SYNCHRONIZATION OF BEADS ON A RING BY FEEDBACK CONTROL

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Abstract. This paper analyzes a discrete-time algorithm to synchronize a group of agents moving back and forth on a ring. Each agent or "bead" changes direction upon encountering another bead moving in the opposite direction. Communication is sporadic: two beads are able to exchange information only when they come sufficiently close. This allows agents to update their state including their velocity and desired sweeping arc on the boundary. Our analysis makes use of consensus algorithms tools and guarantees that for a given set of initial conditions, synchrony is asymptotically reached.

Key Words: synchronization, consensus algorithms, distributed algorithms, stochastic matrices

AMS Subject Classifications: 37N35, 68W15, 34H05, 93C85, 68T40.

1. Introduction. Consider n robotic agents that control their motion on a ring and that communicate when in close proximity of each other. If the n agents control their motion to simulate n beads sliding on a frictionless hoop, then we know that their dynamics is very rich. In fact, in [4], the authors study extensively the case of $n = 3$ and prove the existence of periodic as well as chaotic orbits.

Additionally, it is known that the periodic orbits, referred to as periodic modes, arise only from a thin set of initial conditions. An example periodic mode is described as follows: for $n = 4$, each bead moves back and forth inside a quadrant and impacts its neighboring beads periodically at the boundary of the quadrant. When the robotic agents move along this periodic mode, we say that they are synchronized.

A possible worldly motivation for the study of this class of algorithms is the surveillance of regions in a 2D space. For example, boundary-patrolling algorithms are being proposed in the robotics literature for the monitoring of spreading fires, toxic-area containment and clean-up, and the sensing of sharp temperature gradient surfaces in the sea. These algorithms require sporadic communication among agents, which have to optimally divide the task among themselves without the intervention of a supervisor. On the other hand, the study of the n beads problem can find justification on more fundamental grounds. Namely, the investigation of under what conditions systems subject to impacts and controlled dynamics are robustly stable, and what techniques can be useful helpful in proving such stability properties. Both these aspects have motivated us to consider this synchronization problem.

If the dynamics of the agents emulates one of the oscillatory modes for n beads, then it is guaranteed that every point of the ring gets visited by at least one agent in a bounded time. In particular, among all the modes, the synchronized mode is the one for which any point of the ring is visited after the shortest time interval.

We therefore pose the question: can n intelligent beads, capable of controlling their motion, autonomously organize themselves so that each one sweeps a sector of the ring and impacts with the neighboring beads always at the boundaries of the

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sector? In other words, can they reach a periodic orbit and get in sync? We show that synchronization can indeed be achieved by a simple feedback law. We present an algorithm which requires only occasional communication – two beads exchange information only when they impact. Correctness of this algorithm is proved using the theory of discrete-time consensus algorithms. Moreover, as we discuss later, the algorithm confers certain robustness properties on the emerging synchronized behavior, which is of interest for any control system.

Consensus algorithms have been extensively studied, beginning with the early work on averaging opinions and stochastic matrices in [5]. For the setting of non-degenerate stochastic matrices, [20] gives convergence conditions for consensus algorithms under mild connectivity assumptions. Recent references on average consensus, algebraic graph methods and symmetric stochastic matrices include [12, 8]. Recent surveys [7, 14, 16] discuss attractive properties of these algorithms such as convergence under delays and communication failures, and robustness to communication noise.

Synchronization in itself has been a widely studied problem and has been explored for multi-agent system coordination; e.g. see [21, 6, 13, 11]. In [21] a generalized distributed network of nonlinear dynamic systems with access to global information is considered and synchronization in the network is shown to occur for strong enough coupling strengths. The authors in [6] and [13] present distributed algorithms using which synchronization is achieved in multi-agent systems using event triggered and self triggered control respectively. The authors in [11] draw analogies between impulsive and diffusive synchronization in the weak coupling limit.

References on the problem of perimeter estimation and monitoring by mobile robots include [3, 22, 18, 19]. Patrolling problems have also been studied in [15, 1, 10]. More relevant to this paper are the studies in [2, 9], which make use of the steady-state orbit for even number of synchronized agents described here and referred to as 'balanced' synchronization. In [2] pairs of agents have to be released at particular points, sequentially, and with the same speed. In contrast, in our algorithm the number of agents can be odd, the agents can be released at arbitrary positions and with arbitrary speeds. The distributed algorithm in [9] requires only that the agents move with a fixed speed. However, it can not be easily extended to a perimeter which is a closed curve unless the agents are assumed to have unique identities. Further, we stabilize a broader range of trajectories, namely 'unbalanced' synchronization.

The contributions of this paper can be summarized as follows. We design a distributed algorithm that allows a collection of beads to reach synchronization and that is robust to failure of beads. The algorithm requires the beads to slowdown and speedup immediately prior to and after impact respectively; accordingly, we refer to the algorithm as the "slowdown, impact and speedup algorithm." The definitions of synchronization for both the case of an even and odd number of beads is given. The beads can be deployed with arbitrary initial positions and speeds. At the desired steady state, every bead sweeps a sector of equal length, and neighboring beads meet always at the same point. If n is even, the beads all travel at the same speed, while if n is odd, the beads travel at the same average speed. Two beads exchange information only when they impact. We prove a local convergence result – the agents reach the desired steady state – under some assumptions. Extensive simulations show that synchronization is reached in general, even when the assumptions are not satisfied. The beads are assumed to have some capacities: they can maintain their speed and an uncorrupted version of the coordinates of their sweeping arcs. If there is a small variation in their trajectories, for instance due to the algorithm parameter f described later, we believe that the local stability results will still hold. However, for larger variations, complex phenomena may occur for this impulsive system, as oberved for example in [4]. A preliminary incomplete version of this work appeared in [17].

The paper is organized as follows. Section 2 introduces notation employed and describes in detail what is meant by agent or bead synchronization on a circular boundary. The discrete-time synchronization algorithm is presented in Section 3. A set of preliminary results on which the main theorems build upon is presented in 4. The main results that allow us to analyze the algorithm are included in 5. After this we present simulations in Section 6 showing that convergence of the algorithm is indeed possible in most general cases. Finally, we summarize the results in Section 7.

Notation. On the ring or 1-sphere $\mathbb{S}^1$, by convention, let us define positions as angles measured counterclockwise from the positive horizontal axis. The *counterclockwise distance between two angles* $\mathrm{dist}_{\mathtt{cc}}\colon \mathbb{S}^1 \times \mathbb{S}^1 \to [0, 2\pi)$ is the path length from an angle to the other traveling counterclockwise. The column vector with entries all equal to 1 is $\mathbf{1}_n \in \mathbb{R}^n$. When working with indices in $\{1, \dots, n\}$, we use the identifications $0 \equiv n$ and $n + 1 \equiv 1$.

2. Model and problem statement. Here we model a network of agents moving on a ring and we state our stabilization problem for certain interesting periodic modes.

First, we propose our agents model with motion control, sensing and communication capabilities. The agents are at arbitrary positions $\theta_i \in \mathbb{S}^1$, $i \in \{1, \dots, n\}$, at initial time and ordered counterclockwise.

Each agent controls its motion according to $\dot{\theta}_i(t) = u_i(t)$, where u_i is a bounded control signal. Each agent senses its own position on the ring and senses/distinguishes impacts with its counterclockwise and clockwise neighbors. However, agents do not need to have knowledge of their absolute positions in a global reference frame. Similarly, agents do not need to know the total number of beads n and the circle length. Each agent is equipped with a short-range communication device; for simplicity, we assume two agents communicate only when they are at the same position. In other words, two agents have *communication impacts* when they move to a coincident position. The algorithm can be implemented over anonymous agents; that is, agents lacking an identifier that can distinguish them from each other. However, for simplicity in formulating the problem, we make use of indices $i \in \{1, \dots, n\}$ and make use of coordinates in a global reference frame. Finally, each agent is equipped with a processor, capable of storing quantities in memory and performing computations.

Next, we describe some interesting periodic trajectories for n beads moving on a ring. It is our objective, in the following sections, to design a motion control and communication algorithm to render such trajectories attractive.

Definition 2.1 (Balanced synchronization). *Consider a collection of n beads moving on a ring. The collection of beads is* balanced synchronized *with period T, if (i) any two neighboring beads impact always at the same point, (ii) the time interval between two consecutive impacts, involving the same beads, has duration T, and (iii) all the beads impact simultaneously. In other words, in a balanced synchronized collection, each bead sweeps an arc of length $2\pi/n$ at constant speed $2\frac{2\pi}{nT}$.*

An example of a collection of four beads in sync is shown in Figure 2.1: each bead sweeps an arc at the boundaries of which it impacts with one of its neighbors and the impacts happen simultaneously.

If n is odd, then balanced synchronization cannot be reached. Therefore, we give the following weaker synchronization notion, reachable also for odd n.

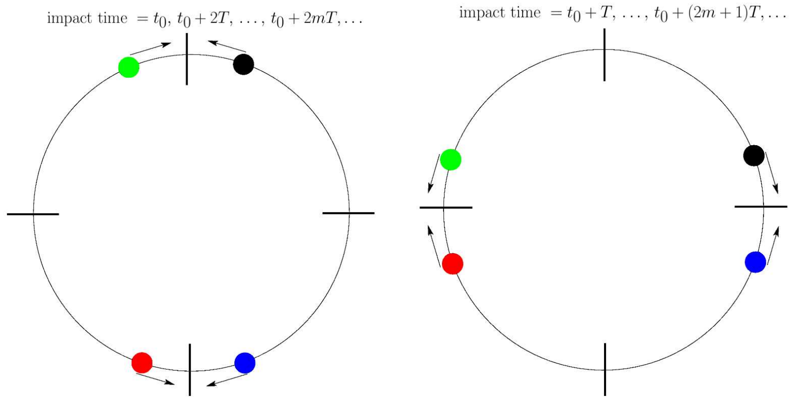

FIG. 2.1. *The figure shows a collection of four beads moving in balanced synchronization.*

DEFINITION 2.2 (Unbalanced synchronization). *Consider a collection of n beads moving on a ring. The collection of beads is* unbalanced synchronized *with period T, if (i) any two beads impact always at the same point and (ii) the time interval between two consecutive impacts, involving the same beads, has duration T. As before, in an unbalanced synchronized collection, each bead sweeps an arc of length $2\pi/n$ at average speed $2\frac{2\pi}{nT}$.*

3. Synchronization algorithm. In this section we describe an algorithm that allows a collection of n agents to achieve balanced synchronization (for n even) and unbalanced synchronization (for n odd). We begin with an informal description for the case when n is even:

> Each agent changes its direction of motion when it impacts another agent with opposing velocity. Each agent maintains an estimate of the arc it eventually sweeps when the network asymptotically achieves balanced synchronization. This estimate is updated according to an averaging law at each communication impact (so that all estimated arcs converge to pairwise contiguous arcs of equal length). A similar averaging law is applied to the agent's speed to ensure that all agents' speeds converge to a common nominal value. To synchronize the back-and-forth motion inside the arcs, each agent travels at nominal speed while inside its arc, slows down when moving away from it, and speeds up when moving towards it after an impact.

We refer to this strategy as to the Slowdown, Impact and Speedup Algorithm, abbreviated as the SIS ALGORITHM. To provide a formal description, we begin by defining all variables that each agent maintains in its memory and we later state how these variables are updated as time evolves and communication impacts take place.

Algorithm Variables. Each agent $i \in \{1, \dots, n\}$ maintains in memory the following tuple:

$$\begin{aligned}
&v_i \in \mathbb{R}_{>0}, && \text{the nominal speed,}\\
&d_i \in \{-1, +1\}, && \text{the direction of motion,}\\
&a_i \in \{-1, +1\}, && \text{the moving-away flag,}\\
&\ell_i \in \mathbb{S}^1, && \text{the arc lower boundary, and}\\
&u_i \in \mathbb{S}^1, && \text{the arc upper boundary.}
\end{aligned}$$

Regarding initialization, we allow $v_i(0)$, $d_i(0)$ to be arbitrary and we set $\ell_i(0) = u_i(0) := \theta_i(0)$, and $a_i(0) := d_i(0)$.

