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On Discrete-time Pursuit-evasion Games with Sensing Limitations

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Abstract

We address discrete-time pursuit-evasion games in the plane where every player has identical sensing and motion ranges restricted to closed discs of given sensing and stepping radii. A single evader is initially located inside a bounded subset of the environment and does not move until detected. We propose a *Sweep-Pursuit-Capture* pursuer strategy to capture the evader and apply it to two variants of the game: the first involves a single pursuer and an evader in a bounded convex environment and the second involves multiple pursuers and an evader in a boundaryless environment. In the first game, we give a sufficient condition on the ratio of sensing to stepping radius of the players that guarantees capture. In the second, we determine the minimum probability of capture, which is a function of a novel pursuer formation and independent of the initial evader location. The Sweep and Pursuit phases reduce both games to previously-studied problems with unlimited range sensing. Thereafter, we demonstrate how capture is achieved using available strategies. We obtain novel upper bounds on the capture time and present simulation studies that suggest robustness to sensing errors.

Index Terms

Pursuit-evasion games, sensing limitations, cooperative control.

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On Discrete-time Pursuit-evasion Games with Sensing Limitations

I. INTRODUCTION

The game of pursuit can be posed as to determine a strategy for a pursuer (a team of pursuers) to capture an evader in a given environment. By *capture,* we mean that the evader and the pursuer (some pursuer) meet at the same location after a finite time. The aim of the pursuer (pursuers) is to capture the evader for any evader trajectory. The evader wins the game if it can avoid capture indefinitely. All the players have identical motion capabilities. Capture strategies are important in surveillance where we would like to detect and capture equally agile intruders. Another application is search-and-rescue operations where a worst-case capture strategy guarantees a rescue, in spite of any unpredictable motion of the victim.

The continuous time version of this game has been studied by Ho *et al.* [1], Lim *et al.* [2] and Pachter [3] to cite a few. Recently, the discrete-time version of the game has received significant attention. Sgall [4] has given sufficient conditions and a strategy for a single pursuer to capture an evader in a semi-open environment. This strategy has been extended by Kopparty and Ravishankar [5] to the case of multiple pursuers in an unbounded environment, to capture a single evader which is inside their convex hull. Alonso *et al.* [6] and Alexander *et al.* [7] propose strategies so that the pursuer can reduce the distance between itself and the evader to a finite, non-zero amount after finite time steps. The game has also been studied in different types of bounded environments, e.g., circular environment by Alonso *et al.* [6], curved environments by LaValle *et al.* [8]. Visibility-based pursuit evasion has been studied by Guibas *et al.* [9] and in polygonal environments by Isler *et al.* [10].

Each of above mentioned works proposes strategies which require that the pursuers have unlimited range sensing capacity. A more realistic assumption is to introduce sensing limitations for the pursuers and the evader. In this context, Gerkey *et al.* [11] have studied a version of visibility limited to an angle, instead of the entire region. Shevchenko [12] has considered a successive pursuit problem in the plane with sensing range limited to a finite disc. Isler *et al.* [13] have considered the problem on a graph, with the visibility of the pursuer limited to nodes adjacent to the current node of a pursuer. A framework which uses probabilistic models for sensing devices for the agents can be found in the works of Hespanha *et al.* [14] and Vidal *et al.* [15].

We address the case of limited range sensing capability: a pursuer and an evader can sense each other only if the distance between them is less than or equal to a given sensing radius. We consider the discrete-time version with one or many pursuers and a single evader in a planar environment. The motion of each player is constrained to a stepping disc around it. The evader is initially located inside a bounded subset of the environment, which we term as the *field*. The players can leave the field but not the environment. The evader follows a *reactive rabbit* model, i.e., does not move until it senses a pursuer [13]. We present an algorithmic approach in the form of a *Sweep-Pursuit-Capture* strategy for the pursuer to capture the evader. We demonstrate this strategy using two variants of the pursuit-evasion game: the first involves a single pursuer and the evader in a bounded convex environment while the second considers multiple cooperating pursuers to capture the evader in a boundaryless environment.

In the first game, the pursuer *sweeps* the environment in a definite path until the evader is sensed, which must necessarily happen in finite time. We then establish how a GREEDY strategy of moving towards the *last-sensed* location of the evader, eventually reduces the present problem to a previously-studied one with unlimited sensing. Finally, we show how capture is achieved using the established LION strategy [4]. Our contributions are as follows: first, we present an analysis which provides a novel upper bound on the time required for the pursuit phase to terminate. Second, we obtain a sufficient condition on the ratio of sensing to stepping radius of the players for capture to take place in a given environment. Finally, we give sufficient conditions and a strategy for the evader to escape against the GREEDY strategy of the pursuer.

The second game is played with at least five cooperative pursuers in a boundaryless environment and the field is a bounded region known to the pursuers. Our contributions are as follows: first, we design a novel pursuer formation and a randomized SWEEP strategy for the pursuers to search the field. They *succeed* when they detect the evader inside a special *capture* region, which we characterize for the pursuer formation. We show that using our SWEEP strategy, the pursuers succeed with a certain probability which is a function of the pursuer formation and independent of the initial evader location. Next, we propose a cooperative pursuit strategy for the pursuers to confine the appropriately-sensed evader within their sensing discs. We show that using this pursuit strategy, the present problem is reduced to a previously-studied one with unlimited sensing. Finally, we show how capture is achieved using the established PLANES strategy [5]. We obtain novel upper bounds on the time for each phase in our strategy. For both problems, we present simulation studies that suggest robustness to sensing errors.

The inspiration for the cooperative strategy proposed in this paper has been derived from aspects of animal behavior. It is well known that predators hunt as a conjoined group, when it is less efficient to hunt alone. This behavior is also observed when the prey is large or can move as fast as the predators [16]. Further, predators show an inclination towards specialized behavior by maintaining a fixed formation during search and capture of preys [17]. Such specializations suggest that there may be configurations that are preferred during group hunting. Also, in the presence of sensing limitations, groups tend to maintain spacing between each other that is regulated by their sensory capabilities [18]. These facts give us additional hints towards designing capture-conducive predator formations. In this context, our analysis sheds light on how the maximum group size of the predators varies with prey availability and with the prey's nutrition value in the present set-up.

The paper is organized as follows: the problem's mathematical model and assumptions are presented in Section II. The individual phases of the *sweep-pursuit-capture* strategies and the corresponding main results for both the problems are presented in Section III. The proofs of the results are presented in Section IV. Simulation results are presented in Section V. Finally, in Section VI, we study the relationship between pursuer group sizes and evader availability and its nutrition value in our set-up.

II. Problem Set-up

We assume a discrete-time model with alternate motion of the evader and the pursuers: the evader moving first. We assume that the players can sense each other precisely only if the distance between them is less than or equal to the sensing radius r_{sens}. Further, we assume that at each time instant, the players take measurements of each other before and after the evader's move, as shown in Figure 1. Define $\mathcal{Q}_\phi := \mathcal{Q} \cup \phi$, where $\mathcal{Q} \subseteq \mathbb{R}^2$ denotes the environment and ϕ is the null element. The null element will be used to denote a lack of measurement in our limited range sensing model. Let $\mathcal{G} \subset \mathcal{Q}$ denote the field, i.e., the region that initially contains the evader. The evader follows a *reactive rabbit* model - moves only after being detected for the first time. We assume that the pursuers know the field $\mathcal{G}$ and the environment $\mathcal{Q}$. The goal of the pursuer(s) is to *capture* the evader, i.e., a pursuer and the evader are at the same position at some finite time. *Evasion* is said to occur if the pursuer cannot capture the evader. We describe the *Sweep-Pursuit-Capture* strategy for the following problems:

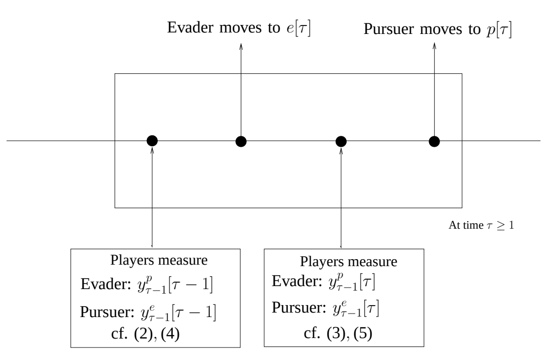


Fig. 1. A snapshot of each time instant $\tau \in \{1, 2, \dots\}$ in our alternate motion model. The players take measurements before and after the evader's move.

A. Single pursuer problem

We have a bounded convex environment $\mathcal{Q} \subset \mathbb{R}^2$ and the field $\mathcal{G} = \mathcal{Q}$. Let $e[t]$ and $p[t]$ denote the absolute positions of the evader and the pursuer respectively, at time $t \in \mathbb{Z}_{\geq 0}$. The discrete-time equations of motion are

$$\begin{aligned} e[t] &= e[t-1] + u^e\Big(e[t-1], \{y^p_{\tau-1}[\tau-1]\}_{\tau=1}^{t}, \{y^p_{\tau-1}[\tau]\}_{\tau=1}^{t-1}\Big), \\ p[t] &= p[t-1] + u^p\Big(p[t-1], \{y^e_{\tau-1}[\tau-1]\}_{\tau=1}^{t}, \{y^e_{\tau-1}[\tau]\}_{\tau=1}^{t}\Big), \end{aligned} \tag{1}$$

where at the τ^{th} time instant, $y^p_{\tau-1}[\tau-1], y^p_{\tau-1}[\tau] \in \mathcal{Q}_\phi$ are the measurements of the pursuer's position taken by the evader before and after its own move, as shown in Figure 1. The parentheses notation $\{y^p_{\tau-1}[\tau-1]\}_{\tau=1}^{t}$ denotes the set $\{y^p_0[0], y^p_1[1], \dots, y^p_{t-1}[t-1]\}$. Due to limited range sensing model, for $\tau \in \{1, \dots, t\}$, we define

$$y^p_{\tau-1}[\tau-1] = \begin{cases} p[\tau-1], & \text{if } \|p[\tau-1] - e[\tau-1]\| \leq r_{\text{sens}}, \\ \phi, & \text{otherwise.} \end{cases} \tag{2}$$

For notational convenience, we define $\{y^p_{\tau-1}[\tau]\}_{\tau=1}^{t-1} = \phi$ for the initial time $t = 1$. For $t \geq 2$ and for $\tau \in \{1, \dots, t-1\}$, we have

$$y^p_{\tau-1}[\tau] = \begin{cases} p[\tau-1], & \text{if } \|p[\tau-1] - e[\tau]\| \leq r_{\text{sens}}, \\ \phi, & \text{otherwise.} \end{cases} \tag{3}$$

Similarly, at the τ^{th} time instant , $y^e_{\tau-1}[\tau-1], y^e_{\tau-1}[\tau] \in \mathcal{Q}_\phi$ are the measurements of the evader's position taken by the pursuer before and after the evader's move respectively, as shown in Figure 1. Due to limited range sensing model, for $\tau \in \{1, \dots, t\}$, we have

$$y^e_{\tau-1}[\tau-1] = \begin{cases} e[\tau-1], & \text{if } \|e[\tau-1] - p[\tau-1]\| \leq r_{\text{sens}}, \\ \phi, & \text{otherwise.} \end{cases} \tag{4}$$

For $\tau \in \{1, \dots, t\}$, we have

$$y^e_{\tau-1}[\tau] = \begin{cases} e[\tau], & \text{if } \|e[\tau] - p[\tau-1]\| \leq r_{\text{sens}}, \\ \phi, & \text{otherwise.} \end{cases} \tag{5}$$

The functions $u^e : \mathcal{Q} \times \underbrace{\mathcal{Q}_\phi \times \cdots \times \mathcal{Q}_\phi}_{2t-1 \text{ times}} \to \mathcal{Q}$ and $u^p : \mathcal{Q} \times \underbrace{\mathcal{Q}_\phi \times \cdots \times \mathcal{Q}_\phi}_{2t \text{ times}} \to \mathcal{Q}$ are termed as *strategies* for the evader and pursuer respectively. The apparent lack of symmetry between the number of arguments in the strategies of the evader and the pursuer is due to the alternate motion model. We assume that both players can move with a maximum step size of r_{step}, that is,

$$\|u^e\| \leq r_{\text{step}}, \quad \|u^p\| \leq r_{\text{step}}. \tag{6}$$

The sensing radius, r_{sens}, is κ times the motion radius, r_{step}. We assume κ is greater than 1, i.e., both players can sense further than they can move. From the reactive rabbit model for the evader, we have $u^e = 0$ until the evader is detected. After this happens, the single pursuer problem consists of *determining u^p that guarantees capture for any evader strategy, u^e*. This problem is described by two key parameters: the ratio of sensing to stepping radius κ and the ratio of the diameter of the environment to the stepping radius $\frac{\mathrm{diam}(\mathcal{Q})}{r_{\text{step}}}$.

