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A Cooperative Homicidal Chauffeur Game

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***Abstract*— We address a pursuit-evasion problem involving an unbounded planar environment, a single evader, and multiple pursuers moving along curves of bounded curvature. The problem amounts to a multi-agent version of the classic *homicidal chauffeur* problem; we focus on parameter ranges in which a single pursuer is not sufficient to capture the evader. We propose a novel cooperative strategy in which the pursuers move in a daisy-chain formation and confine the evader to a bounded region. The proposed policy is inspired by certain hunting and foraging behaviors of various fish species. We characterize the required number of pursuers and the required value of the evader/pursuers speed ratio for which our strategy is guaranteed to lead to confinement.**

I. Introduction

The homicidal chauffeur game has been studied in great detail. Proposed originally by Isaacs [1], this problem involves a single pursuer who wants to overrun an evader, both moving with fixed speeds. The pursuer has greater speed but has constraints on its turning radius while the evader can make arbitrarily sharp turns. The evader is said to be *captured* when the distance between the pursuer and evader becomes less than a specified *capture radius*. We present a multi-agent homicidal chauffeur problem in which a single pursuer is not sufficient to capture the evader. We present a bio-inspired, cooperative strategy for multiple pursuers to confine the evader in a bounded region which the evader cannot leave without being captured.

A. Related Work

The classical homicidal chauffeur problem was proposed and solved by Isaacs [1]. This solution gives a condition on the game parameters, i.e., the speed ratio of the players, the capture radius and the minimum turning radius of the pursuer such that the evader can evade indefinitely. Glizer [2] has considered a generalized capture criterion of forcing the evader within a capture radius and within a prescribed angle with respect to pursuer's direction of motion. Pachter *et al.* [3] have studied a stochastic version of this game, i.e., the effect of noise in the measurements of the evader by the pursuer on the optimal capture strategy. Getz *et al.* [4] have analyzed a two-target version of the homicidal chauffeur problem, i.e., the roles of pursuer and evader are not defined *a priori*.

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In recent past, lot of attention has been received by cooperative control strategies for detection of targets. McLain *et al.* [5] have addressed the problem of cooperative rendezvous in which multiple UAVs are to arrive simultaneously at their targets. Polycarpou *et al.* [6] have presented a cooperative target search using online learning and computing guidance trajectories for the agents. Recently, Tang *et al.* [7], [8] have presented cooperative motion planning methods for multiple UAVs to traverse through slow moving targets and for first-order mobile sensing agents to detect a moving target that lies in a known initial region respectively. Also, McGee *et al.* [9] have proposed guaranteed strategies to search for mobile evaders in a plane.

B. Contributions

We consider the case when the condition for evasion in the classical homicidal chauffeur game is satisfied. This means that one pursuer is not sufficient to capture the evader. We propose a multiple pursuer formation and a novel four-phase, cooperative strategy for the pursuers. In the first two phases of the strategy, we show that the problem is equivalent to having a single pursuer and the task being getting the evader along its direction of motion and at a specified distance, given any set of initial conditions. Once this is achieved, we show that in the last two phases, the pursuers confine the evader within a bounded region, through which there exists no evader trajectory that avoids capture. We characterize the required number of pursuers and the required value of the evader/pursuers speed ratio for which our strategy is guaranteed to lead to confinement. Finally, we characterize a class of confinement strategies and determine a minimum number of pursuers required for any strategy in this class to achieve confinement.

Fig. 1. Daisy-chaining among tarpon fish (source: Florida Tarpon Fishing at http://www.gianttarpon.com/tarpon1.htm)

C. Biological Motivation

The inspirations for the strategies proposed in this paper have been derived from certain aspects of fish behavior. It has been recently reported by Gazda *et al.* [10] that in Cedar Key, Florida, USA, individual dolphins herding slower, more agile prey fish specialize in the roles of driver and barrier. The driver dolphin herds the prey fishes in circles as well as towards the tightly-grouped barrier dolphins. These barrier dolphins are less than one body-length apart and often touching. Pitman *et al.* [11] have reported a herd of killer whales imposing a confinement on pantropical spotted dolphins. The whales cut out up to three dolphins from a school and proceeded to take turns chasing a single dolphin and keeping it within a confined area. A study of American white pelicans by McMahon *et al.* [12] also showed the use a coordinated strategy by a group of birds that maintained position usually in semi-circles and using synchronized bill-dipping to capture fish. These facts give us some hints towards selecting favorable predator formations. The formation that has inspired this work is the daisy-chain, observed among tarpon fish during spawning, shown in Figure 1.

D. Organization

The problem assumptions and its mathematical model are presented in Section II. Pursuer formation, the CONFINE strategy and the main results are presented in Section III. The proofs of the main results along with intermediate results required are presented in Section IV. Conclusions and future directions for this work are summarized in Section V.

II. PROBLEM SET-UP

Our cooperative homicidal chauffeur game is played in an unbounded, planar environment between a single evader and multiple pursuers. We assume that all the players have unlimited sensing capabilities. The pursuers have identical motion abilities and possess greater speed than that of the evader. However, the evader can make arbitrarily sharp turns while the pursuers cannot turn more than a minimum turning radius. We assume that the instantaneous position and velocity of the evader is available to all pursuers.