Given these definitions, it is convenient to introduce the following notation and nomenclature. First, we define the ith processor state $x_i := (v_i, d_i, a_i, \ell_i, u_i)$ and call (θ_i, x_i) the ith agent state. Next, we associate an arc of the ring to each bead. This arc is the fraction of the ring that each bead eventually sweeps when balanced synchrony (as in Definition (2.1)) is asymptotically reached. To each bead i, we associate a *desired sweeping arc* defined by

$$\mathrm{Arc}(\ell_i, u_i) = \{\theta \in \mathbb{S}^1 \mid \mathrm{dist}_{\mathtt{cc}}(\ell_i, \theta) \leq \mathrm{dist}_{\mathtt{cc}}(\ell_i, u_i)\}.$$

This quantity will also be denoted by $\mathcal{D}_i$ for convenience henceforth.

Algorithm Rules. The algorithm rules specify how the agents move in continuous time and how they update their processor states when certain events happen.

First, at all time $t \geq 0$, each bead sets its velocity $\dot{\theta}_i$ depending on whether the bead is traveling inside its desired sweeping arc, or, if outside the sweeping arc, depending on whether it is moving away from or towards the sweeping arc. Specifically, given two scalar gains $\frac{1}{2} < f < 1$ and $h = \frac{f}{2f-1} > 1$, we set

$$\dot{\theta}_i(t) := d_i(t)v_i(t) \cdot \begin{cases} 1, & \text{if } \theta_i(t) \in \mathcal{D}_i,\\ f, & \text{if } \theta_i(t) \notin \mathcal{D}_i \text{ and } d_i(t) = a_i(t),\\ h, & \text{if } \theta_i(t) \notin \mathcal{D}_i \text{ and } d_i(t) = -a_i(t). \end{cases}$$

Second, the ith processor state changes only when one of the following two events occurs: *(Impact Event)* an impact takes place with either bead $i-1$ or with bead $i+1$, or *(Crossing Event)* bead i crosses either ℓ_i or u_i while leaving its desired sweeping arc.

(Impact Event) If at time t an impact occurs for bead i with either bead $i+1$ or $i-1$, then: (1) the two beads exchange through communication their processors states, and (2) with this information, each bead updates its memory as follows. We define an impact between beads i and $i+1$ to be of *head-to-tail type* if $d_i(t) = d_{i+1}(t)$, and of *head-to-head type* if instead $d_i(t) = -d_{i+1}(t)$. The ith processor state is updated

according to:

$$v_i(t^+) := \begin{cases} \frac{1}{2}\big(v_i(t) + v_{i-1}(t)\big), & \text{if the impact occurs with } i-1, \\ \frac{1}{2}\big(v_i(t) + v_{i+1}(t)\big), & \text{if the impact occurs with } i+1, \end{cases} \tag{3.1}$$

$$d_i(t^+) := \begin{cases} -d_i(t), & \text{if the impact is head-to-head type,} \\ d_i(t), & \text{otherwise,} \end{cases} \tag{3.2}$$

$$a_i(t^+) := a_i(t),$$

$$\ell_i(t^+) := \begin{cases} C_i(t) - \frac{1}{2}\operatorname{dist}_{\mathtt{cc}}(C_{i-1}(t), C_i(t)), & \text{if the impact occurs with } i-1, \\ \ell_i(t), & \text{if the impact occurs with } i+1, \end{cases} \tag{3.3}$$

$$u_i(t^+) := \begin{cases} u_i(t), & \text{if the impact occurs with } i-1, \\ C_i(t) + \frac{1}{2}\operatorname{dist}_{\mathtt{cc}}(C_i(t), C_{i+1}(t)), & \text{if the impact occurs with } i+1, \end{cases} \tag{3.4}$$

where the upper-script $+$ indicates the variable value right after the impact, and where we define the *center* $C_i \in \mathbb{S}^1$ of the desired sweeping arc $\mathcal{D}_i$ by $C_i = \ell_i + \operatorname{dist}_{\mathtt{cc}}(\ell_i, u_i)/2$.

Note that, after an impact between beads i and $i-1$, we have $\ell_{i-1}(t^+) = u_i(t^+)$ because they both are defined as the midpoint of the arc from $C_{i-1}(t)$ to $C_i(t)$.

(Crossing Event) The memory of each bead i is updated also when the agent crosses either $\ell_i(t)$ or $u_i(t)$ while leaving its desired sweeping arc. The nominal speed v_i, the direction d_i and the boundary of the sweeping arc ℓ_i and u_i do not change,

$$v_i(t^+) := v_i(t), \quad d_i(t^+) := d_i(t), \quad \ell_i(t^+) := \ell_i(t), \quad u_i(t^+) := u_i(t),$$

The flag a_i is updated as follows:

$$a_i(t^+) := d_i(t).$$

Here the upper-script $+$ indicates the value of the memory right after bead i crosses the boundary of its desired sweeping arc.

Only two-way impacts have been considered in the above algorithm. However, impacts between three or more beads can be assumed to be a sequence of two-way impacts separated by infinitesimal times. By default, impacts between beads with smaller indices can be addressed first. Although the order in which they are addressed affects the subsequent motion of the beads, it does not affect the convergence results of the SIS Algorithm.

4. Preliminary results. In this section we prove some preliminary results before we can prove the correctness of the SIS Algorithm. We begin with an important characterization of initial states.

Definition 4.1 (Admissible balanced and unbalanced configurations). *A state* $\{(\theta_i, x_i)\}_{i\in\{1,\dots,n\}}$ *is*

(i) directionally balanced *if* $\sum_{i=1}^n d_i = 0$

(ii) directionally D-unbalanced *for* $D \in \{-n+1, \dots, n-1\} \setminus \{0\}$, *if* $\sum_{i=1}^n d_i = D$.

Furthermore, a state has an admissible configuration *if in addition to being* directionally balanced *or* D-unbalanced, *for all* $i, j \in \{1, \dots, n\}$ *and* $j \neq i$, $\theta_i \neq \theta_j$. *The set of* admissible balanced configurations, *and* admissible D-unbalanced configurations *are denoted by* $\mathcal{A}_{0-\mathrm{bal}}$, *and* $\mathcal{A}_{D-\mathrm{unbal}}$ *respectively.*

Note that $\{(\theta_i, x_i)\}_{i\in\{1,\dots,n\}} \in \mathcal{A}_{0-\mathrm{bal}}$ if and only if n is even and $n/2$ beads are moving clockwise and $n/2$ are moving counterclockwise.

Next, at each time $t \geq 0$, we define the *impact graph* $\mathcal{G}(t)$ as the undirected graph with vertex set $\{1, \ldots, n\}$ and with edge set defined by the following rule: the pair (i, j) is an edge in $\mathcal{G}(t)$ if the beads i and j collide at time t.

Proposition 4.2 (Uniform connectivity of impact graphs). *Along the trajectories of the* SIS Algorithm, *with* $\{(\theta_i(0), x_i(0))\}_{i\in\{1,\ldots,n\}} \in \mathcal{A}_{0-\mathrm{bal}} \bigcup \mathcal{A}_{D-\mathrm{unbal}}$, *for all* $t_0 \geq 0$ *the graph* $\bigcup_{t\in[t_0,t_0+2\pi/(fv_{\min})]} \mathcal{G}(t)$ *is connected.*

The proof of Proposition 4.2 builds up on the following facts.

Lemma 4.3 (Properties). *Along the trajectories of the* SIS Algorithm, *with* $\{(\theta_i(0), x_i(0))\}_{i\in\{1,\ldots,n\}} \in \mathcal{A}_{0-\mathrm{bal}} \bigcup \mathcal{A}_{D-\mathrm{unbal}}$:

(i) $\sum_{i=1}^n d_i(t)$ *is constant,*

(ii) any two desired sweeping arcs are disjoint or share at most a boundary point, furthermore their label index increases in the counterclockwise direction, i.e., $u_i(t) = \ell_{i+1}(t)$,

(iii) the order of the beads is preserved, i.e., for all i, $j \in \{1, \ldots, n\}$, $j \notin \{i, i+1\}$, *and* $t \geq 0$, *we have* $\mathrm{dist}_{\mathsf{cc}}(\theta_i(t), \theta_{i+1}(t)) \leq \mathrm{dist}_{\mathsf{cc}}(\theta_i(t), \theta_j(t))$. *Therefore, a bead* i *can impact only its immediate neighbors* $i-1$ *and* $i+1$.

Proof. We first prove (i). Let $\sum_{i=1}^n d_i(0) = D$. The only instants at which $\sum_{i=1}^n d_i(t)$ can change is when an impact occurs, as in equation (3.2). If the impact is of head-to-tail type, then the directions of both the beads involved do not change. On the other hand, if the impact is of head-to-head type, then the directions of the beads involved are just swapped, therefore $\sum_{i=1}^n d_i(t) = D$ for any $t \geq 0$.

We now prove (ii). To initialize the algorithm, $\mathcal{D}_i(0) = \ell_i(0) = u_i(0) = \theta_i(0)$, and $\theta_i(0)$ are ordered along the counterclockwise direction. The desired sweeping arc $\mathcal{D}_i$ is updated only when the bead i is involved in an impact according to equations (3.3) and (3.4). It is elementary to show that the update equations for ℓ_i and u_i force $u_i(t^+) = \ell_{i+1}(t^+)$ and $\ell_i(t^+) = u_{i-1}(t^+)$. This clearly implies that the order of the desired sweeping arcs is never changed and that any two desired sweeping arcs can at most share a boundary.

We finally prove (iii). The order of the beads can change only as a consequence of an impact. However, we show next that even after an impact the order of the beads is preserved. If beads i and $i+1$ are involved in an impact of head-to-head type, then after the impacts both beads change their direction so clearly $\mathrm{dist}_{\mathsf{cc}}(\theta_{i-1}(t+s), \theta_i(t+s)) \leq \mathrm{dist}_{\mathsf{cc}}(\theta_{i-1}(t+s), \theta_{i+1}(t+s))$, with $0 \leq s < \overline{s}$ and $t + \overline{s}$ is the time at which i impacts again. If the impact is of head-to-tail type, then the directions of the two beads does not change, but their nominal velocities $v_i(t^+)$ and $v_{i+1}(t^+)$ are equal because of equation (3.1). The impact can occur in $\mathcal{D}_i(t)$, or in $\mathcal{D}_{i+1}(t)$ or in neither, see Figure 4.1. If the impact occurs in $\mathcal{D}_i(t)$ and $d_i(t) = d_{i+1}(t) = +1$, then after the impact $\dot{\theta}_i(t^+) = v_i(t^+)$ while $\dot{\theta}_{i+1}(t^+) = hv_{i+1}(t^+)$. In fact, because of part (ii), $i+1$ is moving towards its desired sweeping arc. If the impact occurs in $\mathcal{D}_i(t)$ and $d_i(t) = d_{i+1}(t) = -1$, then after the impact $\dot{\theta}_i(t^+) = -v_i(t^+)$ and $\dot{\theta}_{i+1}(t^+) = -fv_{i+1}(t^+)$ because $i+1$ is moving away from its desired sweeping arc, again because of part (ii). Recalling that $f < 1$ and $h > 1$ we have that, in both cases, $\mathrm{dist}_{\mathsf{cc}}(\theta_{i-1}(t+s), \theta_i(t+s)) \leq \mathrm{dist}_{\mathsf{cc}}(\theta_{i-1}(t+s), \theta_{i+1}(t+s))$ for any time $0 \leq s < \overline{s}$. An analogous reasoning leads to the conclusion that this property holds also if the impact occurs in $\mathcal{D}_{i+1}(t)$. Now, if the impact occurs in neither $\mathcal{D}_i(t)$ nor $\mathcal{D}_{i+1}(t)$, then the beads are both moving either towards or away their desired sweeping arcs. Therefore, $\dot{\theta}_i(t^+) = \dot{\theta}_{i+1}(t^+) = hv_i(t^+)$ or $\dot{\theta}_i(t^+) = \dot{\theta}_{i+1}(t^+) = fv_i(t^+)$. Again $\mathrm{dist}_{\mathsf{cc}}(\theta_{i-1}(t+s), \theta_i(t+s)) \leq \mathrm{dist}_{\mathsf{cc}}(\theta_{i-1}(t+s), \theta_{i+1}(t+s))$ for any $0 \leq s < \overline{s}$. ☐