B. Multiple pursuer problem

We have a total of $N \geq 5$ pursuers that can communicate among themselves the location of a sensed evader as well as their own position with respect to a fixed, global reference frame. The environment $\mathcal{Q}$ is $\mathbb{R}^2$ and the field $\mathcal{G}$ is a bounded subset of $\mathbb{R}^2$. Define $\mathbb{R}^2_\phi := \mathbb{R}^2 \cup \phi$. Let $p_j[t]$ denote the absolute positions of the j^{th} pursuer at time t for every $j \in \{1, \dots, N\}$. Analogous to (1), the discrete-time equations of motion are

$$
\begin{aligned}
&e[t] = e[t-1] + u^e\Big(e[t-1], \{y^p_{\tau-1}[\tau-1]\}_{\tau=1}^{t}, \{y^p_{\tau-1}[\tau]\}_{\tau=1}^{t-1}\Big), \\
&p_j[t] = p_j[t-1] \\
&\quad + u^{p_j}\left(\left\{\{p_j[\tau]\}_{j=1}^N\right\}_{\tau=1}^{t-1}, \left\{y^e_{\tau-1}[\tau-1]\right\}_{\tau=1}^{t}, \left\{y^e_{\tau-1}[\tau]\right\}_{\tau=1}^{t}\right),
\end{aligned} \tag{7}
$$

where at the τ^{th} time instant, $y^p_{\tau-1}[\tau-1], y^p_{\tau-1}[\tau] \in \underbrace{\mathbb{R}^2_\phi \times \cdots \times \mathbb{R}^2_\phi}_{N \text{ times}}$ denote the sets of measurements of the pursuers' positions taken by the evader before and after its move. Similarly, $y^e_{\tau-1}[\tau-1], y^e_{\tau-1}[\tau] \in \mathbb{R}^2_\phi$ are the measurements of the evader's position taken by the pursuers before and after the evader's move. The measurements are given by expressions analogous to (2)-(5). Akin to the single pursuer problem, the functions $u^e : \mathbb{R}^2 \times \underbrace{\mathbb{R}^2_\phi \times \cdots \times \mathbb{R}^2_\phi}_{(2t-1)N \text{ times}} \rightarrow \mathbb{R}^2$ and $u^{p_j} : \underbrace{\mathbb{R}^2_\phi \times \cdots \times \mathbb{R}^2_\phi}_{(t-1)N \text{ times}} \times \underbrace{\mathbb{R}^2_\phi \times \cdots \times \mathbb{R}^2_\phi}_{2t \text{ times}} \rightarrow \mathbb{R}^2$ for every $j \in \{1, \dots, N\}$, are strategies for the evader and pursuers respectively. The constraint on the maximum step size, given by (6), holds for the evader and every pursuer. Due to the reactive rabbit model for the evader, $u^e = 0$ until it is detected by the pursuers for the first time.

The multiple pursuer problem consists of *designing a pursuer formation and a corresponding strategy that guarantees capture of the evader*. This problem is described by the following key parameters: the ratio of sensing to stepping radius of the players κ, the ratio of the diameter of the field to the stepping radius $\frac{\mathrm{diam}(\mathcal{G})}{r_{\text{step}}}$, and the number of pursuers N.

III. The Sweep-Pursuit-Capture Strategies and Main Results

In this section, we describe the Sweep-Pursuit-Capture strategies for both the problems and the corresponding main results. The proofs are presented in Section IV.

We first introduce the following weak notion of capture.

Definition III.1 (Trap) *The evader is* trapped within the sensing radius (resp. radii) *of the pursuer (resp. pursuers) if for any evader strategy u^e, the motion disc of the evader is completely contained within the sensing disc of the pursuer (resp. union of the sensing discs of the pursuers) after a finite time.*

To be specific, the evader is trapped at time instant T_{trap} if for any evader strategy,

$$
y^e_{T_{\text{trap}}-1}[T_{\text{trap}}-1] = e[T_{\text{trap}}-1], \quad \text{and} \quad y^e_{T_{\text{trap}}-1}[T_{\text{trap}}] = e[T_{\text{trap}}].
$$

The idea behind our Sweep-Pursuit-Capture strategies is to detect the evader and pursue it so as to trap it. Next, we show that the evader remains trapped for all subsequent time instants and that the pursuers achieve capture by using strategies that were developed for the unlimited range sensing version of the game. This principle applies to both versions of the problem.

A. Single pursuer problem

We first present each phase of the strategy for the single pursuer problem.

1) Sweep phase - SWEEP *strategy:* Let $\mathrm{diam}(\mathcal{Q})$ denote the diameter of $\mathcal{Q}$. The SWEEP strategy for the pursuer is to move with maximum step size along a path, as shown in Figure 2(a) such that the union of the sensing discs of the pursuer at the end of each step until the end of this phase contains $\mathcal{Q}$. We term such a path a *sweeping path* for $\mathcal{Q}$. Let t_{sweep} denote the time taken for this strategy to terminate. To determine an upper bound for t_{sweep}, consider placing $\mathcal{Q}$ inside a square region of length $\mathrm{diam}(\mathcal{Q})$ and the pursuer moving along a hypothetical sweeping path for the square region, as shown in Figure 2(b). It is straightforward to show that to achieve coverage, this hypothetical sweeping path is between strips of width $2r_{\text{step}}\sqrt{\kappa^2 - \frac{1}{4}}$, parallel to the side. There are $\left\lceil \frac{\mathrm{diam}(\mathcal{Q})}{2r_{\text{step}}\sqrt{\kappa^2 - \frac{1}{4}}} \right\rceil$ such strips and it takes at most $\left\lceil \frac{\mathrm{diam}(\mathcal{Q})}{r_{\text{step}}} \right\rceil + \left\lceil \sqrt{\kappa^2 - \frac{1}{4}} \right\rceil$ steps to sweep one strip completely and be positioned to sweep through a neighboring strip of this hypothetical sweeping path. We thus obtain the following result.

Lemma III.2 (SWEEP strategy) *In the single pursuer problem with parameters κ and $\frac{\mathrm{diam}(\mathcal{Q})}{r_{\text{step}}}$, the time t_{sweep} taken by the* SWEEP *is at most $\left\lceil \frac{1}{2\sqrt{\kappa^2 - \frac{1}{4}}} \frac{\mathrm{diam}(\mathcal{Q})}{r_{\text{step}}} \right\rceil \left(\left\lceil \frac{\mathrm{diam}(\mathcal{Q})}{r_{\text{step}}} \right\rceil + \left\lceil \sqrt{\kappa^2 - \frac{1}{4}} \right\rceil \right)$ steps.*

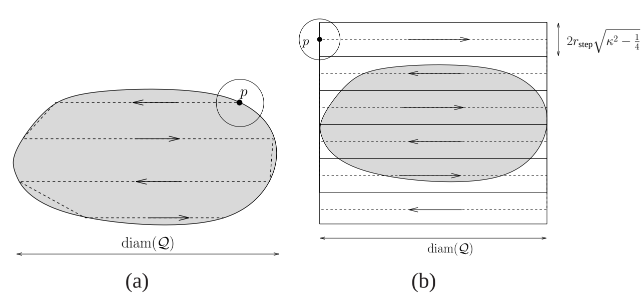


Fig. 2. Single pursuer problem: SWEEP strategy. (a) shows a sweeping path. (b) shows hypothetical sweeping path to determine upper bound on number of steps to detect evader.

2) Pursuit phase - GREEDY *strategy:* Once the evader is detected, the GREEDY strategy for the pursuer is to *move towards the last sensed position of the evader with maximum step size*. This strategy has the property that the pursuer senses the evader's position at every successive time instant. Let t_{trap} denote the *trapping time*, i.e., the time taken by the pursuer to trap the evader after detecting it. We now present our main result for the GREEDY strategy.

Theorem III.3 (GREEDY strategy) *In the single pursuer problem with parameters κ and $\frac{\text{diam}(\mathcal{Q})}{r_{\text{step}}}$, if $\kappa > \sqrt{2 + 2\cos\beta_c}$, where*

$$\beta_c := \frac{1}{2}\left\lceil \frac{\text{diam}(\mathcal{Q})}{\frac{\sqrt{3}}{2} r_{\text{step}}} \right\rceil^{-1} \arctan\frac{1}{4\kappa}, \tag{8}$$

then the GREEDY strategy *has the following properties:*

(i) *the pursuer traps the evader within its sensing radius, and*

(ii) *the trapping time t_{trap} satisfies*

$$t_{\text{trap}} \leq \left(\left\lceil \frac{\log\left(\frac{\sqrt{\kappa^2 - \sin^2\beta_c} - \cos\beta_c - 1}{\kappa - 1}\right)}{\log(1 - \frac{1-\cos\beta_c}{\kappa})} \right\rceil + 1 \right) \left\lceil \frac{\text{diam}(\mathcal{Q})}{\frac{\sqrt{3}}{2} r_{\text{step}}} \right\rceil. \tag{9}$$

Furthermore, if $\kappa > 2$, then as $(\text{diam}(\mathcal{Q})/r_{\text{step}}) \to +\infty$, $t_{\text{trap}} \in O\left((\text{diam}(\mathcal{Q})/r_{\text{step}})^3\right)$.

Theorem III.3 is tight in the sense that if the condition on κ is violated then there exist sufficiently large environments, an evader strategy and initial positions for the players, that lead to evasion against the GREEDY pursuer strategy. This is described by the following result.

Proposition III.4 (Evasion) *Given a single pursuer problem with parameters κ and $\frac{\text{diam}(\mathcal{Q})}{r_{\text{step}}}$ such that $\kappa \leq \sqrt{2 + 2\cos\beta_c}$, where β_c is given by* (8)*, and $\mathcal{Q}$ contains a circle of radius $\frac{r_{\text{step}}}{\sqrt{4-\kappa^2}}$, then there exists an evasion strategy and initial positions of the players for which the pursuer's* GREEDY *strategy fails to trap the evader.*

Figure 3 illustrates this evasion strategy under the conditions required by Proposition III.4.

