_{\mathcal{S}}(\mathrm{v}_i)\,,$$
$$i \in \{1, \ldots, \mu\}, \; j \in \{1, \ldots, \nu\}, \; k \in \{1, \ldots, \eta\}.$$

The application of Theorem 10 requires the knowledge of all cones $T_{\mathcal{S}}(\mathrm{v}_i)$, $i \in \{1, \ldots, \mu\}$. An alternative solution is given by the following corollary in which the Euler approximating discrete-time system of (5) is involved. The proof is analogous to that of Corollary 4.1 in [9].

Corollary 11: The set $\mathcal{S}$ is positive $\mathcal{D}$-invariant for system (5), if and only if there exists $\tau > 0$ such that, for all $\mathrm{v}_i \in \mathrm{vert}(\mathcal{S})$, $\omega_j \in \mathrm{vert}(\mathcal{Q})$ and $r_k \in \mathrm{vert}(\mathcal{D})$,

$$\mathrm{v}_i + \tau(F(w_j)\,\mathrm{v}_i + E(w_j)\,r_k) \in \mathcal{S},$$
$$i \in \{1, \ldots, \mu\},\ j \in \{1, \ldots, \nu\},\ k \in \{1, \ldots, \eta\}. \qquad (6)$$

To overcome the problem of the choice of τ, we introduce Theorem 12 that provides a condition equivalent to (6). The proof is analogous to that of Theorem 2.3 in [8]. Let $\mathcal{C}_i$ be the convex cone defined by the delimiting planes of $\mathcal{S}$ that contain v_i: $\mathcal{C}_i = \{g_h^T s \leq \xi_h,\ \xi_h > 0,\ \text{for every } g_h^T \text{ and } \xi_h\ :\ g_h^T \mathrm{v}_i = \xi_h,\ \mathrm{v}_i \in \mathrm{vert}(\mathcal{S})\}$.

Theorem 12: The set $\mathcal{S}$ is positively $\mathcal{D}$-invariant for system (5), if and only if, for all $\tau > 0$, $\mathrm{v}_i \in \mathrm{vert}(\mathcal{S})$, $\omega_j \in \mathrm{vert}(\mathcal{Q})$ and $r_k \in \mathrm{vert}(\mathcal{D})$: $\mathrm{v}_i + \tau(F(w_j)\,\mathrm{v}_i + E(w_j)\,r_k) \in \mathcal{C}_i$, $i \in \{1, \ldots, \mu\}$, $j \in \{1, \ldots, \nu\}$, $k \in \{1, \ldots, \eta\}$.

If the plane description of $\mathcal{S}$ is available, the next corollary whose proof directly follows from that of Theorem 10, holds.

Corollary 13: The set $\mathcal{S}$ is positively $\mathcal{D}$-invariant for system (5), if and only if, for every $\tau > 0$ and every $\mathrm{v}_i \in \mathrm{vert}(\mathcal{S})$, $\omega_j \in \mathrm{vert}(\mathcal{Q})$,

$$(I_n + \tau\, F(w_j))\,\mathrm{v}_i \in \mathcal{C}_i^\star,\ i \in \{1, \ldots, \mu\},\ j \in \{1, \ldots, \nu\}, \qquad (7)$$

where $\mathcal{C}_i^\star$ is the cone obtained by shifting the planes of $\mathcal{C}_i$ as follows: $\mathcal{C}_i^\star = \{g_h^T s \leq \xi_h - \max_{jk}\{\tau\, g_h^T E(w_j)\, r_k\},\ \omega_j \in \mathrm{vert}(\mathcal{Q}), r_k \in \mathrm{vert}(\mathcal{D}),\ \text{for every } g_h^T : g_h^T \mathrm{v}_i = \xi_h\}$.

Remark 14: According to Theorem 9, conditions (7) and (2) provide us with a set of inequalities in the unknowns K defining the polytope $\mathcal{K}$ of all the state feedback matrices solving the DLCRP.