Lemma 4.4 (Impacts in bounded interval). *Let* $v_{\min} = \min_{i\in\{1,\ldots,n\}} v_i(0)$. *Along*

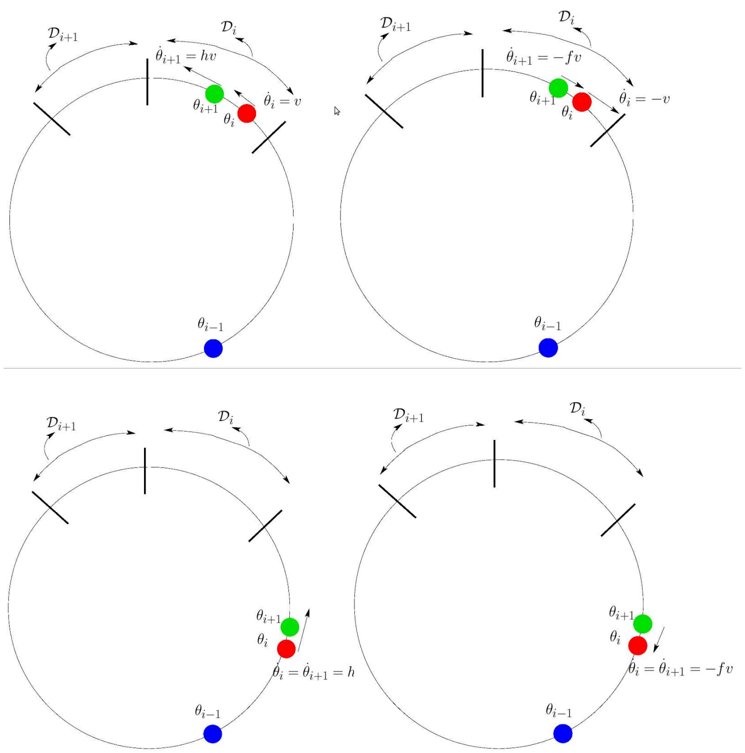

FIG. 4.1. *This figure shows that, regardless from where and with which velocities beads i and $i+1$ impact, the order of the beads is preserved. The velocities in the figure are the velocities after the impact. The speed v is just the average value of v_i and v_{i+1} before the impact.*

the trajectories of the SIS ALGORITHM, *with* $\{(\theta_i(0), x_i(0))\}_{i \in \{1,\ldots,n\}} \in \mathcal{A}_{0-\text{bal}} \bigcup \mathcal{A}_{D-\text{unbal}}$, *for all* $i \in \{1, \ldots, n\}$ *and for all* $t_0 > 0$, *bead* i *impacts at least once with both its neighbors* $i-1$ *and* $i+1$ *across the interval* $[t_0, t_0 + \frac{2\pi}{f v_{\min}}]$.

Proof. Note that $\min_{i \in \{1,\ldots,n\}} v_i(t) \geq \min_{i \in \{1,\ldots,n\}} v_i(0) = v_{\min}$ because of equation (3.1). Therefore for any $t > 0$ the lowest possible speed at which a bead can travel is $f v_{\min}$. We first show that at most after $\frac{\pi}{f v_{\min}}$ any bead has a head-to-head type impact with one of its neighbors. First, any bead i can only impact neighbors $i+1$ and $i-1$ because of Lemma 4.3, part (iii). The necessary time for two beads i and $i+1$ to impact depends on their positions, the directions of motion and the speeds they are traveling with.

In the worst possible case at a time $t = t_0$ all the beads are clustered in a small arc of $\mathbb{S}^1$ of length ϵ, with i and $i+1$ at the opposite ends of the arc (i.e., $\mathrm{dist}_{\mathsf{cc}}(\theta_{i+1}(t_0), \theta_i(t_0)) = \epsilon$), $d_i(t_0) = d_{i+1}(t_0)$, and the speeds have the smallest possible value $|\dot{\theta}_i(t_0)| = |\dot{\theta}_{i+1}(t_0)| = f v_{\min}$. Let us suppose $d_i(t_0) = d_{i+1}(t_0) = +1$. That is, $i+1$ is moving towards the cluster of beads and i is moving away from it. Because of Lemma 4.3, part (i), we have that $\sum_{i=1}^{n} d_i(t_0)| = D < n$ and this implies that $i+1$ can travel at most for $\frac{\epsilon}{2 f v_{\min}}$ before having a head-to-head type impact. So at $t_1 \leq t_0 + \frac{\epsilon}{2 f v_{\min}}$, $d_{i+1}(t_1) = -1$, and $\mathrm{dist}_{\mathsf{cc}}(\theta_{i+1}(t_1), \theta_i(t_1)) \geq \epsilon$. Now, suppose that even after the impact $|\dot{\theta}_{i+1}(t_1)| = f v_{\min}$, then beads i and $i+1$ are moving towards each other and $\mathrm{dist}_{\mathsf{cc}}(\theta_i(t_1), \theta_{i+1}(t_1)) \leq 2\pi - \epsilon$. They then meet at time

$t_2 \leq t_1 + \frac{2\pi-\epsilon}{2fv_{\min}} \leq t_0 + \frac{\epsilon}{2fv_{\min}} + \frac{2\pi-\epsilon}{2fv_{\min}} = t_0 + \frac{\pi}{fv_{\min}}$.

After the impact with $i+1$, $d_i(t_2) = -1$ and, therefore, in its next head-to-head type impact bead i meets $i-1$. Following the same reasoning, we have that at most after $\frac{\pi}{fv_{\min}}$ the two beads i and $i-1$ meet. Hence across the interval $[t_0, t_0 + \frac{2\pi}{fv_{\min}}]$ any bead impacts at least once with both its neighbors. □

Proof. [of Proposition 4.2] Because of Lemma 4.4, for all i and for all t_0 there exist t_1 and $t_2 \in [t_0, t_0 + \frac{2\pi}{fv_{\min}}]$ such that $\mathcal{G}(t_1)$ and $\mathcal{G}(t_2)$ have respectively an edge between vertices i and $i+1$ and between vertices i and $i-1$. Therefore, the graph $\bigcup_{t \in [t_0, t_0 + \frac{2\pi}{f\bar{v}(0)}]} \mathcal{G}(t)$ contains the ring graph. □

5. Convergence analysis. In the first part of this section we prove that the nominal speeds v_i of all the beads will asymptotically be equal to the average of their initial values, and that the desired sweeping arc will asymptotically attain a length $2\pi/N$. In the second and third part of this section we show that SIS Algorithm enables the beads to reach balanced synchrony if n is even and unbalanced synchrony if n is odd. We begin our convergence analysis with a useful result that combines known facts from [7, 20, 12]. Given a symmetric stochastic matrix $F \in \mathbb{R}^{N \times N}$, its *associated graph* has node set $\{1, \dots, N\}$ and edge set defined as follows: (i,j) is an edge if and only if $F_{ij} > 0$.

Theorem 5.1 (Average Consensus Dynamics). *Consider a sequence $\{F(\ell) \mid \ell \in \mathbb{Z}_{\geq 0}\} \subset \mathbb{R}^{N \times N}$ of symmetric stochastic matrices and the dynamical system*

$$x(\ell+1) = F(\ell)x(\ell).$$

Let $G(\ell)$ be the graph associated with $F(\ell)$. Assume that

(A1) $G(\ell)$ has a self loop at each node,

(A2) Each non-zero edge weight $F_{ij}(\ell)$, including the self-loops weights $F_{ii}(\ell)$, is larger than a constant $\alpha > 0$, and

(A3) The graph $\cup_{\tau \geq \ell} G(\tau)$ is connected for all $\ell \in \mathbb{Z}_{\geq 0}$.

Then the system is said to achieve average consensus with

$$\lim_{\ell \to +\infty} x(\ell) = \Big(\frac{1}{N} \sum_{i=1}^{N} x_i(0) \Big) \mathbf{1}_N.$$

We also define some terminology associated with movement of the beads on the ring. Let the k^{th} impact between beads i and $i+1$ occur at the instant I_i^k. Let $I^k = [I_1^k, \dots, I_n^k]^T \in \mathbb{R}^n$. Let us also define the k^{th} *passage time* P_i^k as the instant at which bead i passes by the center of its desired sweeping arc after its k^{th} but before its $(k+1)^{\text{th}}$ impact. Let $P^k = [P_1^k, \dots, P_n^k]^T \in \mathbb{R}^n$.

5.1. Convergence of nominal speed and desired sweeping arc. We start by proving that all nominal speeds v_i converge to being equal to the average of their initial values.

Lemma 5.2 (Speed convergence). *Let $v(t) = [v_1(t), \dots, v_n(t)]^T \in \mathbb{R}^n$. Along the trajectories of the* SIS Algorithm, *with $\{(\theta_i(0), x_i(0))\}_{i \in \{1,\dots,n\}} \in \mathcal{A}_{0-\text{bal}} \bigcup \mathcal{A}_{D-\text{unbal}}$:*

$$\lim_{t \to +\infty} v(t) = \frac{\mathbf{1}_n^T v(0)}{n} \mathbf{1}_n.$$

Proof. For all $i \in \{1, \dots, n\}$, define $A_i \in \mathbb{R}^{n\times n}$ by:

$$[A_i]_{lm} = \begin{cases} \frac{1}{2}, & \text{if } (l,m) \in \{(i,i),(i,i+1),(i+1,i),(i+1,i+1)\}, \\ 1, & \text{if } l = m \text{ and } l \notin \{i, i+1\}, \\ 0, & \text{otherwise}. \end{cases}$$

Because of equation (3.1),

$$v(I_i^k + \varepsilon) = A_i v(I_i^k).$$

where $I_i^k + \varepsilon$ is the time instant just after the impact. This can be extended to account for more than one two-way impacts taking place at the same instant. For example, if two separate impacts occur between beads i and $i+1$ as well as j and $j+1$ at I_i^k, then $v(I_i^k + \varepsilon) = A_i A_j v(I_i^k)$.

Therefore, the dynamics of $v(t)$ is the average consensus dynamics with matrices A_i. Proposition 4.2 ensures that the sequence of impact graphs at impact instants is uniformly jointly connected. Therefore, the assumptions of Theorem 5.1 are satisfied and we know that all velocities $v_i(t)$ converge to $\frac{1}{n}\sum_{i=1}^n v_i(0)$. ∎

We now prove that the desired sweeping arcs converge asymptotically to a stationary configuration in which all sweeping arcs have length $2\pi/n$.

LEMMA 5.3 (Convergence of desired sweeping arc). *Let $L_i(t) = \mathrm{dist}_{\mathtt{cc}}(\ell_i(t), u_i(t))$ be the length of the desired sweeping arc $\mathcal{D}_i$ for $i \in \{1, \dots, n\}$, and $L(t) = [L_1(t), \dots, L_n(t)]^T \in \mathbb{R}^n$. Along the trajectories of the* SIS ALGORITHM*, with $\{(\theta_i(0), x_i(0))\}_{i\in\{1,\dots,n\}} \in \mathcal{A}_{0-\mathrm{bal}} \bigcup \mathcal{A}_{D-\mathrm{unbal}}$, the arc lengths and the arcs converge, that is,*

$$\lim_{t\to+\infty} L(t) = \frac{2\pi}{n}\mathbf{1}_n,$$

and the limits $\lim_{t\to+\infty} \ell_i(t)$ and $\lim_{t\to+\infty} u_i(t)$ exist and are finite.