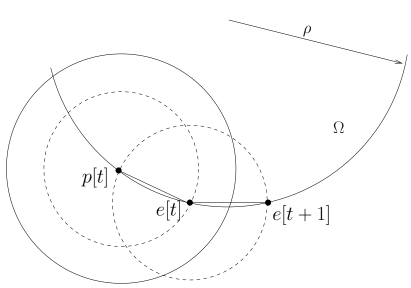


Fig. 3. Illustrating evasion. The dotted circles are the player's motion discs and the solid circle is the pursuer's sensing disc. $e[t]$ and $p[t]$ are on the circle Ω described in Proposition III.4 such that $\|e[t] - p[t]\| = r_{\text{step}}$. Evader chooses to move to $e[t+1]$ on Ω with full step size.

3) Capture phase - LION *strategy:* Once the evader is trapped within the sensing range of the pursuer, the pursuer employs the LION strategy from [4] to complete the capture. For the sake of completeness, we now give a brief description of the LION strategy, adapted to the present problem setting.

The LION strategy can be applied to this phase as follows:

(i) Prior to their $(t+1)^{th}$ move, the pursuer constructs the line $e[t]p[t]$, as shown in Figure 4. Let this line intersect the boundary of the environment at a point $X[t]$ such that $p[t]$ lies between $e[t]$ and $X[t]$.

(ii) The pursuer then also constructs the line $e[t+1]X[t]$ and moves to the intersection of this line with the circle centered at $p[t]$ and of radius r_{step}. Of the two possible intersection points, the pursuer selects the one closer to $e[t+1]$.

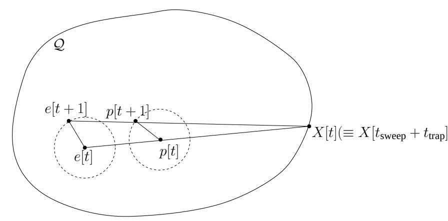


Fig. 4. Single pursuer problem: Using the LION strategy to capture the evader. The dotted circles represent the motion discs of the players.

This construction guarantees that the intersection point $X[t]$ remains the same as the point $X[t_{\text{sweep}} + t_{\text{trap}}]$, for every $t \geq t_{\text{sweep}} + t_{\text{trap}}$, where $t_{\text{sweep}} + t_{\text{trap}}$ is the time at the end of the pursuit phase. Denoting by t_{cap} the time taken by the pursuer to capture the evader after trapping it, we have the following result.

Theorem III.5 (LION strategy [4]) *In the single pursuer problem with parameters κ and $\frac{\text{diam}(\mathcal{Q})}{r_{\text{step}}}$, after trapping the evader within the sensing radius and using the* Lion strategy,

(i) *the distance, $\|p[t] - e[t]\|$, is a non-increasing function of time,*

(ii) *the pursuer captures the evader,*

(iii) *t_{cap} is at most $\left\lceil \left(\frac{\text{diam}(\mathcal{Q})}{r_{\text{step}}}\right)^2 \right\rceil$ steps.*

Thus, our problem with limited sensing is solved because once the evader is trapped within the pursuer's sensing radius, it remains trapped until capture, from part (i) of Theorem III.5. We have also obtained an upper bound on the total time to capture, i.e., $t_{\text{sweep}} + t_{\text{trap}} + t_{\text{cap}}$.

B. Multiple pursuer problem

This section describes the sweep-pursuit-capture strategy for multiple pursuers and the corresponding results. We assume that $\kappa \geq 4$ and $N \geq 5$. We define the following formation for multiple pursuers.

Definition III.6 (Trapping chain) *A group of $N \geq 5$ pursuers $\{p_1, \ldots, p_N\}$ are said to be in a* trapping chain *formation if*

(i) $p_2, \ldots, p_{N-1}$ *are placed counterclockwise on a semi-circle with diameter equal to* $\|p_2 - p_{N-1}\|$,

(ii) *for all* $j \in \{1, \ldots, N-1\}$

$$\|p_j - p_{j+1}\| = r_{\text{step}}\sqrt{4\kappa^2 - 25},$$

and

(iii) p_1, p_2, p_{N-1}, p_N *are on the vertices of a rectangle such that the polygon with vertices* $\{p_1, \ldots, p_N\}$*, in that order, is convex (cf. Figure 5).*

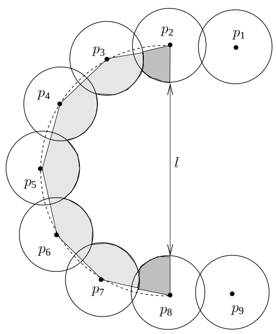


Fig. 5. A trapping chain formation for $N = 9$ pursuers. The circles around the pursuers denote their sensing ranges. The lightly shaded region denotes the capture region and the darkly shaded region along with the lightly shaded one denotes the extended capture region.

We now describe the Sweep-Pursuit-Capture strategy for the multiple pursuer problem.

1) Sweep phase - SWEEP *strategy:* The pursuers begin by placing themselves in a trapping chain formation. We define the *capture region* $\mathcal{S}$ for a trapping chain by

$$\mathcal{S} = \bigcup_{j\in\{3,\ldots,N-2\}} \mathcal{B}_{p_j}(r_{\text{sens}}) \cap \mathring{\mathrm{Co}}\{p_2, \ldots, p_{N-1}\},$$

where $\mathcal{B}_{p_j}(r_{\text{sens}}) \subset \mathbb{R}^2$ denotes the sensing disc of pursuer p_j and $\mathring{\mathrm{Co}}\{p_2, \ldots, p_{N-1}\} \subset \mathbb{R}^2$ denotes the interior of the convex hull of $\{p_2, \ldots, p_{N-1}\}$. The lightly shaded region in Figure 5 is the proposed capture region, $\mathcal{S}$, for the trapping chain. In the sweep phase, pursuers wish to detect the evader within the capture region. We consider a square region of length equal to diameter of the region $\mathcal{G}$ $\mathrm{diam}(\mathcal{G})$ that contains the field $\mathcal{G}$. The pursuers sweep this square region in the direction of the normal to p_1p_N, outward with respect to the convex hull of the pursuers. For a trapping chain shown in Figure 5, we define the *effective* length l as

$$l := \|p_1 - p_N\| - 2r_{\text{sens}} = r_{\text{step}}\left(\frac{\sqrt{4\kappa^2 - 25}}{\sin(\frac{\pi}{2(N-3)})} - 2\kappa\right). \quad (10)$$

As the pursuers move in the direction described earlier, they clear a rectangular strip of length $\mathrm{diam}(\mathcal{G})$ and width $l+4r_{\text{sens}}$. The SWEEP strategy for the pursuers is as follows.

(i) Choose the first rectangular strip at a random distance l_0 from one edge of the square region containing $\mathcal{G}$ and sweep it length-wise. The distance l_0 is a uniform random variable taking values in the interval $[-2r_{\text{sens}}, l + 2r_{\text{sens}}]$. Here, negative l_0 implies that some of the pursuers may begin sweeping from outside the region $\mathcal{G}$.

(ii) Form a sweeping path for the square region and sweep along adjacent strips as shown in Figure 6.

The shaded region in Figure 6 refers to the area that would fall in the proposed capture region $\mathcal{S}$. Now we are interested in determining the probability that an evader falls in the shaded region in Figure 6. That is given by the following result.

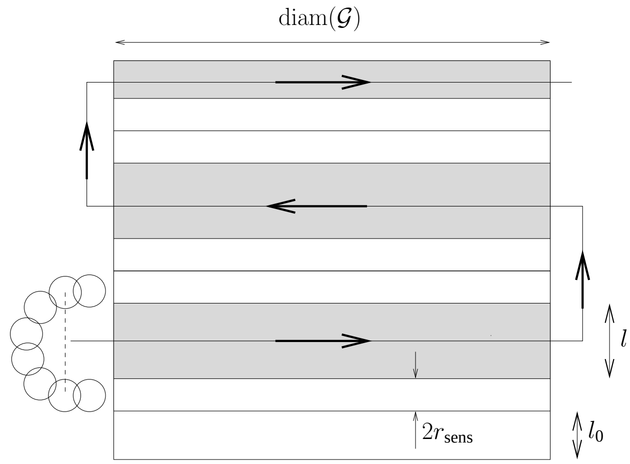


Fig. 6. Multiple pursuer problem: SWEEP strategy. The shaded region represents the region swept by the capture region of the trapping chain.

Theorem III.7 (SWEEP strategy) *In the multiple pursuer problem with parameters* κ, $\frac{\mathrm{diam}(\mathcal{G})}{r_{\text{step}}}$ *and* N*, for any probability distribution for the initial position of the evader with support on* $\mathcal{G}$*, using the* SWEEP *strategy,*

(i) *the probability* P *of detecting the evader inside* $\mathcal{S}$ *for a group of pursuers in a trapping chain, satisfies*

$$P \geq \frac{l}{l + 4r_{\text{sens}}} \geq 1 - \frac{2\pi\kappa}{\left(\sqrt{4\kappa^2 - 25}(N-3) + 2\pi\kappa\right)},$$

and

(ii) *the time* t_{sweep} *taken by the* SWEEP *strategy to terminate satisfies*

$$t_{\text{sweep}} \leq \left\lceil \frac{\mathrm{diam}(\mathcal{G})}{r_{\text{step}}}\left(\frac{\pi/2}{\sqrt{4\kappa^2 - 25}(N-3) + \pi\kappa}\right)\right\rceil \times \left\lceil \frac{\mathrm{diam}(\mathcal{G})}{r_{\text{step}}} + 2\sqrt{4\kappa^2 - 25}N \right\rceil.$$

Remark III.8 The minimum probability P of the pursuers detecting the evader inside the capture region by using the SWEEP strategy is *independent* of the evader's location in $\mathcal{G}$. This means that the best that the evader can do in the present framework is to locate itself initially with a uniform probability in $\mathcal{G}$.

We shall see that the pursuers win when the evader is detected in $\mathcal{S}$ by the pursuers. Otherwise, there exists a path for the evader such that it can avoid being captured indefinitely.

2) Pursuit phase - CIRCUMCENTER *strategy:* If the evader is detected within the proposed capture region at time t_{sweep}, the pursuers need to ensure that they trap the evader within their sensing ranges. Before we describe the strategy for the pursuit phase, consider the following possibility: if the evader steps into the darkly shaded region of the sensing range of p_2 (or of p_{N-1}), then p_2 (resp. p_{N-1}) can use the GREEDY strategy (ref. Section III-A2). By moving towards the evader, the evader's motion disc gets contained inside that pursuer's sensing disc and thus the evader gets trapped. This motivates us to define an *extended capture region* $\mathcal{S}^e$ for the trapping chain by

$$\mathcal{S}^e = \bigcup_{j \in \{2,\dots,N-1\}} \mathcal{B}_{p_j}(r_{\text{sens}}) \cap \mathring{\text{Co}}\{p_2, \dots, p_{N-1}\}.$$

The darkly shaded region along with the lightly shaded region in Figure 5 is the extended capture region $\mathcal{S}^e$ for the trapping chain.

We now present the following pursuit strategy. At each time step $t \geq t_{\text{sweep}}$,

(i) While $e[t+1] \notin \mathcal{S}^e[t]$, the pursuers $p_2, \dots, p_{N-1}$ move towards the *circumcenter*[1] O of the triangle formed by $p_2[t_{\text{sweep}}]$, $e[t_{\text{sweep}}]$ and $p_{N-1}[t_{\text{sweep}}]$ with maximum step. Pursuers p_1 and p_N move parallel to p_2 and p_{N-1} respectively.

(ii) Otherwise, one of the pursuers which senses the evader directly, makes a GREEDY move (ref. Section III-A2) towards the evader and the others move parallel to that pursuer with the maximum step.

One such move is shown in Figure 7. In case (i) of the strategy, note that the pursuers may not sense the evader in every subsequent move. But they will encircle the evader by "closing" the trapping chain around it and then shrink the enclosed region around the evader. We thus have the following result.