III. THE VISIBILITY MAINTENANCE PROBLEM

Let $\Sigma_0 \equiv \{O_0 ; x_0, y_0\}$ be the fixed reference frame in $\mathbb{R}^2$ and $\Sigma_F \equiv \{O_F ; x_F, y_F\}$ and $\Sigma_L \equiv \{O_L ; x_L, y_L\}$ the reference frames attached to a *follower* robot F and a *leader* robot L (see Fig. 1). The robots are supposed to have single integrator dynamics,

$$\dot{p}_F = \sigma_F,\qquad \dot{p}_L = \sigma_L,$$
$$\dot{\theta}_F = \omega_F,\qquad \dot{\theta}_L = \omega_L, \qquad (8)$$

where $p_F = (x_F, y_F)^T$, $p_L = (x_L, y_L)^T$ are the positions, $\sigma_F = (\sigma_F[1], \sigma_F[2])^T$, $\sigma_L = (\sigma_L[1], \sigma_L[2])^T$ the linear velocities and ω_F, ω_L the angular velocities of robots F and L in the frames Σ_F and Σ_L, respectively. We are going to derive a dynamic model describing the relative dynamics of the robots F and L. Referring (8) to the frame Σ_0, we obtain,

$$\dot{p}_F^0 = R_F^0(\theta_F)\,\sigma_F,\qquad \dot{p}_L^0 = R_L^0(\theta_L)\,\sigma_L,$$

where $R_F^0(\theta_F) = \begin{bmatrix} \cos\theta_F & -\sin\theta_F \\ \sin\theta_F & \cos\theta_F \end{bmatrix}$ and $R_L^0(\theta_L)$ is defined analogously. The position of robot L with respect to Σ_F is then given by $p_L^F = R_0^F(\theta_F)(p_L^0 - p_F^0)$. Differentiating this equation,
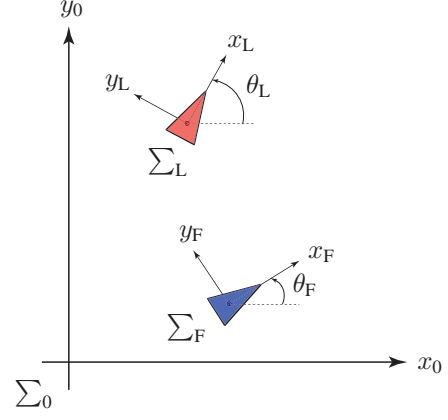

Fig. 1. Leader-follower setup.

we get, $\dot{p}_L^F = \dot{R}_0^F(\theta_F)(p_L^0 - p_F^0) + R_0^F(\theta_F)\big(R_L^0(\theta_L)\,\sigma_L - R_F^0(\theta_F)\,\sigma_F\big)$. Since $\dot{R}_0^F(\theta_F) = \begin{bmatrix} 0 & \omega_F \\ -\omega_F & 0 \end{bmatrix} R_0^F(\theta_F)$, we can rewrite the previous expression as,

$$\dot{p}_L^F = \begin{bmatrix} 0 & \omega_F \\ -\omega_F & 0 \end{bmatrix} p_L^F + R_L^F(\beta)\,\sigma_L - \sigma_F, \qquad (9)$$

where $\beta \triangleq \theta_L - \theta_F$ is a shorthand of β_L^F. Collecting equation (9) and the relative angular dynamics of the robots together, we obtain the system,

$$\begin{bmatrix} \dot{p}_L^F \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -I_2 & \begin{matrix} p_L^F[2] \\ -p_L^F[1] \end{matrix} \\ 0\ \ 0 & -1 \end{bmatrix} \begin{bmatrix} \sigma_F \\ \omega_F \end{bmatrix} + \begin{bmatrix} R_L^F(\beta) & 0 \\ 0 \\ 0\ \ 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_L \\ \omega_L \end{bmatrix}, \qquad (10)$$

where $p_L^F = (p_L^F[1], p_L^F[2])^T$. For the sake of simplicity, we will suppose that robots F and L have,

$$\sigma_F = (1 + v_F,\, 0)^T,\qquad \sigma_L = (1 + v_L,\, 0)^T, \qquad (11)$$

where $|v_F(t)| < 1$, $|v_L(t)| < 1$, for all $t \geq 0$. Substituting (11) in (10), we finally come up with the following system,

$$\begin{bmatrix} \dot{p}_L^F[1] \\ \dot{p}_L^F[2] \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \cos\beta - 1 \\ \sin\beta \\ 0 \end{bmatrix} + \begin{bmatrix} -1 & p_L^F[2] \\ 0 & -p_L^F[1] \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_F \\ \omega_F \end{bmatrix} + \begin{bmatrix} \cos\beta & 0 \\ \sin\beta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_L \\ \omega_L \end{bmatrix}, \qquad (12)$$

with state vector $s = (p_L^F[1], p_L^F[2], \beta)^T \in \mathcal{X} \subset \mathbb{R}^2 \times \mathbb{S}^1$, input vector $u = (v_F, \omega_F)^T \in \mathcal{U} \subset (-1, 1) \times \mathbb{R}$ and disturbance vector $\delta = (v_L, \omega_L)^T \in \mathcal{D} \subset (-1, 1) \times \mathbb{R}$.