Proof. For all $i \in \{1, \dots, n\}$, define $B_i \in \mathbb{R}^{n\times n}$ by:

$$[B_i]_{lm} = \begin{cases} \frac{3}{4}, & \text{if } (l,m) \in \{(i,i),(i+1,i+1)\}, \\ \frac{1}{4}, & \text{if } (l,m) \in \{(i,i+1),(i+1,i)\}, \\ 1, & \text{if } l = m \notin \{i, i+1\}, \\ 0, & \text{otherwise}. \end{cases}$$

From equations (3.3) and (3.4), an impact between i and $i+1$ at time t causes

$$L_i(I_i^k + \varepsilon) = \frac{3}{4} L_i(I_i^k) + \frac{1}{4} L_{i+1}(I_i^k),$$
$$L_{i+1}(I_i^k + \varepsilon) = \frac{1}{4} L_i(I_i^k) + \frac{3}{4} L_{i+1}(I_i^k).$$

Therefore, if at time t an impact between i and $i+1$ occurs and no other impact occurs, then $L(I_i^k + \varepsilon) = B_i L(I_i^k)$. Analogously to the proof of Lemma 5.2, the dynamics of $L(t)$ is the average consensus dynamics with matrices B_i. Proposition 4.2 ensures that the sequence of impact graphs at impact instants is uniformly jointly connected. Therefore, the assumptions of Theorem 5.1 are satisfied and we know that all lengths $L_i(t)$ converge to $\frac{1}{n}\sum_{i=1}^n L_i(0) = \frac{2\pi}{n}$. To prove that the limits of the arc boundaries $\ell_i(t)$ and $u_i(t)$ exist and are finite, it suffices to notice that (i) at each impact the arc boundaries change by an amount proportional to the difference between arc lengths, and (ii) every exponentially decaying sequence is summable. ∎

5.2. Balanced synchrony. We now prove that the SIS ALGORITHM steers the collection of beads to be in balanced synchrony for a set of initial conditions contained in $\mathcal{A}_{0-\mathrm{bal}}$, under certain assumptions. Although convergence to balanced synchronization is proved only locally, simulations shown in Section 6 suggest that indeed the set of initial conditions for which the balanced synchronization is reached is quite large and may be equal to $\mathcal{A}_{0-\mathrm{bal}}$.

THEOREM 5.4 (Balanced synchrony convergence). *Consider an even number n of beads with an initial condition contained in $\mathcal{A}_{0-\mathrm{bal}}$ and executing the* SIS ALGORITHM. *Assume that*

- *(A4) The desired sweeping arcs for each agent are already the desired steady-state regions of equal length $2\pi/n$ and the nominal velocity of each agent has the same value $\overline{v}$. Since the* SIS ALGORITHM *makes sweeping regions and nominal velocities reach these common values for any initial condition in $\mathcal{A}_{0-\mathrm{bal}}$, we can do this without loss of generality.*
- *(A5) $d_{2i}(0) = -d_{2i-1}(0)$ for $i \in \{1, \dots, n/2\}$, i.e., consecutive beads move in opposite directions.*
- *(A6) The initial condition satisfies $|P_i^1 - P_j^1| \leq \delta_{pb}$ for $i,\ j \in \{1, \dots, n\}$, where $\delta_{pb} = \frac{\pi}{n\overline{v}}\left(\ \frac{1+f}{f}\right)$.*

Then

$$\lim_{k\to+\infty} P^k = \frac{\mathbf{1}_n^T P^k}{n}\mathbf{1}_n.$$

Proof. Let us suppose that at time t the beads i and $i+1$, with directions $d_i(t) = -d_{i+1}(t) = +1$, are about to collide after their k^{th} impact. According to Assumption (A4), they also have sweeping arcs that have converged and same nominal velocity $\bar{v}$. Let us assume, without any loss of generality, that the impact between beads i and $i+1$ occurs in $\mathcal{D}_{i+1}$ and precisely at $u_i + \Delta_1$ as shown in Figure 5.1.

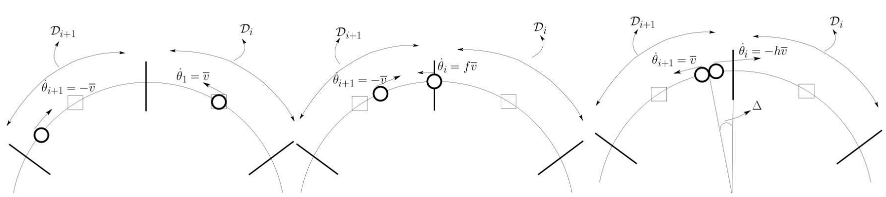

FIG. 5.1. *This figure shows how the speeds of bead i and $i+1$ change while they are traveling towards each other. Note that bead i is early with respect to bead $i+1$.*

The time instant at which beads i and $i+1$ reach the point $u_i+\Delta_1$ simultaneously is:

$$P_i^k + \frac{\pi}{n}\frac{1}{\overline{v}} + \frac{\Delta_1}{f\overline{v}} = P_{i+1}^k + \frac{\pi}{n}\frac{1}{\overline{v}} - \frac{\Delta_1}{\overline{v}}\,. \tag{5.1}$$

Solving (5.1) for Δ_1 we have:

$$\Delta_1 = \overline{v}\frac{f}{1+f}(P_{i+1}^k - P_i^k)\,. \tag{5.2}$$

According to Assumption (A5), beads $i-2$ and $i-1$ are also either going to or have already collided with each other. Let us assume that the impact between them occurs in $\mathcal{D}_{i-1}$ and precisely at $u_{i-2}+\Delta_2$. Following a similar analysis as done for obtaining Δ_1, one can conclude that

$$\Delta_2 = \bar{v}\frac{f}{1+f}(P_{i-1}^k - P_{i-2}^k)\,. \tag{5.3}$$

After the impact between beads i and $i+1$, the directions of both beads change because the impact is of head-to-head type, and they both head towards C_i and C_{i+1}, which they would reach at time P_i^{k+1} and P_{i+1}^{k+1} respectively. In order for the variable P_i^{k+1} to be defined, the bead i should reach the center of its sweeping arc C_i before bead $i-1$ does, after its own k^{th} impact. For this to hold true, the time taken for the former event should be smaller than or equal to the time take for the later event:

$$P_i^k + \frac{2\pi}{n}\frac{1}{\bar{v}} + \frac{\Delta_1}{\bar{v}}\left(\frac{1}{f}+\frac{1}{h}\right) \le P_{i-1}^k + \frac{2}{\bar{v}}\left(\frac{\pi}{n}-\Delta_2\right) + \frac{\pi}{n}\frac{1}{\bar{v}}\left(\frac{1+f}{f}\right) \tag{5.4}$$

Using (5.2) and (5.3) and simplifying,

$$\left(\frac{1-f}{1+f}\right)(P_i^k - P_{i-1}^k) + \left(\frac{2f}{1+f}\right)(P_{i+1}^k - P_{i-2}^k) \le \frac{\pi}{n\bar{v}}\left(\frac{f+1}{f}\right)$$

should hold for P_i^{k+1} to be defined. This is the case, based on Assumption (A6) and the fact that the dynamics of the passage times is average consensus, as will be proved later. The same analysis can be carried out to prove that P_{i+1}^{k+1} is also well-defined. The choice of the impact locations $u_i+\Delta_1$ and $u_{i-2}+\Delta_2$ also accounts for the worst case scenario. Calculating P_i^{k+1} and P_{i+1}^{k+1}:

$$\begin{aligned} P_i^{k+1} &= P_i^k + \frac{2\pi}{n}\frac{1}{\bar{v}} + \frac{\Delta_1}{\bar{v}}\left(\frac{1}{f}+\frac{1}{h}\right), \\ P_{i+1}^{k+1} &= P_{i+1}^k + 2\left(\frac{\pi}{n}-\Delta_1\right)\frac{1}{\bar{v}}\,. \end{aligned}$$

Simplifying:

$$\begin{aligned} P_i^{k+1} &= \frac{1-f}{1+f}P_i^k + \frac{2f}{1+f}P_{i+1}^k + \frac{2\pi}{n\bar{v}}, \\ P_{i+1}^{k+1} &= \frac{2f}{1+f}P_i^k + \frac{1-f}{1+f}P_{i+1}^k + \frac{2\pi}{n\bar{v}}. \end{aligned}$$

Note that $0 < \frac{1-f}{1+f} < 1/3$ and $2/3 < \frac{2f}{1+f} < 1$ since $f \in]0.5, 1[$. Now, let us define the matrices C_{even} and $C_{\text{odd}} \in \mathbb{R}^{n\times n}$ by:

$$\begin{aligned} [C_{\text{even}}]_{lm} &= \begin{cases} \frac{1-f}{1+f}, & \text{if } (l,m) \in \{(i,i+1)\}, \\ \frac{2f}{1+f}, & \text{if } (l,m) \in \{(i,i+2),(j,j)\},\ i \text{ odd}, j \text{ even} \end{cases} \\ [C_{\text{odd}}]_{lm} &= \begin{cases} \frac{1-f}{1+f}, & \text{if } l=m, \\ \frac{2f}{1+f}, & \text{if } (l,m) \in \{(i,i+1),(i+1,i)\},\ i \text{ odd}. \end{cases} \end{aligned}$$

Once again, we use the identification $n+1 \equiv 1$ while working with indices i and j. If the first impact after $t=0$ is between i and $i+1$, and i is even, then the vector P^k evolves as follows:

$$P^{k+1} = \begin{cases} C_{\text{odd}}P^k + \frac{2\pi}{n\bar{v}}\mathbf{1}_n, & \text{if } k \text{ odd}, \\ C_{\text{even}}P^k + \frac{2\pi}{n\bar{v}}\mathbf{1}_n, & \text{if } k \text{ even}. \end{cases} \tag{5.5}$$

If the first impact is between i and $i+1$, and i is odd, then equation (5.5) is still valid as long as the definitions of C_{odd} and C_{even} are exchanged. In any case, the dynamics of the passage times is just the average consensus dynamics with matrices C_{odd} and C_{even}. Therefore, it can be easily proved that $\lim_{k\to+\infty} P^k = \frac{\mathbf{1}_n^T P^k}{n}\mathbf{1}_n$. Further, $\|P^k - \frac{\mathbf{1}_n^T P^k}{n}\mathbf{1}_n\|_2 \geq \|P^{k+1} - \frac{\mathbf{1}_n^T P^{k+1}}{n}\mathbf{1}_n\|_2$ and $\delta \geq \max_{i\{1,\dots,n\}} |P_i^k - P_j^k| \geq \max_{i\{1,\dots,n\}} |P_i^{k+1} - P_j^{k+1}|$. In other words, if the initial conditions of the collection of beads are close to the periodic orbit, i.e., satisfy Assumption (A6), then the resulting trajectory remains close to the periodic orbit. Furthermore, because of Proposition 4.2 and Theorem 5.1, the balanced synchrony, i.e., the consensus, is asymptotically reached. ∎

5.3. Unbalanced synchrony. We now prove that the SIS Algorithm steers the collection of beads to be in unbalanced synchrony for a set of initial conditions contained entirely in $\mathcal{A}_{D-\text{unbal}}$ with $D = \pm 1$. We first start by proving that there exists an orbit along which the beads can reach unbalanced synchrony.