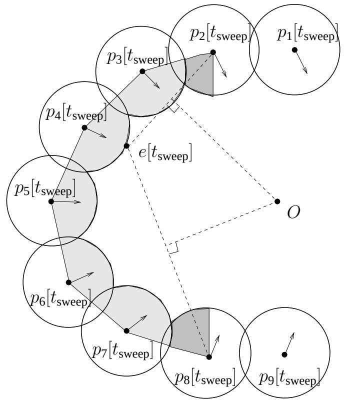


Fig. 7. Multiple pursuer problem: CIRCUMCENTER strategy. At time t_{sweep}, the evader position is sensed by p_4. Pursuers $p_2, \dots, p_8$ move towards O, the circumcenter of triangle formed by p_2, e and p_8. p_1 and p_9 move parallel to p_2 and p_8 respectively. The circles around the pursuers represent their sensing discs.

Theorem III.9 (CIRCUMCENTER strategy) *In the multiple pursuer problem with parameters* κ, $\frac{\text{diam}(\mathcal{G})}{r_{\text{step}}}$ *and* N, *starting from a trapping chain formation, if the evader is detected with* $e[t_{\text{sweep}}] \in \mathcal{S}[t_{\text{sweep}}]$, *then using the* CIRCUMCENTER *strategy,*

(i) *the pursuers trap the evader within their sensing radii,*

(ii) *the* trapping time t_{trap} *satisfies*

$$t_{\text{trap}} \leq \sqrt{4\kappa^2 - 25}N\Big(1 + \frac{1}{2\sin\phi}\Big),$$

where

$$\phi(\kappa) = \frac{\pi}{4} - \arctan\Big(\frac{\kappa}{\sqrt{3\kappa^2 - 25}}\Big),$$

and

(iii) *at that time, the evader is inside the pursuers' convex hull in such a way that*

$$\mathcal{B}_{\frac{r_{\text{step}}}{2}}(e[t_{\text{sweep}} + t_{\text{trap}}]) \subset \text{Co}\{p_1, \dots, p_N\}[t_{\text{sweep}} + t_{\text{trap}}]. \tag{11}$$

[1]The circumcenter of a triangle is the unique point in the plane which is equidistant from all of its three vertices.

The CIRCUMCENTER strategy guarantees trapping of the evader even without pursuers p_1 and p_N. But in that case, the inclusion in (11), which will be required to establish an upper bound on the time for the capture phase that follows, is not guaranteed.

3) The Capture phase - PLANES *strategy:* Once the evader is trapped within the sensing ranges of the pursuers, the pursuers use the PLANES strategy from [5] to capture the evader. Before stating our results, we reproduce this strategy for completeness.

Let the time at the end of the pursuit phase be $t_{\text{sweep}} + t_{\text{trap}}$ and the evader be inside the convex hull of the pursuers as in n (11) (cf. Figure 8). For $t \geq t_{\text{sweep}} + t_{\text{trap}}$ and for every pursuer p_j:

- Draw the line $h_j[t+1]$ through $e[t+1]$, parallel to the line joining $e[t]$ and $p_j[t]$, as shown in Figure 9.
- Move to the point closest to $e[t+1]$ on the line $h_j[t+1]$ with maximum step size.

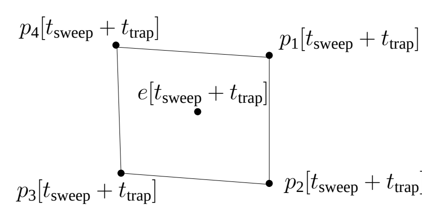


Fig. 8. Multiple pursuer problem: evader trapped inside pursuers' convex hull.

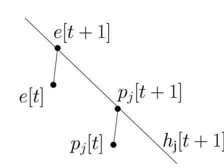


Fig. 9. Multiple pursuer problem: PLANES strategy. Draw the line $h_j[t+1]$ through $e[t+1]$, parallel to the line segment $e[t]p_j[t]$ and move onto it closest to the evader.

Theorem III.9 shows that use of the CIRCUMCENTER strategy in the pursuit phase leads to the evader being trapped and inside the convex hull of the pursuers. Now capture follows from the following theorem, which was partly inspired by the results on the PLANES strategy in [5].

Theorem III.10 (PLANES strategy) *In the multiple pursuer problem with parameters κ, $\frac{\mathrm{diam}(\mathcal{G})}{r_{\text{step}}}$ and N, let the evader be trapped inside the convex hull of the pursuers such that property* (11) *is satisfied. If every pursuer follows the* PLANES *strategy, then*

(i) *the distances, $\|p_j[t] - e[t]\|$ for every $j \in \{1, \dots, N\}$, are non-increasing functions of time,*
(ii) *the pursuers capture the evader and*
(iii) *the time t_{cap} taken in the capture phase is at most $18\kappa\sqrt{4\kappa^2 - 25}N$.*

Item (iii) of Theorem III.10 implies that once the evader is trapped within the sensing ranges of the pursuers, it remains trapped within their sensing ranges at the end of every pursuer move. The capture is now complete and we obtained a novel upper bound on the total time to capture, i.e., $t_{\text{sweep}} + t_{\text{trap}} + t_{\text{cap}}$.

IV. PROOFS OF THE RESULTS

In this section, we formally prove the main results.

A. Single pursuer problem

To prove Theorem III.3, we need some preliminary definitions and results which we present now. In what follows, the notation $\angle ABC$ refers to the smaller of the two angles between segments AB and BC.

Definition IV.1 (Deviation and evasion angles) *Given evader and pursuer at positions $e[\tau]$, $p[\tau]$, for $\tau \in \{t, t+1\}$, define the* deviation angle *$\alpha[t]$ and the evasion angle $\beta[t]$ by:*

$$\alpha[t] := \angle e[t+1]p[t+1]e[t],$$
$$\beta[t] := \alpha[t] + \angle p[t+1]e[t+1]e[t].$$

These angles are illustrated in Figure 10. The following result follows trivially.

Proposition IV.2 *When the pursuer uses the* GREEDY *strategy, for every instant of time t,*

$$|\beta[t]| \geq |\alpha[t]|. \tag{12}$$

Note that equality in (12) only holds when the evader moves away from the pursuer along the line $p[t]e[t]$.

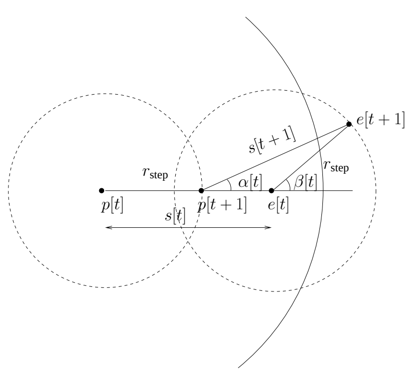


Fig. 10. Relation between the deviation angle $\alpha[t]$ and the evasion angle $\beta[t]$. The dotted circles represent the motion discs of the players. The circle centered at $p[t]$ (shown partially here) is the pursuer's sensing range.

It can be deduced that when the pursuer employs the GREEDY strategy, the distance between the pursuer and evader is a non-increasing function of time. We now define a geometric construction which is useful in the proof.

Definition IV.3 (Cone sector sequence) *Let t_0 denote the time at the end of the sweep phase. Given a time instant $k \in \mathbb{Z}_{\geq 0}$, the sequence $\mathcal{C}_k$ of cone sectors $\mathcal{C}_{k,i}$ for $i \in \mathbb{Z}_{\geq 0}$ is defined as follows:*

(i) *Define the cone sector $\mathcal{C}_{k,0}$ with $p[t_{\text{k}}]$ as its vertex, angle bisector defined by the segment $e[t_{\text{k}}]p[t_{\text{k}}]$ and extended to a point X beyond $e[t_{\text{k}}]$ such that $L_{\text{cone}} := \|p[t_{\text{k}}] - X\| = 2\kappa r_{\text{step}}$, as shown in Figure 11. Let the segment YZ be of length $\frac{r_{\text{step}}}{2}$ and perpendicular to the segment $p[t_{\text{k}}]X$ with X as its midpoint. Accordingly, let $\theta := \angle Yp[t_{\text{k}}]Z = \arctan(1/4\kappa)$ be the* cone angle.
(ii) *For $k, i \geq 0$, denote by t^* the time when the evader leaves the cone sector $\mathcal{C}_{k,i}$. There are two possibilities:*
(a) *the pursuer first constructs a new cone sector $\mathcal{C}_{k,i+1}$ which is a translation of $\mathcal{C}_{k,i}$ having vertex at $p[t^*]$. This is illustrated in Figure 12.*
(b) *If the evader is not inside $\mathcal{C}_{k,i+1}$, then we denote $t_{k+1} := t^*$. The pursuer constructs a new cone sector sequence $\mathcal{C}_{k+1}$.*

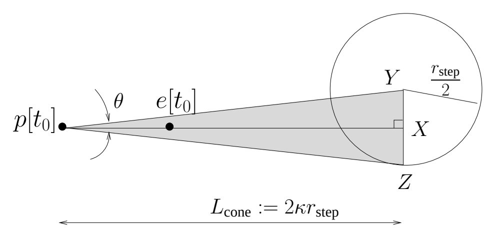


Fig. 11. Construction of cone $\mathcal{C}_{0,0}$. Choose X on the line $e[t_0]p[t_0]$ such that $\|p[t_0] - X\| = 2\kappa r_{\text{step}}$. YZ has length $\frac{r_{\text{step}}}{2}$ and is perpendicular to segment $p[t_0]X$ with X as its midpoint. θ is the cone angle.

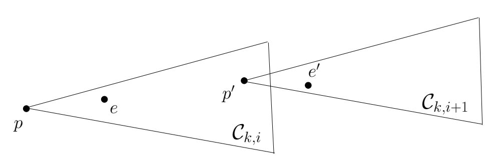


Fig. 12. Construction of cone sector $\mathcal{C}_{k,i+1}$. Translate cone sector $\mathcal{C}_{k,i}$ to have its vertex at p'.

The cone sector sequence described above has the following property.

Proposition IV.4 (Cone sector sequence) *Given a cone sector sequence $\mathcal{C}_k$, the maximum number of steps N^* for which the evader can remain inside it without being captured satisfies*

$$N^* \leq \frac{4\kappa}{\sqrt{3}} \left\lceil \frac{\mathrm{diam}(\mathcal{Q})}{2\kappa r_{\text{step}}} \right\rceil. \tag{13}$$

Proof: We compute an upper bound on the number of steps a pursuer in any cone sector while using the GREEDY strategy. From the definition of a cone sector, this is also an upper bound on the number of steps the evader can remain inside a cone sector. Construct a rectangle with length L_{cone} and width $\frac{r_{\text{step}}}{2}$ such that it contains a cone sector, as shown in Figure 13. Orient a frame of reference such that its X axis is parallel to the angle bisector of the cone sector. Let $p[t_k]-P_1-\cdots-P_4$ denote the path as a result of the pursuer's GREEDY strategy while in the cone sector $\mathcal{C}_{k,0}$. We now construct a path with step size equal to r_{step} at each time instant, that has length no lesser than any such greedy pursuer paths. Select a point P_1' between A and B such that $\|p[t_k]-P_1'\| = r_{\text{step}}$. Then select another point P_2' on between C and D such that $\|P_1'-P_2'\| = r_{\text{step}}$. Of the two possible points, select that point which is farther from $p[t_k]$ as P_2'. Selecting odd P_i''s between A and B and even P_i''s between A and B until it is not possible to select any more of the P_i's on segments AB and CD. This is illustrated in Figure 13. This construction leads to the property that the X coordinates of the P_i''s are greater than those of the corresponding P_i's. Thus the number of P_i''s are no lesser than the number of P_i's. Thus, the path $p[t_k]-P_1-\cdots-P_7$ has its length at least equal to that of $p[t_k]-P_1-\cdots-P_4$. Since the length of segment AB is L_{cone}, the number of steps of such a path is at most equal to L_{cone} divided by the difference in X coordinates of any two consecutive P_i''s, i.e., $\lceil \frac{L_{\text{cone}}}{\frac{\sqrt{3}}{2}r_{\text{step}}} \rceil$. Since $\mathcal{Q}$ has a finite diameter, there can be at most $\lceil \frac{\text{diam}(\mathcal{Q})}{L_{\text{cone}}} \rceil$ cone sectors in any cone sector sequence. Thus, the upper bound (13) is established. ■

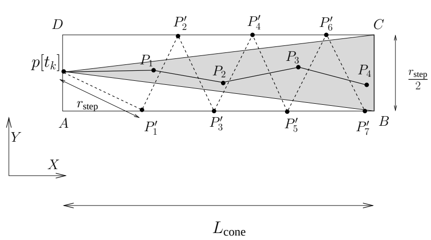


Fig. 13. Upper bound on the number of steps a pursuer can be inside a cone sector. The cone sector $\mathcal{C}_{k,0}$ is illustrated here. The dotted path shows a hypothetical pursuer path that takes the maximum number of steps before leaving a cone sector.