In the forthcoming analysis, we will suppose that robot F is equipped with a sensor (e.g., a panoramic camera, a laser range finder, etc.) with limited sensing range. We will call *visibility set* of robot F any compact and convex polyhedral set $\mathcal{S} \subset \mathcal{X}$ containing the origin in its interior. Note that the visibility set generalizes the notion of *sensor footprint* since it embeds not only position but also angle information.

We suppose that robot L moves along an arbitrary trajectory and the aim of robot F is to keep L always inside its visibility set $\mathcal{S}$, while respecting the control bound $\mathcal{U}$. By referring to system (12), we can formalize this problem as follows.

Problem 15 (Visibility maintenance problem (VMP)):
Let $\mathcal{S}$ be the visibility set of robot F and let $s(0) \in \mathcal{S}$. Find a control $u(t)$ such that for all $\delta(t) \in \mathcal{D}$, the conditions $s(t) \in \mathcal{S}$ and $u(t) \in \mathcal{U}$ are fulfilled for $t > 0$.

If we rewrite system (12) in the linear parametric form (5), then the VMP simply reduces to the DLCRP introduced in Sect. II and suitable solvability conditions can be derived using conditions (7) and (2). After simple matrix manipulations in (12), we obtain,

$$
\begin{bmatrix} \Delta \dot{p}_L^F[1] \\ \dot{p}_L^F[2] \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\cos\beta - 1}{\beta} \\ 0 & 0 & \frac{\sin\beta}{\beta} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta p_L^F[1] \\ p_L^F[2] \\ \beta \end{bmatrix} +
$$
$$
+ \begin{bmatrix} -1 & p_L^F[2] \\ 0 & -\Delta p_L^F[1] - d \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_F \\ \omega_F \end{bmatrix} + \begin{bmatrix} \cos\beta & 0 \\ \sin\beta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_L \\ \omega_L \end{bmatrix},
$$
(13)

which can be written in the form (5) with,

$$
A(q) = \begin{bmatrix} 0 & 0 & q_2 \\ 0 & 0 & 1+q_1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B(q) = \begin{bmatrix} -1 & q_4 \\ 0 & -d-q_3 \\ 0 & -1 \end{bmatrix}, \quad E(q) = \begin{bmatrix} 1+q_5 & 0 \\ q_6 & 0 \\ 0 & 1 \end{bmatrix},
$$
(14)

$q_1 = \frac{\sin\beta}{\beta} - 1$, $q_2 = \frac{\cos\beta - 1}{\beta}$, $q_3 = \Delta p_L^F[1]$, $q_4 = p_L^F[2]$, $q_5 = \cos\beta - 1$ and $q_6 = \sin\beta$. We made the following change of variables in system (13): $(p_L^F[1], p_L^F[2], \beta)^T \to (\Delta p_L^F[1], p_L^F[2], \beta)^T$, $\Delta p_L^F[1] = p_L^F[1] - d$, where d is a strictly positive constant. There are two main reasons for this transformation: first of all, if robot F is able to keep L always inside a visibility set displaced of d with respect to its center (with $d > \max\{\frac{1}{2}\|s_1 - s_2\| : s_1, s_2 \in \text{vert}(\mathcal{S})\}$), then this automatically guarantees the collision avoidance between the robots. Second, this choice simplifies the study of the VMP when multiple leaders are considered.

Note that A_0, B_0 and E_0 in (14) (recall the notation used in Sect. II) correspond to the constant matrices obtained by linearizing system (12) around the equilibrium $s_{eq} = (d, 0, 0)^T$, $u_{eq} = (0, 0)^T$, $\delta_{eq} = (0, 0)^T$.