Theorem 5.5 (Existence of periodic orbit for 1-unbalanced collections: sufficiency). *Given $D \in \{-1,+1\}$, assume that $\{(\theta_i(0), x_i(0))\}_{i\in\{1,\dots,n\}} \in \mathcal{A}_{D-\text{unbal}}$, $\frac{1}{2} < f < \frac{n}{1+n}$, and that, for $i \in \{1,\dots,n\}$, $v_i(t) = v_i(0) = \overline{v}$, $\ell_i(t) = \ell_i(0)$, $u_i(t) = u_i(0)$ with $\ell_i(0) = u_{i-1}(0)$ and with $\operatorname{dist}_{\texttt{cc}}(\ell_i(0), \ell_{i+1}(0)) = \frac{2\pi}{n}$. Then*
- *(i) there exists a periodic orbit for the* SIS Algorithm *in which the beads are in unbalanced synchrony with period $2\frac{2\pi}{n}\frac{1}{\overline{v}}$; and*
- *(ii) along this orbit each bead i impacts its neighboring bead $i-1$ at position $\ell_i(0) + D\delta$, where $\delta = \frac{2\pi}{n^2}\frac{f}{1-f} < \frac{2\pi}{n}$.*

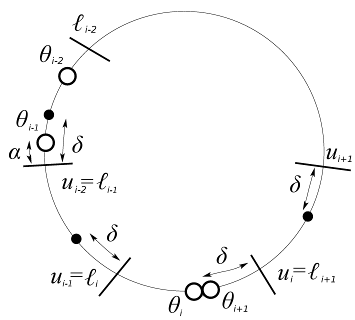

Fig. 5.2. *This figure shows the periodic orbit described in Theorem 5.5. The white circles are the positions of beads. The black dots are the locations of the impacts for any two neighboring beads. Note that bead $i-1$ and $i-2$ are moving towards each other and so are beads i and $i+1$.*

Remark 1 (Impacts order in 1-unbalanced synchrony). *It is useful to take note of the order in which the impacts happen in a D-unbalanced collection of beads that reach unbalanced synchrony, where $D \in \{-1,+1\}$. As we will see in the proof of Theorem 5.5, if $\sum_{i=1}^n d_i(0) = -1$ and i and $i+1$ have just met, then the next impact will be between $i-1$ and $i-2$ and so on until i meets $i+1$ again and the periodic orbit is complete. More concisely, if the first two beads to impact are i and $i+1$, then the k^{th} impact happens between $(i-3Dk) \mod n$ and $(i+1-3Dk) \mod n$. Therefore, if $\sum_{i=1}^n d_i(0) = -1$, then the impacts happen in a counterclockwise fashion; on the other hand, if $\sum_{i=1}^n d_i(0) = +1$, then the impacts happen in a clockwise fashion. Let us illustrate the idea using a the graph $\mathcal{G}(t)$ introduced in Proposition 4.2. We recall that the graph $\mathcal{G}(t)$ has as vertex set $\{1,\dots,n\}$ and edge from i to $i+1$ if and only if* *the beads i and $i+1$ collide at time t. Figure 5.3 shows $\mathcal{G}(t)$ for $t \in [I_{1,2}, I_{1,2} + 2\frac{2\pi}{n}\frac{1}{\overline{v}}]$ and the time at which the impacts happen for $n = 5$.* •

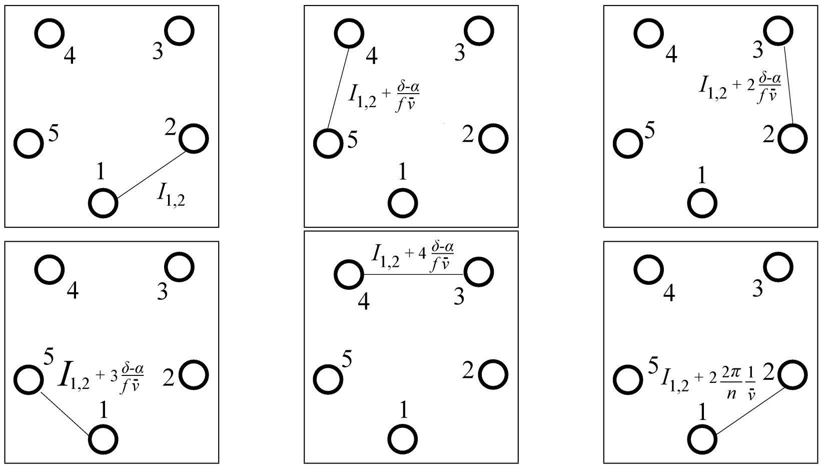

FIG. 5.3. *This figure illustrates $\mathcal{G}(t)$ for $t \in [I_{1,2}, I_{1,2} + 2\frac{2\pi}{n}\frac{1}{\overline{v}}]$ and the time at which each edge appears for $n = 5$ and $\sum_{i=1}^{n} d_i(0) = -1$ when unbalanced synchrony is reached.*

Proof. [of Theorem 5.5] We now prove the theorem by constructing the periodic orbit. Without loss of generality let us suppose that $\sum_{i=1}^{n} d_i(0) = -1$. Let I_i^1 be the time at which bead i and bead $i+1$ impact at $u_i(0) - \delta \equiv \ell_{i+1}(0) - \delta$. Let us suppose that $\theta_{i-1}(I_i^1) = \ell_{i-1}(0) - \alpha$ and that $\theta_{i-2}(I_i^1)$ is such that:

$$I_{i-2}^1 = I_{i-1}^1 + \frac{\delta - \alpha}{f\overline{v}}, \tag{5.6}$$

with $\delta < \frac{2\pi}{n}$ and $\alpha < \delta$ (see Figure 5.2). Recalling (5.6) and by symmetry we have:

$$I_2^1 = I_1^1 + \frac{n-1}{2}\frac{\delta - \alpha}{f\overline{v}}, \tag{5.7}$$

$$I_n^1 = I_1^1 + \frac{n+1}{2}\frac{\delta - \alpha}{f\overline{v}}. \tag{5.8}$$

For beads 1 and 2 to meet again at $u_1(0) - \delta \equiv \ell_2(0) - \delta$, the following must hold:

$$I_2^1 + \left(\frac{2\pi}{n} - \delta\right)\frac{1}{\overline{v}} + \frac{\delta}{f\overline{v}} = I_n^1 + \frac{\delta}{h\overline{v}} + \left(\frac{2\pi}{n} - \delta\right)\frac{1}{\overline{v}}. \tag{5.9}$$

In fact, after impacting with bead 3, bead 2 travels along the arc $\mathrm{Arc}(\ell_2(0), u_2(0) - \delta)$ with velocity $-\overline{v}$ since it is in its desired sweeping arc. After crossing $\ell_2(0)$, the speed of bead 2 becomes $-f\overline{v}$ because it is moving away from its arc. For bead 1 the dual is true. After impacting with bead n, bead 1 travels along the arc $\mathrm{Arc}(\ell_1(0) - \delta, \ell_1(0))$ with speed $h\overline{v}$ since it is moving towards its desired sweeping arc. After crossing $\ell_1(0)$, the speed of bead 1 becomes $\overline{v}$ because it is in its arc (see Figure 5.4).

Recalling (5.7) and (5.8), we have:

$$I_1^1 + \frac{n-1}{2}\frac{\delta - \alpha}{f\overline{v}} + \left(\frac{2\pi}{n} - \delta\right)\frac{1}{\overline{v}} + \frac{\delta}{f\overline{v}} = I_1^1 + \frac{n+1}{2}\frac{\delta - \alpha}{f\overline{v}} + \frac{\delta}{h\overline{v}} + \left(\frac{2\pi}{n} - \delta\right)\frac{1}{\overline{v}}.$$

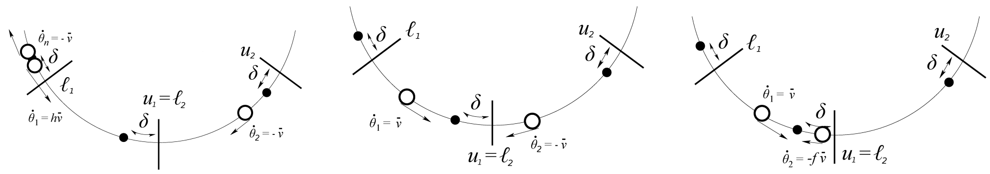

FIG. 5.4. *This figure shows how the speeds of bead* 1 *and* 2 *change as they are traveling towards each other, shortly after bead* 1 *meets bead* n*.*

Rearranging all the terms and solving for α:

$$\alpha = \delta(2f - 1). \tag{5.10}$$

In order to be a periodic orbit we need to impose that beads 1 and 2 meet again after a period:

$$I_1^1 + \frac{n-1}{2}\frac{\delta - \alpha}{f\overline{v}} + \left(\frac{2\pi}{n} - \delta\right)\frac{1}{\overline{v}} + \frac{\delta}{f\overline{v}} = I_1^1 + 2\frac{2\pi}{n}\frac{1}{\overline{v}}. \tag{5.11}$$

Substituting (5.10) in (5.11) and solving for δ, we have:

$$\delta = \frac{2\pi}{n^2}\frac{f}{1-f}.$$

Recalling the assumption of f we have:

$$f < \frac{n}{1+n} \qquad \Longrightarrow \qquad \delta = \frac{2\pi}{n^2}\frac{f}{1-f} < \frac{2\pi}{n}.$$

□

It turns out that $f < \frac{n}{1+n}$ is not only sufficient but also necessary for the existence of the periodic orbit described in part (ii) of Theorem 5.5.

THEOREM 5.6 (Existence of periodic orbit for 1-unbalanced collections: necessity). *Given* $D \in \{-1, +1\}$*, assume that* $\{(\theta_i(0), x_i(0))\}_{i\in\{1,\dots,n\}} \in \mathcal{A}_{D-\mathrm{unbal}}$*, and that, for* $i \in \{1, \dots, N\}$*,* $v_i(t) = v_i(0) = \overline{v}$*,* $\ell_i(t) = \ell_i(0)$*,* $u_i(t) = u_i(0)$ *with* $\ell_i(0) = u_{i-1}(0)$ *and with* $\mathrm{dist}_{\texttt{cc}}(\ell_i(0), \ell_{i+1}(0)) = \frac{2\pi}{N}$*. If along the trajectories of the* SIS ALGORITHM *the unbalanced synchrony is reached, that is, beads* i *and* $i-1$ *always meet at* $\ell_i(t) + D\delta$ *with* $\delta < \frac{2\pi}{n}$ *and the period of the orbit is* $2\frac{2\pi}{n}\frac{1}{\overline{v}}$*, then* $f < \frac{n}{1+n}$*.*

Proof. Let us assume, with no loss of generality, that $\sum_{i=1}^{n} d_i(0) = -1$. Let t^+ be the time spent by each bead traveling along the positive direction, and t^- be the time spent by each bead traveling along the negative direction in a period of the periodic orbit. In other words, if $\delta < \frac{2\pi}{n}$, then $t^- = (\frac{2\pi}{n} - \delta)\frac{1}{\overline{v}} + \frac{\delta}{f\overline{v}}$, and $t^+ = \frac{\delta}{h\overline{v}} + (\frac{2\pi}{n} - \delta)\frac{1}{\overline{v}}$, as in (5.9). Clearly $t^- + t^+ = 2\frac{2\pi}{n}\frac{1}{\overline{v}}$, which is the period of the orbit, and $t^- > t^+$, that is each bead spends more time traveling along the negative direction than along the positive. At every instant of time only one bead is unbalanced and $t^- - t^+$ is the time each bead is unbalanced during a period. By symmetry we can then conclude that $n(t^- - t^+)$ must be equal to a period:

$$2\frac{2\pi}{N}\frac{1}{\overline{v}} = n(t^- - t^+)\,. \tag{5.12}$$

Recalling the expressions for t^- and t^+, we have:

$$2\frac{2\pi}{n}\frac{1}{\overline{v}} = n2\frac{\delta}{\overline{v}}\frac{f}{1-f}\,,$$

and solving for δ

$$\delta = \frac{2\pi}{n^2}\frac{f}{1-f}\,.$$

By assumption $\delta < \frac{2\pi}{n}$, therefore:

$$\delta = \frac{2\pi}{n^2}\frac{f}{1-f} < \frac{2\pi}{n} \qquad \Longrightarrow \qquad f < \frac{n}{1+n}.$$

□

A natural question to ask is if there exists a periodic orbit for the SIS ALGORITHM when $\{(\theta_i(0), x_i(0))\}_{i\in\{1,\ldots,N\}} \in \mathcal{A}_{D-\mathrm{unbal}}$ and $|D| > 1$. To answer this question, we extend the result of Theorem 5.6 to the more general case of D-unbalanced collections of beads.