We now state two additional results needed to prove Theorem III.3.

Proposition IV.5 (Maximum evasion angle) *If the pursuer uses the* GREEDY *strategy and if $(\kappa^2-1)r_{\text{step}}^2 \geq s^2[t]$, where $s[t] = \|p[t]-e[t]\|$, then define*

$$\beta_{\text{max}}[t] := \arccos\left(\frac{(\kappa^2-1)r_{\text{step}}^2 - s^2[t]}{2s[t]r_{\text{step}}}\right). \tag{14}$$

If at some time t, $\beta[t] \geq \beta_{\text{max}}[t]$, then the pursuer moves towards $e[t+1]$ and traps the evader.

This result is obtained by applying the cosine rule to $\triangle p[t]e[t]e[t+1]$, where the notation $\triangle ABC$ stands for triangle formed by points A, B and C, as shown in Figure 14. .

Lemma IV.6 (Constraint on maximum evasion angle) *For the evader to move out of a cone sector sequence*

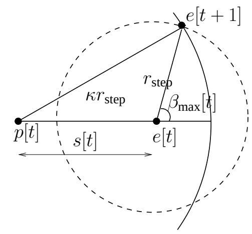


Fig. 14. Constraint on maximum evasion angle. The dotted circle represent the evader's motion disc. The circle centered at $p[t]$ (shown partially here) is the pursuer's sensing range.

$\mathcal{C}_k$, described in Definition IV.3, there exists a time instant $t \in [t_k, t_{k+1}[$ (ref. Definition IV.3) such that

$$|\beta[t]| > \frac{\theta}{2N^*} =: \beta_{\text{c}}, \tag{15}$$

where N^* is defined in Proposition IV.4.

Proof: For the evader to move out of the cone sector sequence $\mathcal{C}_k$, the sum of the angles of deviation for the pursuer must exceed half of the cone angle θ, i.e.,

$$\sum_{t=t_k}^{t_{k+1}} |\alpha[t]| > \frac{\theta}{2}.$$

Geometrically, this condition implies that the angle between the vectors $e[t_{k+1}]-p[t_{k+1}]$ and $e[t_k]-p[t_k]$ must be at least $\frac{\theta}{2}$. This is illustrated in Figure 15. From Proposition IV.2, it implies that

$$\sum_{t=t_k}^{t_{k+1}} |\beta[t]| > \frac{\theta}{2}.$$

Equation (15) now follows from the fact that $t_{k+1}-t_k \leq N^*$, for every k, since there exists a maximum number of time steps N^* for which the evader can remain inside any cone sector sequence, as derived in Proposition IV.4. ■

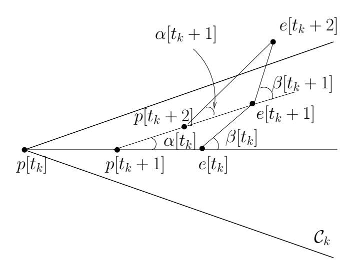


Fig. 15. Illustrating Lemma IV.6. This is a case of the evader moving out of the cone sector sequence $\mathcal{C}_k$ by moving out of $\mathcal{C}_{k,0}$.

We are now ready to prove Theorem III.3.

Proof of Theorem III.3: Two cases need to be considered:

(i) *Evader does not move out of a cone sector sequence*: Capture follows from the construction of the cone sector sequence and from Proposition IV.4.

(ii) *Evader moves out of a cone sector sequence*: In this case, we seek to show that the evader cannot keep moving out of an arbitrarily large number of cone sector sequences. If the evader leaves the cone sector sequence $\mathcal{C}_k$, then for some

$\tau \in \{t_k, \ldots, t_{k+1}-1\}, \beta[\tau] > \beta_{\mathrm{c}}$. Applying the cosine rule to $\triangle p[\tau]e[\tau]e[\tau+1]$, we obtain

$$s^2[\tau+1] = r_{\text{step}}^2 + (s[\tau]-r_{\text{step}})^2 + 2r_{\text{step}}(s[\tau]-r_{\text{step}})\cos\beta[\tau],$$
$$\implies \quad s^2[\tau]-s^2[\tau+1] = 2r_{\text{step}}(s[\tau]-r_{\text{step}})(1-\cos\beta[\tau]).$$

Using equation (15) and since

$$s[\tau]+s[\tau+1] \leq 2\kappa r_{\text{step}},$$

we obtain

$$s[\tau+1]-r_{\text{step}} \leq \left(1 - \frac{(1-\cos(\frac{\theta}{2N^*}))}{\kappa}\right)(s[\tau]-r_{\text{step}}). \quad (16)$$

Defining $\chi_k := s[t_k] - r_{\text{step}}$, we conclude that

$$\begin{aligned}
\chi_{k+1} &\leq s[\tau+1]-r_{\text{step}}, \\
&\leq \left(1 - \frac{(1-\cos(\frac{\theta}{2N^*}))}{\kappa}\right)(s[\tau]-r_{\text{step}}), \\
&\leq \left(1 - \frac{(1-\cos(\frac{\theta}{2N^*}))}{\kappa}\right)\chi_k, \qquad (17)
\end{aligned}$$

where the first and third inequalities follow from the fact that distance $s[t]$ is non-increasing in the GREEDY strategy and the second inequality follows from equation (16). Recall that $\kappa > 1$ by assumption and hence the term in the parenthesis is positive and strictly less than 1. Thus, $\chi_k \to 0$ asymptotically, i.e., the distance between the pursuer and evader tends to r_{step} asymptotically. Moreover, for $\kappa > 2$, the distance reduces to $(\kappa-1)r_{\text{step}}$ after a finite time. Thus, the motion disc of the evader will become completely contained within the sensing disc of the pursuer. Hence, the result follows.

The case of $\kappa = 2$: We have seen that the distance $s[t]$ between the pursuer and evader tends asymptotically to r_{step}. From Proposition IV.5, we obtain that as $s[t] \to r_{\text{step}}$, the angle $\beta_{\max} \to 0$. So, after some finite time,

$$\beta_{\max} < \frac{\theta}{2N^*} =: \beta_{\mathrm{c}}.$$

Thus, the evader becomes confined to the present cone sector sequence according to Lemma IV.6 and from Proposition IV.4, and we can see from part (i) of this proof that the pursuer traps the evader within its sensing radius.

If $\kappa < 2$: We have seen that at each time step t, there is a maximum value $\beta_{\max}[t]$ of the evasion angle $\beta[t]$, so that the evader's next step $e[t+1]$ is not in the pursuer's sensing disc centered at $p[t]$. This is shown in Figure 16. The key idea of this part of the proof is that if we ensure that for all subsequent times after a certain time t^*, $\beta_{\max}[t]$ is less than the minimum value β_c (cf. Lemma IV.6) needed for the evader to leave a cone sector sequence, then the evader is forced to remain inside a final cone sector sequence and trapping follows from part (i). In previous cases, we have seen that the GREEDY strategy reduces the distance $s[t]$ asymptotically to r_{step}. Thus after a finite time t^*, $s[t^*]$ attains a value such that the maximum evasion angle is less than or equal to $(1+\delta)\gamma$, where γ is the maximum evasion angle if the evader is at $e'[t^*]$, which satisfies $\|p[t^*]-e'[t^*]\| = r_{\text{step}}$ and δ is a given positive

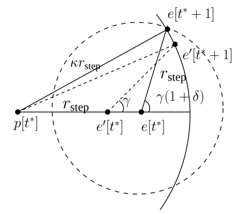


Fig. 16. Illustrating parameters in Equation (18). $e'[t^*]$ is a point such that $\|p[t^*]-e'[t^*]\| = r_{\text{step}}$. γ is the value of the maximum evasion angle if the evader were at $e'[t^*]$. The circle of radius κr_{step} around the pursuer (shown partially here) is the pursuer's sensing disc which the dotted circle around $e[t^*]$ is the evader's motion disc.

number. At this time instant t_{final}, let the pursuer construct a new cone sector sequence, $\mathcal{C}_{\text{final}}$. So, if

$$(1+\delta)\gamma N^* = \frac{\theta}{2}, \quad (18)$$

where N^* and θ are defined in Proposition IV.4 and in the definition of a cone respectively, then for some $\tau \in \{t_{\text{final}}, \ldots, t_{\text{final}}+N^*\}$, $\beta[\tau] \geq (1+\delta)\gamma = \beta_{\mathrm{c}}$ for the evader to leave $\mathcal{C}_{\text{final}}$. This means that the evader is forced to step inside the sensing disc of the pursuer or to remain inside the final cone $\mathcal{C}_{\text{final}}$. In both cases, the pursuer traps the evader within its sensing radius. From equation (18),

$$\gamma < \frac{\theta}{2N^*} = \beta_{\mathrm{c}}.$$

Applying the cosine rule to $\triangle p[t]e'[t]e'[t+1]$,

$$\kappa = \sqrt{2+2\cos\gamma} > \sqrt{2+2\cos\beta_{\mathrm{c}}}.$$

Thus, we have shown that if $\kappa > \sqrt{2+2\cos\beta_{\mathrm{c}}}$, then the pursuer's GREEDY strategy guarantees that the evader is trapped.