Assumption 16: For the sake of simplicity, hereafter we will suppose that,

$$
\mathcal{U} = \{(v_F, \omega_F)^T : -V_F \leq v_F \leq V_F, -\Omega_F \leq \omega_F \leq \Omega_F\},
$$
$$
\mathcal{D} = \{(v_L, \omega_L)^T : -V_L \leq v_L \leq V_L, -\Omega_L \leq \omega_L \leq \Omega_L\},
$$
(15)

where $V_F < 1$, $V_L < 1$, Ω_F and Ω_L are strictly positive constants. We will also restrict our attention to the following visibility set,

$$
\mathcal{S} = \{(\Delta p_L^F[1], p_L^F[2], \beta)^T : -a \leq \Delta p_L^F[1] \leq a,
$$
$$
-a \leq p_L^F[2] \leq a, -b \leq \beta \leq b\},
$$
(16)

where $a > 0$ and $b > 0$, (see Fig. 2).

Constraint (15) is motivated by the presence of saturation bounds on the driving motors of the robots. The set (16) has been chosen because is computationally simple to handle and because its cross section is a coarse approximation of a disk sensor footprint (e.g., due to a omnidirectional camera or a 360° laser scanner).
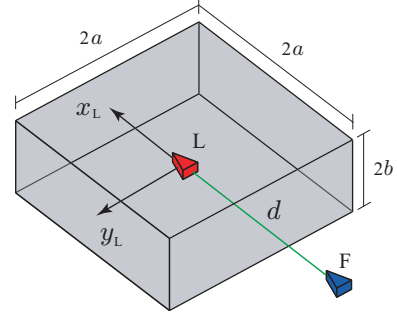

Fig. 2. The visibility set (16) and the pose of the robots L and F for $(\Delta p_L^F[1], p_L^F[2], \beta)^T = (0, 0, 0)^T$, $d > a$.

Since the state $(\Delta p_L^F[1], p_L^F[2], \beta)^T$ is constrained in (16), the polyhedron $\mathcal{Q} \subset \mathbb{R}^6$ of system (13) is defined by,

$$
q_1 \in \left[\frac{\sin b}{b} - 1, 0\right], \quad q_2 \in \left[\frac{\cos b - 1}{b}, \frac{1 - \cos b}{b}\right], \quad q_3 \in [-a, a],
$$
$$
q_4 \in [-a, a], \quad q_5 \in [\cos b - 1, 0], \quad q_6 \in [-\sin b, \sin b].
$$
(17)

We are now ready to state the main result of this section.

Theorem 17 (Solvability of the VMP): Choose $\mathcal{U}$, $\mathcal{D}$ and $\mathcal{S}$ as in Assumption 16 and let $d > a$, $0 < b \leq \pi/2$. The VMP for the robots F and L has a solution if the following conditions are satisfied,

$$
V_F \geq V_L \left(1 + \frac{a \sin b}{d - a}\right) + 1 - \cos b + \frac{a b}{d - a},
$$
(18)

$$
\Omega_L \leq \frac{(1 - V_L) \sin b}{d + a}, \quad \frac{V_L \sin b + b}{d - a} \leq \Omega_F.
$$
(19)

Proof: Let us apply Corollary 13 to system (13). By selecting $\tau = 1$ in (7), we obtain,

$$
\begin{bmatrix} 1 - k_{11} + q_4 k_{21} & -k_{12} + q_4 k_{22} & q_2 - k_{13} + q_4 k_{23} \\ -(d+q_3)k_{21} & 1 - (d+q_3)k_{22} & 1 + q_1 - (d+q_3)k_{23} \\ -k_{21} & -k_{22} & 1 - k_{23} \end{bmatrix} v_i \in \mathcal{C}_i^\star.
$$
(20)

Condition (20) must be evaluated only on the vertices $v_1 = (a, a, b)^T, v_2 = (a, a, -b)^T, v_3 = (a, -a, b)^T, v_4 = (a, -a, -b)^T$ since the set (16) is symmetric with respect to the origin. Because of the special structure of $B(q)$ in (14), we can select a simplified state feedback matrix $K = \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & k_{23} \end{bmatrix}$, that allows us to the decouple the control inputs v_F and ω_F (and visualize the polytope $\mathcal{K} \subset \mathbb{R}^3$ of all the feasible gain matrices). Rewriting (20) in a simplified form, the following set of linear inequalities in the variables k_{11}, k_{22}, k_{23} is obtained,