THEOREM 5.7 (Existence of a periodic orbit: necessity). *Let* $\{(\theta_i(0), x_i(0))\}_{i\in\{1,\ldots,n\}} \in \mathcal{A}_{D-\mathrm{unbal}}$ *and* $|D| > 1$*. If along the trajectories of the* SIS ALGORITHM *the unbalanced synchrony is reached and bead* i *meets bead* $i-1$ *at location* $\ell_i(t) + \frac{D}{|D|}\delta$ *with* $\delta < \frac{2\pi}{n}$*, then* $f < \frac{n/|D|}{1+n/|D|}$*.*

Proof. The proof parallels the one of Theorem 5.6. Without loss of generality let us assume $\sum_{i=1}^{n} d_i(t) = D < -1$. At every instant of time $|D|$ beads are unbalanced and $t^- - t^+$ is the time each bead is unbalanced during a periodic orbit. By symmetry we can then conclude that $n\frac{(t^- - t^+)}{|D|}$ must be equal to a period, therefore equation (5.12) becomes:

$$2\frac{2\pi}{n}\frac{1}{\bar{v}} = n\frac{(t^- - t^+)}{|D|}\,,$$

where $t^- - t^+ = 2\frac{\delta}{\bar{v}}\frac{f}{1-f}$. Solving for δ we have:

$$\delta = |D|\frac{2\pi}{n^2}\frac{f}{1-f}\,.$$

Imposing the constraint $\delta < \frac{2\pi}{n}$ we can calculate the necessary condition for the existence of the periodic orbit in a D-unbalanced collection of beads:

$$f < \frac{n/|D|}{1+n/|D|}\,.$$

Note that the higher the ratio $|D|/n$ is, the smaller f needs to be so that each bead spends enough time outside of its desired sweeping arc $\mathrm{Arc}(\ell_i(t), u_i(t))$ but it does not get too far from it. □

We now prove that the SIS ALGORITHM steers the collection of beads to be in unbalanced synchrony for a set of initial conditions contained in $\mathcal{A}_{D-\mathrm{unbal}}$, under certain assumptions.

In particular we prove that the interval between two consecutive times each bead passes by a point while moving in the same direction asymptotically approaches $2\frac{2\pi}{n}\frac{1}{\bar{v}}$, which is the period of the periodic orbit. This is just a consequence of the definition of unbalanced synchrony.

THEOREM 5.8 (1-unbalanced synchrony convergence). *Consider* n *beads executing the* SIS ALGORITHM*, with* n *being odd. Let* $\delta = \frac{2\pi}{n^2}\frac{f}{1-f} < \frac{2\pi}{n}$*, and* $\tilde{C}_i$ *be the center of the counterclockwise arc* $\mathrm{Arc}(\ell_i(0) + D\delta, u_i(0) + D\delta)$ *for all* $i \in \{1,\ldots,n\}$*. Further, assume that*

(A7) The desired sweeping arcs for each agent are already the desired steady-state regions of equal length $2\pi/n$ and the nominal velocity of each agent has the same value $\overline{v}$. Since the initial condition is in $\mathcal{A}_{D-\mathrm{unbal}}$, we can do this without loss of generality.

(A8) $D \in \{-1, +1\}$

(A9) The initial condition is such that $|P_i^1 - P_j^1| \leq \delta_{pub}$ for $i, j \in \{1, \dots, n\}$ where $\delta_{pub} = \frac{1}{\overline{v}}(\ \delta + \frac{\pi}{n})\ (\ \frac{1-f}{f})\ .$

Then, along the trajectories of the SIS Algorithm*:*

$$\lim_{k \to +\infty} P^{2k} - P^{2(k-1)} = \mathbf{1}_n \frac{2}{\overline{v}} \frac{2\pi}{n} ,$$

that is, the collection of beads asymptotically reaches unbalanced synchrony.

Proof. *Case (i)* Let us suppose $\delta < \frac{\pi}{n}$, and $\sum_{i=1}^n d_i(0) = -1$. According to Assumption (A7), the beads have sweeping arcs which have converged and same nominal velocity $\bar{v}$. Let us suppose that bead $i-1$ and bead i are moving towards each other and let P_{i-1}^k and P_i^k be the last time they passed by $\tilde{C}_{i-1}$ and $\tilde{C}_i$ with directions $d_{i-1} = +1$ and $d_i = -1$. If the two beads are not in unbalanced sync, they will not meet at $u_{i-1} - \delta$ but at $u_{i-1} - \delta - \Delta$, as shown in Figure 5.5.

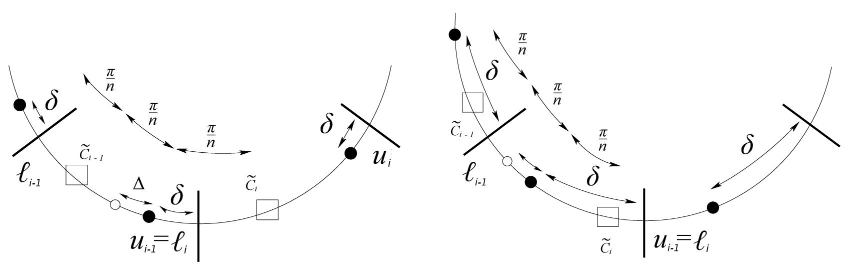

Fig. 5.5. *From top to bottom, the figure illustrates the position of $\tilde{C}_{i-1}$, $\tilde{C}_i$, and of $u_{i-1} - \delta - \Delta$ for $\delta < \frac{\pi}{n}$ and $\delta > \frac{\pi}{n}$.*

In order to calculate where and when the beads impact we need to impose that i and $i-1$ reach simultaneously $u_{i-1} - \delta - \Delta$:

$$P_{i-1}^k + (\frac{\pi}{n} - \Delta)\frac{1}{\overline{v}} = P_i^k + (\frac{\pi}{n} - \delta)\frac{1}{\overline{v}} + \frac{(\delta + \Delta)}{f\overline{v}} .$$

Note that the speeds of the beads are decided based on their location with respect to the sweeping arcs. According to Assumption (A8), these are shifted by an amount δ from the desired sweeping arcs $\mathcal{D}_i$ defined earlier. The direction of shift is determined by the sign of D. Solving for Δ we have:

$$\Delta = \frac{-f}{f+1}\overline{v}(P_i^k - P_{i-1}^k) + \frac{f-1}{f+1}\delta. \tag{5.13}$$

Note that requiring i and $i-1$ to be in unbalanced sync is equivalent to imposing $\Delta = 0$ which implies $P_i^k - P_{i-1}^k = \frac{f-1}{f}\frac{\delta}{\overline{v}}$. After impacting at $u_{i-1} - \delta - \Delta$, beads $i-1$

and i change directions and head back towards $\tilde{C}_{i-1}$ and $\tilde{C}_i$, that they will reach at time P_{i-1}^{k+1} and P_i^{k+1}:

$$\begin{aligned} P_{i-1}^{k+1} &= P_{i-1}^{k} + 2(\frac{\pi}{n} - \Delta)\frac{1}{\bar{v}}\,, \\ P_{i}^{k+1} &= P_{i}^{k} + 2(\frac{\pi}{n} + \Delta)\frac{1}{\bar{v}}\,. \end{aligned}$$

Recalling equation (5.13) and rearranging the terms we have:

$$\begin{bmatrix} P_{i-1}^{k+1} \\ P_i^{k+1} \end{bmatrix} = M \begin{bmatrix} P_{i-1}^{k} \\ P_i^{k} \end{bmatrix} + \frac{2\delta}{\bar{v}}\frac{1-f}{f}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + \frac{1}{\bar{v}}\frac{2\pi}{n}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\,,$$

where

$$M = \begin{bmatrix} 1 - \frac{2f}{f+1} & \frac{2f}{f+1} \\ \frac{2f}{f+1} & 1 - \frac{2f}{f+1} \end{bmatrix}\,. \tag{5.14}$$

Note that the dynamics matrix M is doubly stochastic since $f \in \,]0.5, 1[$.

Before proceeding, we note that for P_i^{k+1} to be defined, we must impose that bead i reaches $\tilde{C}_i$ before bead $i+1$ does, i.e.

$$P_i^{k+1} \leq P_{i+1}^{k} + (\ \frac{\pi}{n} + \delta)\ \frac{1}{\bar{v}} + (\ \frac{\pi}{n} - \delta)\ \frac{1}{f\bar{v}}$$

The term on the left hand side of this inequality is the time required for $i+1$ to reach $\tilde{C}_i$ after its k^{th} impact. This inequality can be simplified further:

$$(\ \frac{2f}{f+1})\ P_{i-1}^{k} + (\ \frac{1-f}{1+f})\ P_i^{k} - P_{i+1}^{k} \leq \frac{1}{\bar{v}}(\ \delta + \frac{\pi}{n})\ (\ \frac{1-f}{f})$$

This is true according to Assumption (A9), and the convergence properties of the passage times which will be proved later.

Returning back to the dynamics of the passage times, any time an impact between $i-1$ and i occurs, if Assumption (A9) is satisfied, the beads pass again by the centers of their cells at:

$$\begin{bmatrix} P_1^k \\ \vdots \\ P_{i-1}^{k+1} \\ P_i^{k+1} \\ \vdots \\ P_n^k \end{bmatrix} = E_{i-1}\begin{bmatrix} P_1^k \\ \vdots \\ P_{i-1}^{k} \\ P_i^{k} \\ \vdots \\ P_n^k \end{bmatrix} + \frac{2\delta}{\bar{v}}\frac{1-f}{f}u_{i-1} + \frac{1}{\bar{v}}\frac{2\pi}{n}w_{i-1}\,,$$

where

$$E_{i-1} = \begin{bmatrix} 1 & 0 & \dots & & & 0 \\ 0 & \ddots & & & & 0 \\ \vdots & & M_{11} & M_{12} & & \vdots \\ \vdots & & M_{21} & M_{22} & & \vdots \\ & & & & \ddots & \\ 0 & 0 & \dots & & & 1 \end{bmatrix}, \quad u_{i-1} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ -1 \\ \vdots \\ 0 \end{bmatrix}, \quad w_{i-1} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 0 \end{bmatrix},$$

and M_{ij} are the entries of the matrix M defined in equation (5.14). After any bead has met both its two neighbors once, the vector P^{k+2} can be calculated in closed form:

$$P^{k+2} = \tilde{E}P^k + \frac{2\delta}{\overline{v}}\frac{1-f}{f}\tilde{U} + \frac{1}{\overline{v}}\frac{2\pi}{n}\tilde{W}, \tag{5.15}$$

where $\tilde{E} = \prod_{m=1}^{n} E_{j_m}$, $j_m \in \{1,\dots,n\}$ (the value of j_m depends on the order of the impacts), $\tilde{U} = \sum_{r=1}^{n}\left(\prod_{m=1+r}^{n} E_{j_m}\right) u_{j_r}$, and $\tilde{W} = \sum_{r=1}^{n}\left(\prod_{m=1+r}^{n} E_{j_m}\right) w_{j_r}$.