Computing an upper bound on the trapping time: We have seen that when the pursuer uses the GREEDY strategy, the evader cannot leave an arbitrarily large number of cone sector sequences. Thus, to compute an upper bound on the trapping time, we compute an upper bound on the number of cone sector sequences that the evader can leave. We have seen that using the GREEDY strategy, $\beta_{\max} \leq \beta_{\mathrm{c}}$ after finite time. From (14), we can determine that distance s_{c} for which $\beta_{\max} = \beta_{\mathrm{c}}$, so that subsequently, the evader is confined to the final cone sequence:

$$s_{\mathrm{c}} = (\sqrt{\kappa^2 - \sin^2\beta_{\mathrm{c}}} - \cos\beta_{\mathrm{c}})r_{\text{step}}.$$

If k is the final cone sequence index, then using equation (17),

$$s_{\mathrm{c}} - r_{\text{step}} \leq \chi_k \leq \lambda\chi_{k-1} \leq \cdots \leq \lambda^k(\kappa-1)r_{\text{step}},$$

where $\lambda = 1 - \frac{1-\cos(\frac{\theta}{2N^*})}{\kappa}$ and the worst-case $\chi_0 = (\kappa-1)r_{\text{step}}$. Upon simplifying, we obtain

$$k \leq \left\lceil \frac{\log\left(\frac{\sqrt{\kappa^2-\sin^2\beta_{\mathrm{c}}}-\cos\beta_{\mathrm{c}}-1}{\kappa-1}\right)}{\log(\lambda)} \right\rceil$$

The result now follows from the fact that for the case of $\kappa < 2$, we construct an extra final cone sequence and the maximum number of steps in each cone sequence can be at most N^*. The asymptotic result follows by routine simplifications. ■

Proof of Proposition III.4: We prove this result by determining a set of initial conditions and an evader strategy that leads to evasion. Suppose at time t, the pursuer and the evader are on a circle Ω with radius $\rho = \frac{r_{\text{step}}}{\sqrt{4-\kappa^2}}$, such that $\|e[t]-p[t]\| = r_{\text{step}}$ as shown in Figure 3. The evader is not trapped as its motion disc is not completely contained inside the pursuer's sensing disc. An evader strategy is to choose a point $e[t+1]$ on Ω such that $\|e[t]-e[t+1]\| = r_{\text{step}}$. Since $\rho = \frac{r_{\text{step}}}{\sqrt{4-\kappa^2}}$, $e[t+1]$ lies outside the pursuer's sensing disc before its move at time $t+1$. By the GREEDY strategy, $p[t+1] = e[t]$. Thus, $\|e[t+1]-p[t+1]\| = r_{\text{step}}$ and the evader can avoid getting trapped. ■

B. Multiple pursuer problem

We first state a property of the effective length of the trapping chain.

Proposition IV.7 *The effective length of the trapping chain satisfies*

$$\frac{2(\sqrt{4\kappa^2-25}(N-3)-\pi\kappa)}{\pi} < \frac{l}{r_{\text{step}}} < \sqrt{4\kappa^2-25}N-2\kappa.$$

Proof: The left hand side of the inequality follows from the fact that the circumference of the circle passing through the vertices of the trapping chain is greater than the sum of the distances of neighboring vertices. The right hand side follows from repeated use of the triangle inequality. ■

Proof of Theorem III.7: Let the evader be located at a point $Y_e \in \mathcal{G}$ and let its distance from the edge AB be y, as shown in Figure 17. Note that the distance of the evader from the edge AD does not play any role in what follows. The main idea behind this proof is as follows: Y_e would lie in the shaded region in Figure 17 if $l_0+2r_{\text{sens}} < y$ or $l_0-2r_{\text{sens}} > y$. This is equivalent to choosing a length equal to the effective length l of the trapping chain from a total length $l+4r_{\text{sens}}$, if we let l_0 take a uniformly random value from $[-2r_{\text{sens}}, l+2r_{\text{sens}}]$. Thus, the probability of success for the pursuers is at least the ratio of l to $l+4r_{\text{sens}}$.

To be more specific, let the spatial probability density of the Y coordinate of the evader inside $\mathcal{G}$ be $p(y)$. Thus, $\int_0^{\operatorname{diam}\mathcal{G}} p(y)dy = 1$. The probability that the evader is detected inside the capture region $\mathcal{S}$ is given by

$$P(e\in\mathcal{S}) = \int_{-2r_{\text{sens}}}^{l+2r_{\text{sens}}} P(e\in\mathcal{S}|l_0=k)P(l_0=k)dl_0,$$

where $k \in [-2r_{\text{sens}}, l+2r_{\text{sens}}]$. Assuming the pursuers have no information about the evader's location inside $\mathcal{G}$, l_0 is chosen uniformly randomly from $[-2r_{\text{sens}}, l+2r_{\text{sens}}]$. Hence, we have

$$\begin{aligned}P(e\in\mathcal{S}) &\geq \int_{-2r_{\text{sens}}}^{l+2r_{\text{sens}}} P(e\in\mathcal{S}|l_0=k)\frac{1}{l+4r_{\text{sens}}}dl_0\\ &=\frac{1}{l+4r_{\text{sens}}}\int_{-2r_{\text{sens}}}^{l+2r_{\text{sens}}}\Big(\sum_{j=0}^{n-1}\int_{l_0+2r_{\text{sens}}+j(l+2r_{\text{sens}})}^{l_0+2r_{\text{sens}}+j(l+2r_{\text{sens}})+l} p(y)dy\\ &\quad+\int_{l_0+2r_{\text{sens}}+n(l+4r_{\text{sens}})+l}^{\operatorname{diam}\mathcal{G}} p(y)dy\Big)dl_0,\end{aligned}$$

where $n := \lceil \operatorname{diam}\mathcal{G}/(l+4r_{\text{sens}})\rceil$ is the number of times the pursuers sweep to clear the entire environment. Since the variables l_0 and y are independent (cf. Figure 18 for the region of integration), changing the order of integration gives

$$\begin{aligned}P(e\in\mathcal{S}) &\geq \frac{1}{l+4r_{\text{sens}}}\sum_{j=0}^{n-1}\int_{j(l_0+4r_{\text{sens}})}^{(j+1)(l_0+4r_{\text{sens}})} p(y)f(y,l)dy\\ &= \frac{1}{l+4r_{\text{sens}}}\int_0^{n(l_0+4r_{\text{sens}})} p(y)f(y,l)dy,\end{aligned}$$

where

$$f(y,l) := \begin{cases}\int_{-2r_{\text{sens}}}^{y-2r_{\text{sens}}} dl_0 + \int_{y+2r_{\text{sens}}}^{l+2r_{\text{sens}}} dl_0, & \text{for } y\leq l,\\ \int_{y-2r_{\text{sens}}-l}^{y-2r_{\text{sens}}} dl_0, & \text{otherwise.}\end{cases}$$

In both cases, $f(y,l) = l$. Thus, the minimum probability of success is $l/(l+4r_{\text{sens}})$, since $\int_0^{n(l_0+4r_{\text{sens}})} p(y)dy = \int_0^{\operatorname{diam}\mathcal{G}} p(y)dy + \int_{\operatorname{diam}\mathcal{G}}^{n(l_0+4r_{\text{sens}})} p(y)dy$, and $p(y) = 0$ outside of $\mathcal{G}$. The second inequality in part (i) follows by use of the left hand inequality in Proposition IV.7. The reason why this is a lower bound on the required probability is that if the pursuers had some information about the evader's location, then they could choose l_0 randomly from a smaller interval than the current one and thus increase the probability of detecting the evader in the capture region.

From the SWEEP strategy, the width of each strip swept is $l+4r_{\text{sens}}$. So the maximum number of strips after which the sweep phase terminates is $\lceil\frac{\operatorname{diam}(\mathcal{G})}{l+4r_{\text{sens}}}\rceil$. It takes at most $\frac{\operatorname{diam}(\mathcal{G})+2(l+2r_{\text{sens}})}{r_{\text{step}}}$ time steps for the pursuers to clear a strip followed by aligning themselves parallel to the adjacent strip. The result now follows using Proposition IV.7. ■

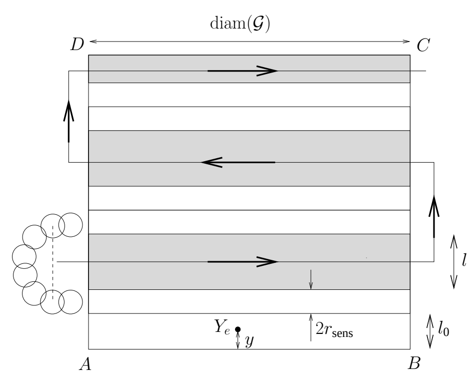


Fig. 17. Illustrating the proof of Theorem III.7.

To prove Theorem III.9, we first establish the following properties of a trapping chain. In what follows, given points $a, b, c \in \mathbb{R}^2$, the notation $\operatorname{dist}(a, bc)$ is the distance of point a from the line bc.

Lemma IV.8 (Trapping chain properties) *If* $e[t] \in \mathcal{S}[t]$, *then the following statements hold:*

(i) *If* $\operatorname{dist}(e[t], p_j[t]p_{j+1}[t]) > \frac{3}{2}r_{\text{step}}$, *for all* $j \in \{1,\dots,N-1\}$, *then the evader cannot step outside* $\operatorname{Co}\{p_1[t],\dots,p_N[t]\}$ *at time* $t+1$ *by crossing* $p_j[t]p_{j+1}[t]$.

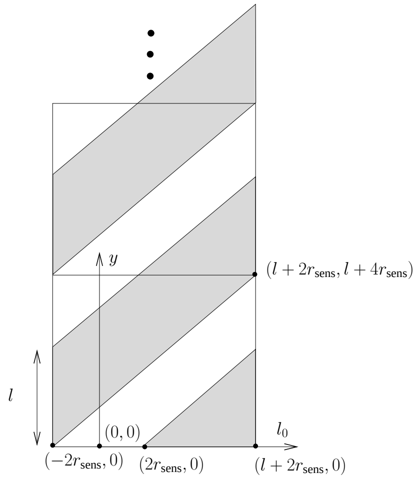


Fig. 18. The region of integration in determining $P(e \in \mathcal{S})$ in the proof of Theorem III.7. Fix a value of l_0 in the interval $(-2r_{\text{sens}}, l+2r_{\text{sens}})$ to get the values of y that correspond to $P(e \in \mathcal{S})$.

(ii) *If* $\operatorname{dist}(e[t], p_j[t]p_{j+1}[t]) \leq \frac{3}{2}r_{\text{step}}$ *or* $\operatorname{dist}(e[t+1], p_j[t]p_{j+1}[t]) \leq \frac{3}{2}r_{\text{step}}$, *for some* $j \in \{1,\dots,N-1\}$, *then the evader is* trapped within the sensing radii *of pursuer* p_j *or* p_{j+1} *or of both* p_j *and* p_{j+1}.

(iii) *There exists a* $\phi > 0$, *independent of* N, *such that for every point* $q \in \bigcup_{j\in\{3,\dots,\lfloor\frac{N}{2}\rfloor+1\}} \mathcal{B}_{p_j}(r_{\text{sens}}) \cap \mathring{\mathrm{Co}}\{p_2,\dots,p_{N-1}\}$,

$$\angle q p_2 p_{N-1} > \phi,$$

Proof: Parts (i) and (ii) follow trivially from the definitions of trapping within sensing radii and from the construction of the trapping chain. For part (iii), we can see that for $j^* = \lfloor\frac{N}{2}\rfloor + 1$ if q (in the specified set) is the point of intersection of the tangent from p_2 to the sensing disc of p_{j^*}, then the angle $\angle q p_2 p_{N-1}$ is minimized. This follows from the fact that the line $p_2 p_{j^*}$ is parallel to $p_3 - p_{j^*-1}$. This angle is minimum when $N = 5$. Thus, given a $\kappa \geq 4$, from trigonometry, we obtain

$$\phi = \frac{\pi}{4} - \arctan\Big(\frac{\kappa}{\sqrt{3\kappa^2 - 25}}\Big).$$

■

The use of the CIRCUMCENTER strategy in the pursuit phase and the geometry of the trapping chain gives us the following result.