$$
-k_{11} + q_4 k_{22} + \frac{b}{a} q_4 k_{23} \leq -\frac{b}{a} q_2 - \frac{V_L}{a},
$$
$$
-(d+q_3) k_{22} - \frac{b}{a} (d+q_3) k_{23} \leq -\frac{b}{a} (1+q_1) - \frac{V_L \sin b}{a},
$$
$$
-k_{11} + q_4 k_{22} - \frac{b}{a} q_4 k_{23} \leq \frac{b}{a} q_2 - \frac{V_L}{a},
$$
$$
-(d+q_3) k_{22} + \frac{b}{a} (d+q_3) k_{23} \leq \frac{b}{a} (1+q_1) - \frac{V_L \sin b}{a},
$$
$$
-\frac{a}{b} k_{22} - k_{23} \leq -\frac{\Omega_L}{b}, \quad \frac{a}{b} k_{22} - k_{23} \leq -\frac{\Omega_L}{b},
$$
$$
-k_{11} - q_4 k_{22} + \frac{b}{a} q_4 k_{23} \leq -\frac{b}{a} q_2 - \frac{V_L}{a},
$$
$$
-k_{11} - q_4 k_{22} - \frac{b}{a} q_4 k_{23} \leq \frac{b}{a} q_2 - \frac{V_L}{a}.
$$
(21)

The admissibility condition (2) leads to the additional constraints,

$$k_{11} \leq \frac{V_F}{a}, \quad k_{22} + \frac{b}{a}k_{23} \leq \frac{\Omega_F}{a}, \quad k_{22} - \frac{b}{a}k_{23} \geq -\frac{\Omega_F}{a},$$
$$k_{11} \geq -\frac{V_F}{a}, \quad k_{22} - \frac{b}{a}k_{23} \leq \frac{\Omega_F}{a}, \quad k_{22} + \frac{b}{a}k_{23} \geq -\frac{\Omega_F}{a}. \tag{22}$$

Applying the Fourier-Motzkin elimination method [11] to the inequalities (21)-(22) with the assumption that $d > a$ (in order to fix the sign of the coefficients of k_{22} and k_{23} in the second and fourth expression in (21)) we obtain the following conditions on the variables a, b, d, V_F, V_L, Ω_F, Ω_L and uncertain parameters $q_1, \ldots, q_4$: $\Omega_L \leq \frac{b(1+q_1)-V_L \sin b}{d+q_3}$, $\Omega_F \geq \frac{b(1+q_1)+V_L \sin b}{d+q_3}$. $V_F \geq V_L\left(1+\frac{q_4 \sin b}{d+q_3}\right)+b\left(q_2+\frac{q_4(1+q_1)}{d+q_3}\right)$ and $V_F \geq V_L\left(1+\frac{q_4 \sin b}{d+q_3}\right)-b\left(q_2+\frac{q_4(1+q_1)}{d+q_3}\right)$, for $q_4 > 0$. $V_F \geq V_L + b q_2$, for $q_4 = 0$. $V_F \geq V_L\left(1-\frac{q_4 \sin b}{d+q_3}\right)+b\left(q_2+\frac{q_4(1+q_1)}{d+q_3}\right)$ and $V_F \geq V_L\left(1-\frac{q_4 \sin b}{d+q_3}\right)-b\left(q_2+\frac{q_4(1+q_1)}{d+q_3}\right)$, for $q_4 < 0$. An appropriate selection of the parameters $q_1, \ldots, q_4$ on the extremes of the intervals (17), leads us to (18) and (19). ∎

Note that conditions (18) and (19) are necessary and sufficient for the linear uncertain system (13). From (19), we see that $\Omega_F \geq \Omega_L$.

Once fixed the variables a, b, d, V_F, V_L, Ω_F, Ω_L according to (18) and (19), the polytope $\mathcal{K}$ of all the feasible state feedback matrices is simply given by (21)-(22). By evaluating (21)-(22) on the 64 vertices of the polyhedron $\mathcal{Q}$, we see that $\mathcal{K}$ is defined by a set of 392 inequalities, only a small number of which (see for example Fig. 3(a)) is active.

Remark 18: Since the polytope $\mathcal{K}$ contains infinite gain matrices, we may use an optimal criterion to select K, such as, e.g., minimizing any matrix norm. In the simulations in Sect. IV, we have chosen the matrix $K = \begin{bmatrix} k_{11} & 0 & 0 \\ 0 & k_{22} & k_{23} \end{bmatrix}$ with minimum 2-norm.