For all $k \in \mathbb{N}$ the dynamics matrix $\tilde{E}$ is actually constant because by assumption the order of the impacts is just like in Figure 5.3. Since the dynamics (5.15) is time invariant we can write the trajectory in closed-form:

$$P^{2k+1} = \tilde{E}^k P^1 + \left(\sum_{j=1}^{k-1}\tilde{E}^j\right)\left(\frac{2\delta}{\overline{v}}\frac{1-f}{f}\tilde{U} + \frac{1}{\overline{v}}\frac{2\pi}{n}\tilde{W}\right).$$

We can then calculate:

$$P^{2k+1} - P^{2(k-1)+1} = (\tilde{E}^k - \tilde{E}^{k-1})P^1 + \tilde{E}^{k-1}\left(\frac{2\delta}{\overline{v}}\frac{1-f}{f}\tilde{U} + \frac{1}{\overline{v}}\frac{2\pi}{n}\tilde{W}\right).$$

Now, note that:

$$P^{2(k+1)+1} - P^{2k+1} = (\tilde{E}^{k+1} - \tilde{E}^{k})P^1 + \tilde{E}^{k}\left(\frac{2\delta}{\overline{v}}\frac{1-f}{f}\tilde{U} + \frac{1}{\overline{v}}\frac{2\pi}{n}\tilde{W}\right),$$

therefore we can write:

$$P^{2(k+1)+1} - P^{2k+1} = \tilde{E}(P^{2k+1} - P^{2(k-1)+1}).$$

Since $\tilde{E}$ is doubly stochastic, $\tilde{E}\mathbf{1}_n = \mathbf{1}_n$ and therefore:

$$P^{2(k+1)+1} - P^{2k+1} - \mathbf{1}_n\frac{2}{\overline{v}}\frac{2\pi}{n} = \tilde{E}\Big(P^{2k+1} - P^{2(k-1)+1} - \mathbf{1}_n\frac{2}{\overline{v}}\frac{2\pi}{n}\Big).$$

This implies that $\|(P^{2k+1} - P^{2(k-1)+1}) - \mathbf{1}_n\frac{2}{\overline{v}}\frac{2\pi}{n}\|_2 \geq \|(P^{2(k+1)+1} - P^{2k+1}) - \mathbf{1}_n\frac{2}{\overline{v}}\frac{2\pi}{n}\|_2$ and that $\max_{i\{1,\dots,n\}}|(P_i^{2k+1} - P_i^{2(k-1)+1}) - \mathbf{1}_n\frac{2}{\overline{v}}\frac{2\pi}{n}| \geq \max_{i\{1,\dots,n\}}|(P^{2(k+1)+1} - P^{2k+1}) - \mathbf{1}_n\frac{2}{\overline{v}}\frac{2\pi}{n}|$. Therefore, if the initial conditions of the collection of beads are close to the periodic orbit, then the resulting trajectory remains close to the periodic orbit. We now prove that the collection of beads asymptotically reaches unbalanced synchrony. Since $\tilde{E}$ is doubly stochastic and its associated graph is connected, $\lim_{k\to+\infty}\tilde{E}^k = \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}$ (see [12]), and therefore:

$$\begin{aligned}
\lim_{k\to+\infty} P^{2k+1} - P^{2(k-1)+1} &= \left(\frac{\mathbf{1}_n\mathbf{1}_n^T}{n} - \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\right)P^1 + \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\left(\frac{2\delta}{\overline{v}}\frac{1-f}{f}\tilde{U} + \frac{1}{\overline{v}}\frac{2\pi}{n}\tilde{W}\right)\\
&= \frac{\mathbf{1}_n\mathbf{1}_n^T}{n}\sum_{r=1}^{n}\left(\frac{2\delta}{\overline{v}}\frac{1-f}{f}\prod_{m=1+r}^{n} E_{j_m}u_{j_r} + \frac{1}{\overline{v}}\frac{2\pi}{n}\prod_{m=1+r}^{n} E_{j_m}w_{j_r}\right)\\
&= \frac{2\delta}{\overline{v}}\frac{1-f}{f}\sum_{r=1}^{n}\left(\frac{\mathbf{1}_n\mathbf{1}_n^T}{n}u_{j_r}\right) + \frac{1}{\overline{v}}\frac{2\pi}{n}\sum_{r=1}^{n}\left(\frac{\mathbf{1}_n\mathbf{1}_n^T}{n}w_{j_r}\right)\\
&= 0 + \frac{1}{\overline{v}}\frac{2\pi}{n}\sum_{r=1}^{n} 2\frac{\mathbf{1}_n}{n}\\
&= \frac{2}{\overline{v}}\frac{2\pi}{n}\mathbf{1}_n\,.
\end{aligned}$$

The third equality holds because $\mathbf{1}_n^T E_{j_m} = \mathbf{1}_n^T$ for all $j_m \in \{1, \ldots, n\}$ since E_{j_m} is doubly stochastic, while the fourth equality holds because $\mathbf{1}_n^T u_{j_r} = 0$ and $\mathbf{1}_n^T w_{j_r} = 2$ for all $j_r \in \{1, \ldots, n\}$.

Case (ii) Let us now suppose $\delta \geq \frac{\pi}{n}$. To calculate where beads $i-1$ and i will impact we need to solve (see Figure 5.5):

$$P_{i-1}^k + (\delta - \frac{\pi}{n})\frac{1}{h\overline{v}} + (\frac{2\pi}{n} - \delta - \Delta)\frac{1}{\overline{v}} = P_i^k + (\frac{\pi}{n} + \Delta)\frac{1}{f\overline{v}},$$

solving for Δ we have:

$$\Delta = \frac{-f}{f+1}\overline{v}(P_i^k - P_{i-1}^k) + \frac{f-1}{f+1}\delta, \tag{5.16}$$

just like for case (i). After impacting at $u_{i-1} - \delta - \Delta$ beads $i-1$ and i change directions and head back towards $\tilde{C}_{i-1}$ and $\tilde{C}_i$. We can now calculate P_{i-1}^{k+1} and P_i^{k+1}:

$$\begin{aligned} P_{i-1}^{k+1} &= P_i^k + 2(\frac{\pi}{n} + \Delta)\frac{1}{\overline{v}}, \\ P_i^{k+1} &= P_i^k + 2(\frac{\pi}{n} - \Delta)\frac{1}{\overline{v}}. \end{aligned}$$

The dynamics of P_{i-1} and P_i are just like in case (i), therefore the analysis and conclusion of case (i) are valid also for case (ii). □

6. Simulations. In this section we present numerical simulations obtained by implementing the SIS Algorithm on balanced and unbalanced collection of beads. Based on the simulations we formulate four conjectures.

6.1. Balanced collection of beads. As we have seen in Section 5.2, it can be proved that the SIS Algorithm allows the beads to get in sync if for all $i \in \{1, \ldots, N\}$, $v_i(0) = \overline{v} > 0$, $\mathrm{dist}_{\mathsf{cc}}(\ell_i(0), \ell_{i+1}(0)) = \frac{2\pi}{n}$, $\mathrm{dist}_{\mathsf{cc}}(\ell_i(0), u_i(0)) = \frac{2\pi}{n}$, and $d_i(0) = -d_j(0)$ for $j \in \{i-1, i+1\}$. Extensive simulations suggest that the basin of attraction of the periodic orbit is indeed much larger; we state this observation as a conjecture.

Conjecture 1 (Balanced collection: global basin of attraction). *Given initial conditions $\{(\theta_i(0), x_i(0))\}_{i \in \{1,\ldots,n\}} \in \mathcal{A}_{0-\mathrm{bal}}$, let P_i^k be the last instant at which bead i passed by the center of its desired sweeping arc before time t and let $P^k = [P_1^k, \ldots, P_n^k]^T \in \mathbb{R}^n$. Then, along the trajectories of the* SIS Algorithm*:*

$$\lim_{k \to +\infty} P^k = \frac{\mathbf{1}_n^T P^k}{n}\mathbf{1}_n.$$

In what follows we present the simulation results obtained by implementing the SIS Algorithm with $n = 8$ beads, when beads are randomly positioned on $\mathbb{S}^1$, $v_i(0)$ uniformly distributed in $]0, 1]$, $d_1(0) = d_2(0) = d_4(0) = d_6(0) = +1$ and $f = 0.7$.

Figure 6.1(a) shows the positions of the eight beads vs time. Consecutive beads do not move in opposite directions initially, as is assumed in Assumption (A5) for the validity of Theorem 5.4. They also do not possess same initial nominal speeds, as is necessary according to Assumption (A4). Since beads $i = 3, 4, 6$ and 7 impact neighboring beads even before they pass through the centers of their respective desired sweeping arcs after their first impacts, clearly Assumption (A6) is also not satisfied.

In spite of none of the assumptions being satisfied, each bead meets its neighbor at the same location on the circle asymptotically, reaching synchrony. The beads also attain the same nominal speed asymptotically. In Figure 6.1(b), the positions and the desired sweeping arc boundaries for bead $i = 5$ are illustrated. The solid line represents $\theta_5(t)$, the dash-dot line represents $\ell_5(t)$, and the thicker solid line represents $u_5(t)$. The distance $\mathrm{dist}_{\mathtt{cc}}(\ell_5(t), u_5(t))$ asymptotically approaches $360/N = 45$ degrees.

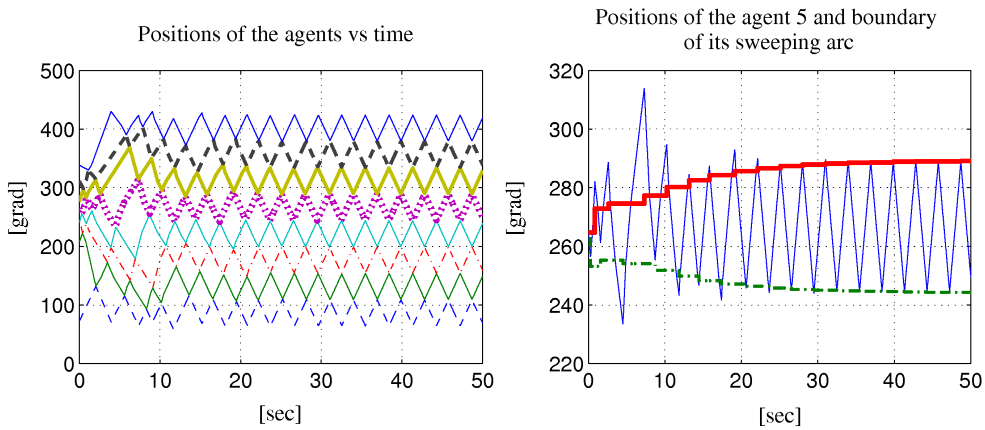

Fig. 6.1. *The* SIS Algorithm *is implemented with $n = 8$ beads, which are randomly positioned on $\mathbb{S}^1$, $v_i(0)$ is uniformly distributed in $]0, 1]$, $d_1(0) = d_2(0) = d_4(0) = d_6(0) = +1$, and $f = 0.7$. (a) shows positions of beads vs time. Beads $2, 4, 6, 8$ are represented by solid lines, while the dash line, dash-dot line, point line, and thicker dash line represent the positions of beads $1, 3, 5, 7$. (b) shows $\theta_5(t)$ (solid line), $u_5(t)$ (thicker solid line), and $\ell_5(t)$ (dash-dot line).*

6.2. Unbalanced collection of beads. In Theorem 5.8 we have proved that if $\{(\theta_i(0), x_i(0))\}_{i\in\{1,\dots,n\}} \in \mathcal{A}_{D-\mathrm{unbal}}$ with $D \in \{-1, +1\}$, and if the collection of beads is close to unbalanced synchrony, then the SIS Algorithm steers the collection to synchrony. Also in this case, extensive simulations suggest that the basin of attraction of the periodic orbit is larger.

Conjecture 2 (1-unbalanced collection: global basin of attraction). *Given initial conditions $\{(\theta_i(0), x_i(0))\}_{i\in\{1,\dots,n\}} \in \mathcal{A}_{D-\mathrm{unbal}}$ with $D \in \{-1, +1\}$, let $\delta = \frac{2\pi}{n^2}\frac{f}{1-f} < \frac{2\pi}{n}$, and let $\tilde{C}_i(t)$ be the center of the counterclockwise arc* $\mathrm{Arc}(\ell_i(t) + D\delta, u_i(t) + D\delta)$ *for all $i \in \{1, \dots, n\}$. Let P_i^k be the instant at which bead i passes by $\tilde{C}_i$ for the k^{th} time and let $P^k = [P_1^k, \dots, P_n^k]^T \in \mathbb{R}^n$. Then, along the trajectories of the* SIS Algorithm*:*

$$\lim_{k\to+\infty} P^{2k} - P^{2(k-1)} = \mathbf{1}_n \frac{2}{\bar{v}}\frac{2\pi}{n},$$

that is, the collection of beads asymptotically reaches unbalanced synchrony.