Lemma IV.9 *If the evader is trapped within the union of the sensing radii of pursuers at time* t_{trap}, *for every* $j \in \{1,\dots,N-1\}$, *then*

$$\operatorname{dist}(e[t_{\text{trap}}], p_j[t_{\text{trap}}]p_{j+1}[t_{\text{trap}}]) > \frac{r_{\text{step}}}{2}.$$

Proof: Let the evader be trapped at time t_{trap} in the sensing radii of the pursuers. From part (ii) of Lemma IV.8, at time $t_{\text{trap}} - 1$ and for every $j \in \{1,\dots,N-1\}$,

$$\operatorname{dist}(e[t_{\text{trap}}-1], p_j[t_{\text{trap}}-1]p_{j+1}[t_{\text{trap}}-1]) > \frac{3}{2}r_{\text{step}}.$$

Thus, immaterial of where the evader decides to step, its distance from $p_j[t_{\text{trap}}-1]p_{j+1}[t_{\text{trap}}-1]$ is greater than $\frac{r_{\text{step}}}{2}$. Two cases are possible:

(a) *the evader steps inside the sensing disc of some pursuer* p_j*:* There are two further possibilities. If $\operatorname{dist}(e[t_{\text{trap}}], p_j[t_{\text{trap}}-1]p_{j+1}[t_{\text{trap}}-1]) \leq \frac{3}{2}r_{\text{step}}$, then the evader is trapped by part (ii) of Lemma IV.8 and the present lemma is proven. Else, we have $\operatorname{dist}(e[t_{\text{trap}}], p_j[t_{\text{trap}}-1]p_{j+1}[t_{\text{trap}}-1]) > \frac{3}{2}r_{\text{step}}$. Now, p_j uses part (ii) of the CIRCUMCENTER strategy. Even in this case, $\operatorname{dist}(e[t_{\text{trap}}], p_j[t_{\text{trap}}]p_{j+1}[t_{\text{trap}}]) > \frac{r_{\text{step}}}{2}$, for every $j \in \{1,\dots,N-1\}$.

(b) *the evader steps outside the sensing disc of every pursuer:* In a trapping chain, the overlap between the sensing discs of any two neighboring pursuers has the property that length of the common chord of these discs is greater than $\frac{3}{2}r_{\text{step}}$. This means that even if any two neighboring pursuers p_j and p_{j+1} happen to move parallel to each other, we have $\operatorname{dist}(e[t_{\text{trap}}], p_j[t_{\text{trap}}]p_{j+1}[t_{\text{trap}}]) > \frac{r_{\text{step}}}{2}$. ■

We now present the proof of Theorem III.9.

Proof of Theorem III.9: We first look at a case in which $\operatorname{dist}(e[t_{\text{sweep}}], p_j[t_{\text{sweep}}]p_{j+1}[t_{\text{sweep}}]) \leq \frac{3}{2}r_{\text{step}}$ for some $j \in \{1,\dots,N-1\}$. In this case, the evader is already trapped within the sensing radii of the pursuers, from part (ii) of Lemma IV.8 and the result holds.

Now let $\operatorname{dist}(e[t_{\text{sweep}}], p_j[t_{\text{sweep}}]p_{j+1}[t_{\text{sweep}}]) > \frac{3}{2}r_{\text{step}}$, for every $j \in \{1,\dots,N-1\}$. There are two possibilities: if $e[t+1] \in \mathcal{S}^e[t]$, for any $t \geq t_{\text{sweep}}$, then there is a pursuer p_j for which $y^e[t+1] = e[t+1]$. This pursuer uses part (ii) of the CIRCUMCENTER strategy and the evader is trapped within the sensing radius of p_j. Part (iii) of the result follows using Lemma IV.9.

So, let $e[t_{\text{sweep}}+1] \notin \mathcal{S}[t_{\text{sweep}}]$. Now, the pursuers compute the circumcenter O of $\triangle p_2[t_{\text{sweep}}]e[t_{\text{sweep}}]p_{N-1}[t_{\text{sweep}}]$. Lemma IV.8 implies that the evader cannot step out of the pursuers' convex hull by crossing line $p_j[t]p_{j+1}[t]$, for any $j \in \{1,\dots,N-1\}$. Thus, it suffices to show that the evader cannot leave the pursuers' convex hull by crossing line $p_1[t]p_N[t]$. In fact, we show that at the end of every pursuer move, the evader remains on the same side of p_2p_{N-1} until it gets trapped. We argue this as follows. As illustrated in Figure 19, any point on lines $p_2[t_{\text{sweep}}]O$ and $p_{N-1}[t_{\text{sweep}}]O$ is reached faster by p_2 and p_{N-1} respectively than by the evader. Thus, the motion of the evader is confined to the convex hull of $\{O, p_2, \dots, p_{N-1}\}$, which reduces to the point O in a number of time steps upper bounded by

$$\max_{j\in\{2,\dots,N-1\}} \left\lceil \frac{\|p_j[t_{\text{sweep}}] - O\|}{r_{\text{step}}} \right\rceil,$$

which is essentially the time taken by the furthest pursuer to reach O. Thus,

$$t_{\text{trap}} \leq \frac{R + l + 2\kappa r_{\text{step}}}{r_{\text{step}}},$$

where R denotes the circumradius of $\triangle p_2[t_{\text{sweep}}]e[t_{\text{sweep}}]p_{N-1}[t_{\text{sweep}}]$. From elementary geometry, at time t_{sweep} we have

$$\begin{aligned} R &= \frac{\|p_2 - e\| \|p_{N-1} - e\| \|p_2 - p_{N-1}\|}{4\,\text{Area}(\triangle p_2 e p_{N-1})}, \\ &\leq \frac{l + 2\kappa r_{\text{step}}}{2 \sin \angle e p_2 p_{N-1}}, \\ &\leq \frac{l + 2\kappa r_{\text{step}}}{2 \sin \phi}, \end{aligned}$$

where the second and third inequalities follow from part (iii) of Lemma IV.8. Thus, part (ii) of the theorem follows from the use of right hand inequality in Proposition IV.7.

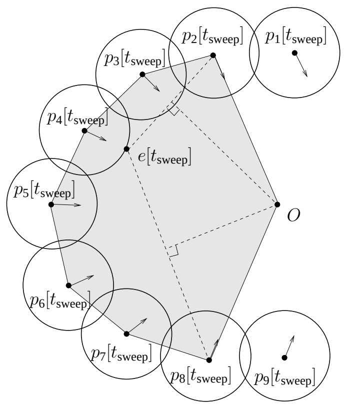


Fig. 19. A move of the CIRCUMCENTER strategy. The evader is confined to the shaded region. The circles around the pursuers represent their sensing discs.

To prove part (iii), recall that pursuers p_1 and p_N move parallel to p_2 and p_{N-1}, respectively. Since the evader remains inside the convex hull of pursuers $p_2, \ldots, p_{N-1}$, the distance of the evader from line $p_1 p_N$ is always greater than $\frac{r_{\text{step}}}{2}$, until it gets trapped. From this fact and Lemma IV.9, part (iii) now follows. ■

Proof of Theorem III.10: Part (i) follows the PLANES strategy. Thus, once the evader is trapped, it remains trapped at all successive time instants when the pursuers use the PLANES strategy. Thus, the problem is reduced to one with unlimited sensing for the pursuers. To show that the algorithm leads to capture in finite time, we refer the reader to [5].

We now determine an upper bound on the time taken for the capture phase in terms of the trapping chain parameters. Referring back to the proof of correctness of the PLANES strategy, since the evader is in the convex hull of the pursuers, let v_j denote vectors of magnitude r_{step} in the direction of $p_j[t_{\text{trap}}] - e[t_{\text{trap}}]$. We now wish to seek a lower bound on the radius ϵ of the largest circle centered at the origin that can be inscribed inside the convex hull of the vectors v_j. This is equivalent to determining what is the largest of the angles $\angle p_i[t_{\text{trap}}] e[t_{\text{trap}}] p_j[t_{\text{trap}}]$. Due to the property (11) and to the fact that the distance between any two adjacent pursuers in the trapping chain is non-increasing during the CIRCUMCENTER strategy, the angle $\angle p_i[t_{\text{trap}}] e[t_{\text{trap}}] p_j[t_{\text{trap}}]$ is the greatest when i and j are adjacent and the evader is equidistant from both of them and at a distance of $\frac{r_{\text{step}}}{2}$ from $p_i[t_{\text{trap}}] p_j[t_{\text{trap}}]$. This is shown in Figure 20. This gives, $\epsilon = r_{\text{step}} \frac{r_{\text{step}}/2}{\kappa r_{\text{step}}} = \frac{r_{\text{step}}}{2\kappa}$. Now, following [5], we observe that there exist three pursuers which contain the evader within their convex hull at the end of the pursuit phase, such that the sum of the distances of these pursuers to the evader decreases by at least $\frac{\epsilon}{3}$ at the end of every pursuer move. The result now follows from the fact that the distance between any one of the three pursuers and the evader is at most $l + 2\kappa r_{\text{step}}$ at the end of the pursuit phase and the use of the right hand side inequality in Proposition IV.7. ■

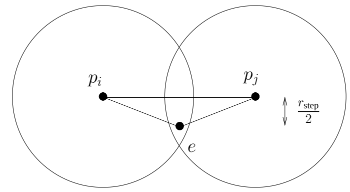


Fig. 20. Illustrating proof of Theorem III.10. Given the evader to be at a distance greater than $\frac{r_{\text{step}}}{2}$ from the line $p_i p_j$, the angle $\angle p_i[t_{\text{trap}}] e[t_{\text{trap}}] p_j[t_{\text{trap}}]$ is the greatest when the evader is equidistant from both of them

V. SIMULATION RESULTS

We now present simulation studies to investigate the robustness of the algorithms to sensing errors. The simulations were run using MATLAB®.

A. Single pursuer problem

We assume zero-mean additive Gaussian measurement errors in the position of the evader with a standard deviation given by

$$\sigma[t] = \epsilon \|p[t] - e[t]\|,$$

for some constant $\epsilon > 0$. This means that the uncertainty in the location of the evader increases with its distance from the pursuer. Under this noisy sensor model:

- The SWEEP strategy remains unchanged. It terminates when an evader measurement is available.
- For the GREEDY and LION strategies, the pursuer uses the noisy measurements of the evader position instead of the true position $e[t]$ to compute its next position.

The evader is defined to be captured if the probability of the evader being inside the motion disc of the pursuer before the pursuer's move is more than a certain threshold. In other words, for some t

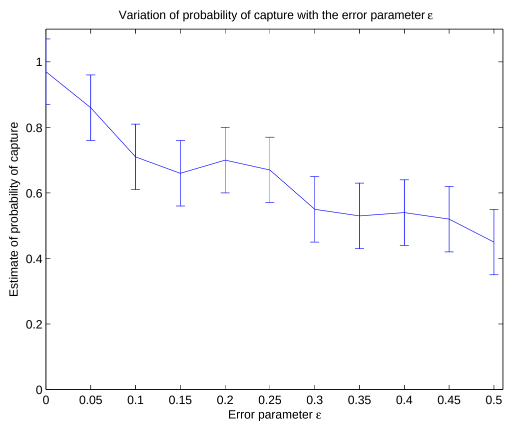
$$\mathcal{B}_{\sigma[t]}(y^e_{t-1}[t]) \subset \mathcal{B}_{r_{\text{step}}}(p[t-1]),$$

where $\mathcal{B}_{\sigma[t]}(y^e_{t-1}[t])$ denotes the circular region of radius $\sigma[t]$ centered at $y^e_{t-1}[t]$. For the evader's motion, we assume that it moves away from the pursuer with some randomization, while avoiding the boundary. Specifically,

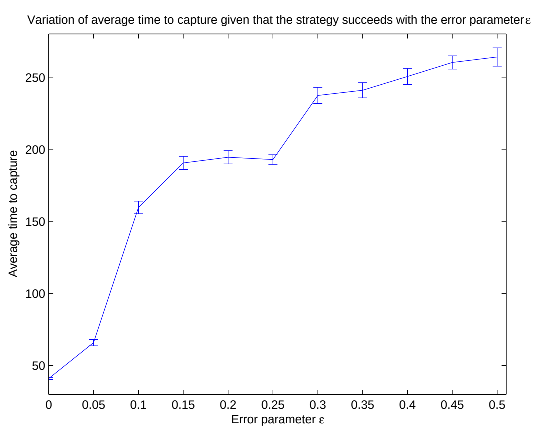
- if the evader is not close to the boundary of the environment, then it chooses to move to a point on its motion circle, selected uniformly randomly in a sector with angle 0.2 radians. This sector is placed symmetrically along the line $e[t]p[t]$ and away from the pursuer.
- If the boundary is visible to the evader, then it chooses to move to a point $e[t+1]$ on its motion circle such that $\angle e[t+1]e[t]p[t] = \pi - 0.2$. Of the two points possible,

the evader chooses that point which is further away from the boundary. In other words, when the evader reaches the boundary, it chooses to move to a point that is away from the pursuer as well as not very close to the boundary.