A. Extension: rejection of UBB disturbances

Consider the following system,

$$\begin{bmatrix} \Delta \dot{p}_L^F[1] \\ \dot{p}_L^F[2] \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{\cos\beta-1}{\beta} \\ 0 & 0 & \frac{\sin\beta}{\beta} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Delta p_L^F[1] \\ p_L^F[2] \\ \beta \end{bmatrix} +$$
$$+ \begin{bmatrix} -1 & 0 & p_L^F[2] \\ 0 & -1 & -\Delta p_L^F[1]-d \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_F \\ h_F \\ \omega_F \end{bmatrix} + \begin{bmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v_L \\ h_L \\ \omega_L \end{bmatrix}. \tag{23}$$

With respect to system (13), two new components, h_F and h_L, are present in the vectors u and δ. They are *unknown but bounded* (UBB) disturbances acting on the robots F and L (e.g., lateral wind). Our aim here is to solve the VMP in the presence of the disturbances h_F, h_L. We can collect all the perturbations acting on the nominal system (i.e., v_L, ω_L, h_F and h_L), in the last term of (23). Let $\mathcal{U}$ be given in (15) and define,

$$\mathcal{D} = \{(v_L, \omega_L, h_F, h_L)^T : -V_L \leq v_L \leq V_L, -\Omega_L \leq \omega_L \leq \Omega_L,$$
$$-H_F \leq h_F \leq H_F, -H_L \leq h_L \leq H_L\}, \tag{24}$$

where H_F, H_L are strictly positive constants.

Corollary 19 (Solvability of the VMP with disturbances): Choose $\mathcal{U}$, $\mathcal{D}$ as in (15), (24) and let $d > a$, $0 < b \leq \pi/2$.
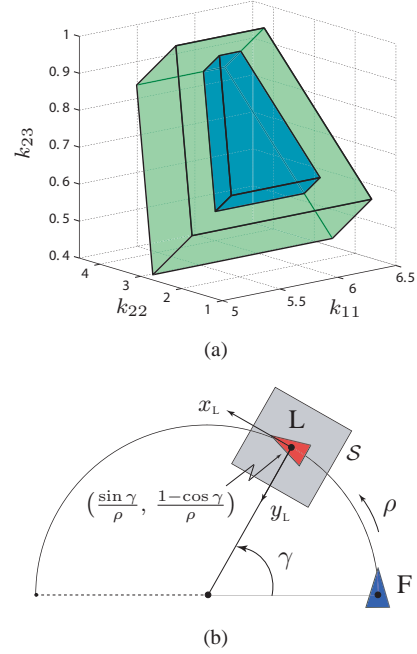

Fig. 3.   (a) Polytopes $\mathcal{K}$ for a set of given parameters: *blue*, with UBB disturbances; *green*, without UBB disturbances. (b) VMP on a circle (top-down view): the pose of the robots L and F for $(\Delta p_L^F[1], \Delta p_L^F[2], \Delta\beta)^T = (0, 0, 0)^T$ is shown.

The VMP for the robots F and L in the presence of the UBB disturbances h_F, h_L has a solution if the following conditions are satisfied,

$$V_F \geq V_L\left(1+\frac{a\sin b}{d-a}\right) + 1 - \cos b + \frac{a(H_F+H_L+b)}{d-a} + H_L \sin b, \tag{25}$$

$$\Omega_L \leq \frac{(1-V_L)\sin b - (H_F+H_L)}{d+a}, \quad \frac{V_L \sin b + b + (H_F+H_L)}{d-a} \leq \Omega_F. \tag{26}$$

Note that because of the additional terms H_F and H_L, conditions (25) and (26) are stricter than (18) and (19) and then the polytope $\mathcal{K}$ is smaller in this case. This is evident in Fig. 3(a), where the polytope $\mathcal{K}$ (blue) obtained for $a = 0.15$ m, $b = \pi/3$ rad, $d = 1.6$ m, $V_F = 0.95$ m/s, $V_L = 0.1$ m/s, $\Omega_F = \pi/2$ rad/s, $\Omega_L = \pi/20$ rad/s and $H_F = 0.2$ m/s, $H_L = 0.1$ m/s is compared with the polytope (green) obtained with $H_L = H_F = 0$ m/s.

B. Extension: VMP on a circle

Let us consider the following change of variables in system (12): $(p_L^F[1], p_L^F[2], \beta)^T \rightarrow (\Delta p_L^F[1], \Delta p_L^F[2], \Delta\beta)^T$, $(v_L, \omega_L)^T \rightarrow (v_L, \Delta\omega_L)^T$, $(v_F, \omega_F)^T \rightarrow (v_F, \Delta\omega_F)^T$ where $\Delta p_L^F[1] = p_L^F[1] - \frac{\sin\gamma}{\rho}$, $\Delta p_L^F[2] = p_L^F[2] - \frac{1-\cos\gamma}{\rho}$, $\Delta\beta = \beta - \gamma$, $\Delta\omega_L = \omega_L - \rho$, $\Delta\omega_F = \omega_F - \rho$. Parameters $0 < \gamma < \pi/2$ and $\rho > 0$ define the pose of robot L with respect to the frame of robot F (see Fig. 3(b)). Following the same procedure detailed above, we can obtain the solvability conditions for the VMP on a circle.