In what follows we present the simulation results obtained by implementing the SIS Algorithm with $n = 7$ beads, the beads are randomly positioned on $\mathbb{S}^1$, $v_i(0)$ uniformly distributed in $]0, 1]$, $d_1(0) = d_4(0) = d_5(0) = d_7(0) = -1$, that is the collection of beads is D-unbalanced with $D = -1$, and $f = 0.6$. Note that $f < \frac{n}{1+n} = \frac{7}{8}$. Figure 6.2(a) shows the positions of the seven beads vs time. Clearly, asymptotically each bead meets its neighbor at the same location on the circle, reaching synchrony. In Figure 6.2(b), the positions and the desired sweeping arc boundaries for bead $i = 3$

are illustrated. The solid line represents $\theta_3(t)$, the dash-dot line represents $\ell_3(t)$, and the thicker solid line represents $u_3(t)$. The distance $\mathrm{dist}_{\mathtt{cc}}(\ell_3(t), u_3(t))$ asymptotically approaches $360/n \approx 51.42$ degrees.

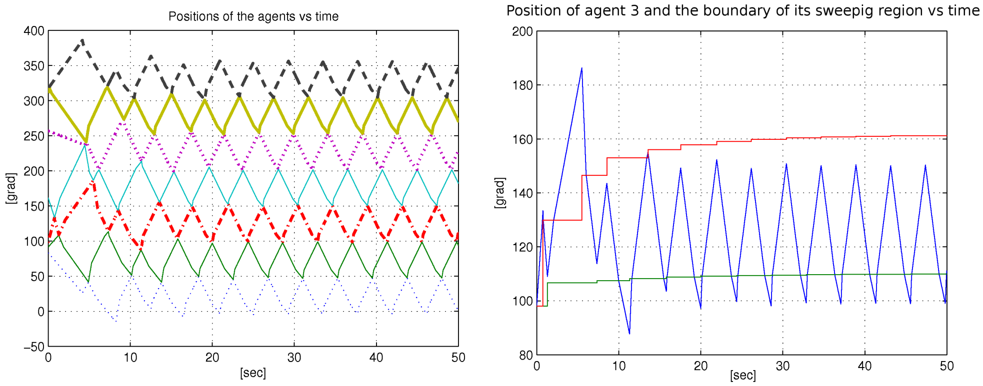

FIG. 6.2. *The* SIS ALGORITHM *is implemented for $n = 7$ beads. The beads are randomly positioned on $\mathbb{S}^1$, $v_i(0)$ is uniformly distributed in $]0,1]$, $d_1(0) = d_4(0) = d_5(0) = d_7(0) = -1$, and $f = 0.6$. (a) shows θ_i vs time. Beads $2, 4, 6$ are represented by solid lines, while the dash line, dash-dot line, point line, and thicker dash line represent the positions of beads $1, 3, 5, 7$. (b) shows $\theta_3(t)$ (solid line), $u_3(t)$ (thicker solid line), and $\ell_3(t)$ (dash-dot line).*

For the more general case of D-unbalanced collections with $n > |D| > 1$, Theorem 5.7 states that $f < \frac{n/|D|}{1+n/|D|}$ is just a necessary condition for the existence of a period orbit, along which, i and $i-1$ meet always at $\ell_i + \frac{D}{|D|}\delta$, with $\delta < \frac{2\pi}{n}$. We conjecture that (i) $f < \frac{n/|D|}{1+n/|D|}$ is also sufficient for the existence of a periodic orbit in the most general case of $|D| > 1$, and (ii) the SIS ALGORITHM steers the collection of D-unbalanced beads to synchrony.

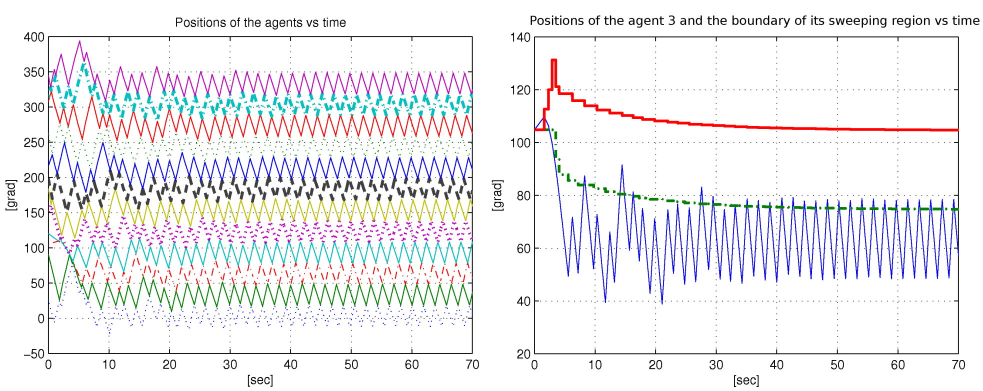

FIG. 6.3. *The* SIS ALGORITHM *is implemented for $n = 12$ beads. The beads are randomly positioned on $\mathbb{S}^1$, $v_i(0)$ is uniformly distributed in $]0,1]$, $d_1(0) = d_2(0) = d_4(0) = d_6(0) = d_7(0) = d_9(0) = d_{12}(0) = -1$, and $f = 0.84$. (a) shows positions of the beads vs time. Beads $2, 4, 6, 8, 10, 12$ are represented by solid lines, while the dash line, dash-dot line, point line, and thicker dash line represent the positions of beads $1, 3, 5, 7, 9, 11$. (b) shows $\theta_3(t)$ (solid line), $u_3(t)$ (thicker solid line), and $\ell_3(t)$ (dash-dot line).*

CONJECTURE 3 (D-unbalanced collection: existence of periodic orbit). *Assume $\{(\theta_i(0), x_i(0))\}_{i\in\{1,\dots,n\}} \in \mathcal{A}_{D-\mathrm{unbal}}$, $v_i(t) = \overline{v}$, $\mathrm{dist}_{\mathtt{cc}}(\ell_i(0), \ell_{i+1}(0)) = \frac{2\pi}{n}$ for all*

$i \in \{1, \ldots, n\}$. *The following two statements are equivalent:*

(i) $\frac{1}{2} < f < \frac{n}{1+n}$,

(ii) there exists a periodic orbit along which each bead i *impacts with its previous bead* $i-1$ *always at position* $\ell_i(0) + D\delta$, *where* $\delta = \frac{2\pi}{n^2}\frac{f}{1-f} < \frac{2\pi}{n}$.

CONJECTURE 4 (D-unbalanced collection: global basin of attraction). *Assume* $\{(\theta_i(0), x_i(0))\}_{i\in\{1,\ldots,n\}} \in \mathcal{A}_{D-\text{unbal}}$ *with* $n > |D| > 1$, $\delta = \frac{2\pi}{n^2}\frac{f}{1-f} < \frac{2\pi}{n}$, *and let* $\tilde{C}_i(t)$ *be the center of the counterclockwise arc* $\text{Arc}(\ell_i(t) + D\delta, u_i(t) + D\delta)$ *for all* $i \in \{1, \ldots, N\}$. *Let* P_i^k *be the instant at which bead* i *passed by* $\tilde{C}_i$ *for the* k^{th} *time and let* $P^k = [P_1^k, \ldots, P_n^k]^T \in \mathbb{R}^n$. *Then, along the trajectories of the* SIS ALGORITHM*:*

$$\lim_{k\to+\infty} P^{2k} - P^{2(k-1)} = \mathbf{1}_n \frac{2}{\bar{v}}\frac{2\pi}{n},$$

that is, the collection of beads asymptotically reaches unbalanced synchrony.

In what follows we present the results of two simulations (Figures 6.3 and 6.4) obtained by implementing the SIS ALGORITHM with a collection of $N = 12$ beads which are D-unbalanced with $D = -2$, the beads are randomly positioned on $\mathbb{S}^1$, $v_i(0)$ uniformly distributed in $]0, 1]$. Note that according to our conjectures $f < \frac{n/|D|}{1+n/|D|} = \frac{6}{7} \approx 0.857$ has to hold in order to reach unbalanced synchrony. In the first simulation $f = 0.84$, while in the second simulation $f = 0.87$, therefore we expect to the collection of beads to be in sync asymptotically in the first simulation but not in the second one.

Figure 6.3(a) shows the positions of the 12 beads vs time with $f = 0.84$. Clearly, asymptotically each bead meets its neighbor at the same location on the circle, reaching synchrony. In Figure 6.3(b), the positions and the desired sweeping arc boundaries for bead $i = 3$ are illustrated. The solid line represents $\theta_3(t)$, the dash-dot line represents $\ell_3(t)$, and the thicker solid line represents $u_3(t)$. The distance $\text{dist}_{\mathtt{cc}}(\ell_3(t), u_3(t))$ asymptotically approaches $360/n = 30$ degrees. Figure 6.4 shows the positions of the 12 beads vs time when $f = 0.87$. Clearly synchrony is not reached as expected.

7. Concluding discussion. We presented and analyzed the SIS ALGORITHM that synchronizes a collection of n agents or beads, moving on a ring, so that each bead patrols only a sector of the ring. The algorithm is distributed and requires that two agents exchange information only when they meet. We have established that the proposed algorithm renders locally attractive the periodic modes corresponding to balanced and unbalanced synchrony. Simulations indicate that convergence to the desired periodic modes takes places for a large set of initial conditions.

Without providing a formal analysis, we mention here a few properties of the proposed algorithm. The SIS ALGORITHM (1) adapts smoothly to arrival and departures of agents throughout execution time, including adapting to switches between odd and even numbers of agents, (2) handles smoothly measurement noise and control disturbances, (3) has memory requirements and message sizes independent of n, (4) is truly distributed and does not require agents to have unique identifiers, and (5) is invariant under rotations and reflections.

Furthermore, our algorithm may be implemented even on robotic agents that do not have access to their position with respect to a global reference frame on the ring, i.e., even if they do not agree upon the position of the absolute 0 angle. To be specific, assume that each agent can only measure the angular distances that it travels and that, at communication impacts, the agent transmits its travel distance from its arc center to the impact position. Then, it is easy to see that this "relative angle" information suffices to implement the update rules (in equations (3.3) and (3.4)).

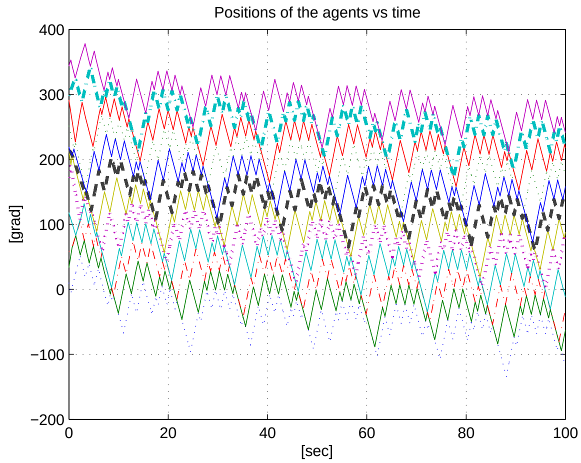

Fig. 6.4. *This figure shows θ_i vs time, obtained by implementing the* SIS Algorithm *with $n = 12$ beads, the beads are randomly positioned on $\mathbb{S}^1$, $v_i(0)$ uniformly distributed in $]0, 1]$, $d_1(0) = d_4(0) = d_6(0) = d_7(0) = d_8(0) = d_9(0) = d_{10}(0) = -1$, and $f = 0.87$. The positions of the beads $2, 4, 6, 8, 10, 12$ are represented by solid lines, while the dash line, dash-dot line, point line, and thicker dash line represent the positions of beads $1, 3, 5, 7, 9, 11$.*

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