We study how the average capture time after evader detection varies with ϵ. We assume the environment is a circle with diameter 40 units. We assume $\kappa = 5$ units and $r_{\text{step}} = 1$ unit. The initial position of the evader was chosen uniformly randomly in the environment. An upper limit of $2,000$ time steps was set to decide whether the strategy terminated in a success. The plots of probability of success of the strategy and average capture times after detection (given that the strategy terminates with capture) versus ϵ are shown in Figures 21(a) and 21(b).



(a) Estimate of capture probability versus noise parameter ϵ. The vertical bars give a 95% confidence interval about the probability estimate $P(\epsilon)$ which is given by $\left[P(\epsilon) - 2\sqrt{\frac{0.25}{n}}, P(\epsilon) + 2\sqrt{\frac{0.25}{n}}\right]$, where $n = 100$ is the number of trials [19].



(b) Average capture time, given that the strategy succeeds versus noise parameter ϵ. The vertical bars give a 95% confidence interval about the average capture time $T(\epsilon)$ which is given by $\left[T(\epsilon) - 2\sqrt{\frac{SD(\epsilon)}{nP(\epsilon)}}, T(\epsilon) + 2\sqrt{\frac{SD(\epsilon)}{nP(\epsilon)}}\right]$, where $SD(\epsilon)$ is the standard deviation in the capture time, $P(\epsilon)$ is the estimated probability of success and $n = 100$ is the number of trials [19].

Fig. 21. Simulation results for the single pursuer problem. For a particular evader strategy, we study how the average capture time varies with the noise parameter ϵ.

B. *Multiple pursuer problem*

We assume zero-mean additive independent Gaussian measurement errors in the position of the evader with standard deviations given by

$$\sigma_j[t] = \epsilon \| p_j[t] - e[t] \|,$$

for some noise parameter $\epsilon > 0$. Under this noise model:

- The SWEEP strategy remains unchanged. It terminates when an evader measurement is available.
- For the CIRCUMCENTER and PLANES strategies, the team of pursuers use the average of the available evader measurements $\tilde{y}_{\text{t-1}}[t-1]$ and $\tilde{y}_{\text{t-1}}[t]$, to compute their next positions.

The evader is defined to be captured if for some pursuer $j \in \{1, \dots, N\}$,

$$\mathcal{B}_{\sigma_j[t]}(\tilde{y}_{\text{t-1}}[t]) \subset \mathcal{B}_{r_{\text{step}}}(p_j[t-1]).$$

For the sake of simulations, we assume $N = 7$ pursuers with $\kappa = 5$ units and $r_{\text{step}} = 1$ unit. We assume a square field of edge length 100 units, where the evader is initially placed at a uniformly randomly selected location. Upon detection, we assume that the evader moves away from the closest pursuer with some randomization. Specifically, it moves to a point on its motion circle, selected uniformly randomly in a sector of angle equal to 0.2 radians. This sector has its vertex at $e[t]$ and angle bisector parallel to the line $e[t_{\text{sweep}}]O$, where t_{sweep} is the time when the evader is detected and O is the circumcenter of the triangle $p_2[t_{\text{sweep}}]$, $p_6[t_{\text{sweep}}]$ and $e[t_{\text{sweep}}]$. We study how the average capture time after detection varies with ϵ. An upper limit of 1000 time steps was set to decide whether the strategy terminated in a failure. The plots of probability of success of the strategy and average capture times after detection (given that the strategy terminates with capture) versus ϵ are shown in Figures 22(a) and 22(b).

In both the problems, we observe a decreasing trend in the probability of success of the strategies with increasing values of the error parameter ϵ. The average capture times after detection, given that the strategies lead to capture is observed to increase with ϵ.

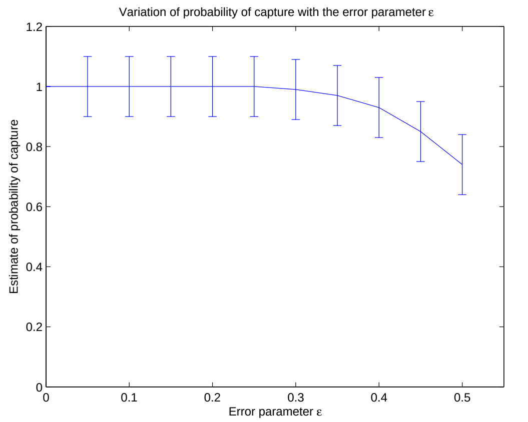
VI. BIOLOGICAL INTERPRETATIONS

Our analysis in the previous sections can shed light on the trade-offs that predators face when deciding upon the group size. Based on our results from Section III-B, we now study how the group size of the pursuers varies with the evader availability in the multiple pursuer problem.

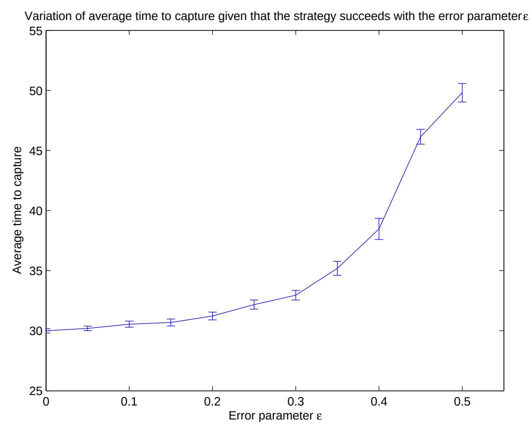
For simplicity, we assume a square field where the evader is initially located and denote by $\delta := \frac{1}{\text{diam}^2(\mathcal{G})}$ the evader density. From the results in Section III-B, an upper bound on the total time taken by the pursuers in all three phases of the strategy is given by

$$\frac{1}{\delta(aN+b)} + \frac{cN}{\sqrt{\delta}(aN+b)} + kN,$$

where $a := 2r_{\text{step}}^2\sqrt{4\kappa^2 - 25}/\pi$, $b := (2\pi\kappa - 6\sqrt{4\kappa^2 - 25})/\pi$, $c := 2r_{\text{step}}\sqrt{4\kappa^2 - 25}$ and $k := \sqrt{4\kappa^2 - 25}(1 + 1/\sin\phi) + 18\kappa\sqrt{4\kappa^2 - 25}$ are constants independent of N or δ.



(a) Estimate of capture probability versus noise parameter ϵ. The vertical bars give a 95% confidence interval for the probability estimate is defined identically as in Figure 21(a).



(b) Average capture time, given that the strategy succeeds versus noise parameter ϵ. The vertical bars give a 95% confidence interval about the average capture time is defined identically as in Figure 21(b).

Fig. 22. Simulation results for the multiple pursuer problem. For $N = 7$ and for a particular evader strategy, we study how the average capture time varies with noise parameter ϵ.

From part (i) of Theorem III.7, we observe that when all other variables are kept constant, the lower bound on successful detection probability of the SWEEP strategy increases with N. However, a higher N results into a greater time to capture the evader. This suggests a trade-off on the group size N which we analyze as follows.

Let ν denote the nutritional content of the evader. We quantify the energy spent by each pursuer as the time taken to capture the evader. The energy gain from the pursuit is quantified as the amount of nutrition each participating pursuer receives from the evader. For a self-sustaining pursuit, we must have that the energy gained by each pursuer is at least equal to the energy spent during the hunt. Thus,

$$\frac{\nu}{N} \geq \frac{1}{\delta(aN+b)} + \frac{cN}{\sqrt{\delta}(aN+b)} + kN.$$

From this relation, we observe that for a given value of δ, there exists an upper limit on the number of pursuers in the group for which it is advantageous for the pursuers to engage in a pursuit with the prospect of gaining energy. A plot of the upper limit on the group size N versus the evader density δ is shown in Figure 23.

This analysis shows that for higher values of δ, a larger number of pursuers can be accommodated in the trapping chain. This is consistent with observations in the biology literature by Caraco and Wolf [20] that have reported higher group size in foraging lions during the wet season (prey abundance), than in the dry season, (prey scarcity). Further, from our analysis, it also follows that for a given evader density, the higher the prey nutrition value ν, the higher is the upper limit on the number of pursuers in the trapping chain. This is consistent with the observations reported by Griffiths [21] regarding how the size of hunting packs relate to the size of the prey relative to that of the predators.

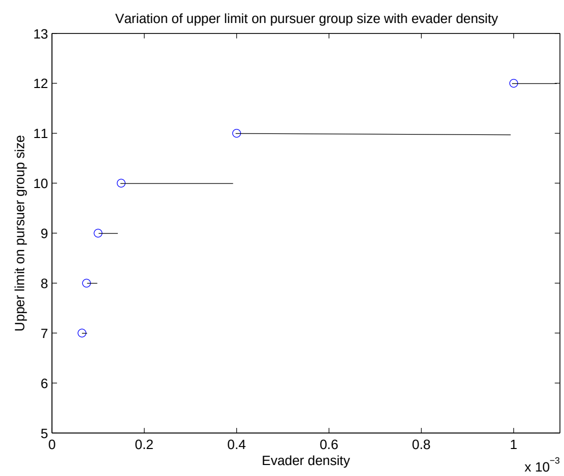


Fig. 23. Plot of maximum group size of pursuers that can be sustained versus the evader density δ, for $\kappa = 5$, $r_{\text{step}} = 1$, $\nu = 10000$.

VII. CONCLUSIONS AND FUTURE DIRECTIONS

We have addressed discrete-time pursuit-evasion problems in the plane with sensing capabilities restricted to a finite disc. We consider two variants of the pursuit-evasion in discrete-time. The first involves a single pursuer and an evader in a bounded convex environment. The second is a formation design problem for multiple communicating pursuers to capture a single evader in a boundaryless environment. In both problems, the evader is initially located inside a bounded subset of the environment and moves only when detected. We propose a Sweep-Pursuit-Capture strategy for both problems.

In the first problem, we give sufficient conditions on the range of values taken by the ratio of sensing to stepping radius of the players so that the GREEDY pursuer strategy of moving towards the last-sensed evader position leads to the evader being trapped within the pursuer's sensing disc and finally to capture. We also give conditions under which there exist locations from which the evader can escape. In the second problem, we have shown that the pursuers capture the evader with a certain probability that is independent of the initial evader location in a bounded region. We give novel upper bounds on the total time taken to capture for both problems. We also present simulation studies that suggest robustness with respect to sensing errors. Finally, on the basis of the obtained upper bound on the total capture time, we provide an upper bound on the group size of the pursuers for which the pursuit is advantageous from the point of view of gaining energy. The conclusions based on our analysis are consistent with observations reported in ecology literature.

In the future, it would be interesting to study the effects of communication losses or errors in the multiple pursuer problem. Another interesting direction for future research would be to consider more sophisticated sensing models for the players.

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