Theorem 20 (Solvability of the VMP on a circle): Let $(v_F, \Delta\omega_F)^T \in [-V_F, V_F] \times [-\Omega_F, \Omega_F]$, $(v_L, \Delta\omega_L)^T \in [-V_L, V_L] \times [-\Omega_L, \Omega_L]$, $(\Delta p_L^F[1], \Delta p_L^F[2], \Delta\beta)^T \in [-a, a]^2 \times [-b, b]$ and let $1 - \cos\gamma > \rho a$, $0 \leq b \pm \gamma \leq \pi/2$.
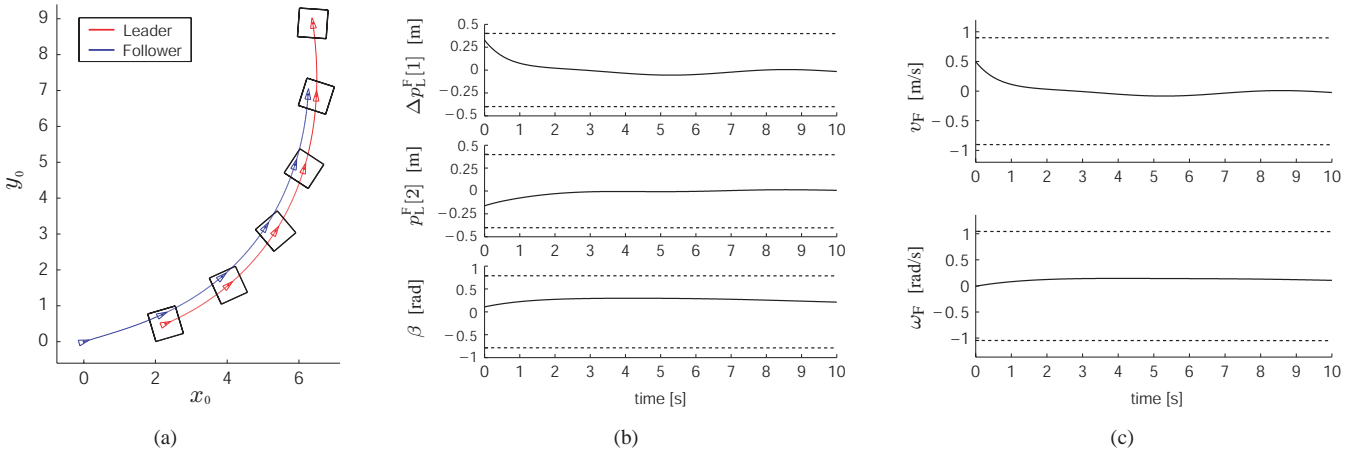
Fig. 4. (a) Trajectory of robots L, F and visibility set $\mathcal{S}$ (top-down view); (b) $\Delta p_L^F[1]$, $p_L^F[2]$, β (solid) and bounds $\pm a$, $\pm a$, $\pm b$ (dash); (c) v_F, ω_F (solid) and bounds $\pm V_F$, $\pm \Omega_F$ (dash).

The VMP on a circle has a solution if the following conditions are satisfied,

$$V_F \geq V_L \left(\cos(b - \gamma) + \frac{\sin(b+\gamma)(1-\cos\gamma-\rho\,a)}{\sin\gamma+\rho a} \right) + \cos\gamma + \rho a$$
$$- \frac{1-\cos\gamma-\rho\,a}{\sin\gamma+\rho a} \left(\sin(b+\gamma) - \sin\gamma + \rho a \right) - \cos(b+\gamma),$$

$$\Omega_L \leq \rho \left(\frac{(1-V_L)\sin(b+\gamma)}{\sin\gamma+\rho a} - 1 \right),$$

$$\Omega_F \geq \frac{\rho\,(V_L \sin(b+\gamma) + \sin(b-\gamma) + \sin\gamma + \rho a)}{\sin\gamma - \rho a}.$$

IV. SIMULATION EXPERIMENTS

In the simulation results reported in this section, robot L moves with velocities $v_L(t) = 0.05 \sin(t)$ m/s, $\omega_L(t) = \frac{\pi}{20} \cos(t/10)$ rad/s. We set $V_L = 0.1$ m/s, $\Omega_L = \pi/15$ rad/s, $V_F = 0.9$ m/s, $\Omega_F = \pi/3$ rad/s, $a = 0.4$ m, $b = \pi/4$ rad and $d = 2$ m, according to the conditions of Theorem 17. We chose the gain matrix in $\mathcal{K}$ with minimum 2-norm: $K = \begin{bmatrix} 1.5173 & 0 & 0 \\ 0 & 0.3707 & 0.4925 \end{bmatrix}$. Note that since K is in the interior of $\mathcal{K}$, the asymptotic stability is assured. System (13) has been initialized with $(\Delta p_L^F[1](0), p_L^F[2](0), \beta(0))^T = (0.3285, -0.1626, 0.1071)^T$. Fig. 4(a) reports the trajectory of robot L (red) and F (blue) and the visibility set $\mathcal{S}$ (black). Fig. 4(b) shows that $\Delta p_L^F[1]$, $p_L^F[2]$, β (solid), keep inside the respective bounds $\pm a$, $\pm a$, $\pm b$ (dash), as expected. Finally, Fig. 4(c) shows that the control inputs v_F, ω_F (solid), respect the velocity bounds $\pm V_F$, $\pm \Omega_F$ (dash).

V. CONCLUSIONS AND FUTURE WORK

The paper proposes an original solution to the *visibility maintenance problem* (VMP) for leader-follower Dubins-like vehicles, based on the notion of controlled invariance. By interpreting the nonlinear model describing the relative dynamics of the robots as a linear system with parameter uncertainty, the VMP is reformulated as a *linear constrained regulation problem with additive disturbances* (DLCRP). General conditions for the positive $\mathcal{D}$-invariance of linear uncertain systems with parametric disturbance matrix are derived and used to study the feasibility of the VMP when box bounds on the state, input and disturbance are considered.

Future research lines include the extension of our results to vehicles with more involved dynamics and to general robotic networks described by directed graphs. The integration of our visibility conditions in the existing rendezvous, coverage or deployment algorithms is also a subject of future investigation.

REFERENCES

[1] R.M. Murray. Recent Research in Cooperative Control of Multivehicle Systems. *J. Dyn. Syst. Meas. Contr.*, 129(5):571–583, 2007.

[2] H. Ando, Y. Oasa, I. Suzuki, and M. Yamashita. Distributed Memoryless Point Convergence Algorithm for Mobile Robots with Limited Visibility. *IEEE Trans. Robot. Autom.*, 15(5):818–828, 1999.

[3] K. Savla, G. Notarstefano, and F. Bullo. Maintaining limited-range connectivity among second-order agents. *SIAM J. Contr. Optim.*, 2008, to appear.

[4] B.P. Gerkey, S. Thrun, and G. Gordon. Visibility-based Pursuit-evasion with Limited Field of View. *Int. J. Robot. Res.*, 25(4):299–315, 2006.

[5] A. Ganguli, J. Cortes, and F. Bullo. Visibility-based multi-agent deployment in orthogonal environments. In *Proc. American Contr. Conf.*, pages 3426–3431, 2007.

[6] A. Ganguli, J. Cortes, and F. Bullo. Multirobot Rendezvous with Visibility Sensors in Nonconvex Environments. *IEEE Trans. Robot.*, 2008, conditionally accepted.

[7] A. Tiwari, J. Fung, R. Bhattacharya, and R.M. Murray. Polyhedral Cone Invariance Applied to Rendezvous of Multiple Agents. In *Proc. 43rd IEEE Int. Conf. Dec. Contr.*, pages 165–170, 2004.

[8] F. Blanchini. Feedback Control for Linear Time-Invariant Systems with State and Control Bounds in the Presence of Disturbances. *IEEE Trans. Automat. Contr.*, 35(11):1231–1234, 1990.

[9] F. Blanchini. Constrained Control for Uncertain Linear Systems. *J. Optim. Theor. Appl.*, 71(3):465–484, 1991.

[10] F. Blanchini. Set invariance in control. *Automatica*, 35(11):1747–1767, 1999.

[11] A. Schrijver. *Theory of Linear and Integer Programming*. John Wiley, New York, 1986.