Let $e(t)$ and $p_k(t)$, for $k \in \{1, \ldots, N\}$, denote the positions of the evader and the k^{th} pursuer in $\mathbb{R}^2$ at time t, as shown in Figure 2. Let $v_{\rm e}$ and $v_{\rm p}$ denote the speeds of the evader and all the pursuers, respectively. Let $\overline{v}_{\rm e}$ and $\overline{v}_{{\rm p},k}$ denote the velocity vectors of the evader and the k^{th} pursuer, respectively. Given a *minimum turning radius* $R > 0$, the mathematical model for this problem can be described as follows [1].

$$\begin{aligned} \text{For evader: } \dot{e}_x(t) &= v_{\rm e} \cos\theta_{\rm e}(t), \\ \dot{e}_y(t) &= v_{\rm e} \sin\theta_{\rm e}(t). \\ \text{For pursuers: } \dot{p}_{k,x}(t) &= v_{\rm p} \cos\theta_{{\rm p},k}(t), \\ \dot{p}_{k,y}(t) &= v_{\rm p} \sin\theta_{{\rm p},k}(t), \\ \dot{\theta}_{{\rm p},k} &= \frac{v_{\rm p}}{R} u_{{\rm p},k}, \end{aligned} \tag{1}$$

where $\theta_{\rm e}(t)$ and $\theta_{{\rm p},k}(t)$ are respectively the angles made by the velocity vectors of the evader and of the k^{th} pursuer with reference to a global X axis. $u_{{\rm p},k}$ is the control applied by the k^{th} pursuer and satisfies the constraint

$$\|u_{{\rm p},k}\| \leq 1.$$

We define the *evader/pursuer speed ratio* $\gamma := v_{\rm e}/v_{\rm p}$ and assume $\gamma < 1$. Given a *capture radius* $c > 0$, the evader is said to be *captured* if, at some time t and for some k,

$$\|p_k(t) - e(t)\| \leq c.$$

In what follows, without loss of generality, we set the capture radius c and the pursuers speed $v_{\rm e}$ to 1. In summary, our cooperative homicidal chauffeur game is described by the number of pursuers $N \in \mathbb{N}$, the minimum curvature radius $R \in \mathbb{R}_{>0}$, and the evader/pursuers speed ratio $\gamma \in]0, 1[$.

Next, we introduce the notion of confinement as follows.

Definition II.1 (Confinement) *The evader is said to be* confined *to a bounded region $\mathcal{G} \subset \mathbb{R}^2$ at time t^* if $e(t^*) \in \mathcal{G}$ and there exist pursuer trajectories $p_k : [t^*, +\infty[\to \mathbb{R}^2$ satisfying equation* (1) *such that the evader cannot leave $\mathcal{G}$ without being captured.*

A set of functions $\{u_{{\rm p},k}\}$, for $k \in \{1, \ldots, N\}$, leading to evader confinement is termed as a *confinement strategy*. In the case of a single pursuer and single evader, Isaacs [1] has shown that there exists an evasion policy if $R > 2/(\pi - 2)$ and if the evader/pursuer speed ratio $\gamma \geq \gamma_{\min}(R)$, where $\gamma_{\min} :]2/(\pi - 2), +\infty[\to]0, 1[$ is the unique solution to

$$\frac{1}{x} = \sqrt{1 - \gamma_{\min}(x)^2} + \gamma_{\min}(x) \arcsin(\gamma_{\min}(x)) - 1. \tag{2}$$

We seek cooperative, deterministic multiple-pursuer strategies which guarantee a confinement of the evader when $\gamma \in [\gamma_{\min}(R), 1[$ for $R > 2/(\pi - 2)$.

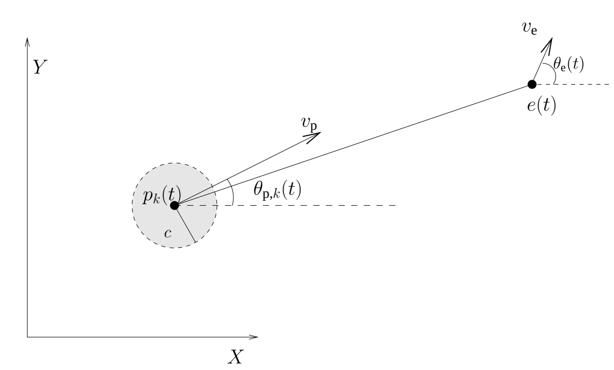

Fig. 2. Variables in the homicidal chauffeur game. The shaded region around the pursuer indicates its capture disc.

III. THE CONFINE STRATEGY

In this section, we describe our proposed CONFINE strategy for the pursuers to confine the evader and the corresponding main results. We first propose the following arrangement for the pursuers.

Definition III.1 (Pursuers daisy-chain formation) *Given an* inter-pursuer separation $s_{\text{ip}} > 0$, *the set* $\{p_1, \dots, p_N, \overline{v}_{\text{p},1}, \dots, \overline{v}_{\text{p},N}\}$ *is said to be in a* daisy-chain formation *if, for every* $k \in \{2, \dots, N\}$, *there exists a solution* $\eta : [0, s_{\text{ip}}] \to \mathbb{R}^2$ *of equation* (1) *satisfying*

$$\eta(0) = p_k, \qquad \dot{\eta}(0) = \overline{v}_{\text{p},k},$$
$$\eta(s_{\text{ip}}) = p_{k-1}, \qquad \dot{\eta}(s_{\text{ip}}) = \overline{v}_{\text{p},k-1}.$$

A daisy-chain formation has the following property.

Proposition III.2 (A Daisy-chain property) *Assume the pursuers are in a daisy-chain formation with separation* s_{ip}. *Any feasible path taken by the first pursuer can be exactly traversed by the* k^{th} *pursuer, for* $k \in \{2, \dots, N\}$, *in the daisy-chain after a time delay of* $(k-1)s_{\text{ip}}$.

Figure 3 shows an example of a daisy-chain formation.

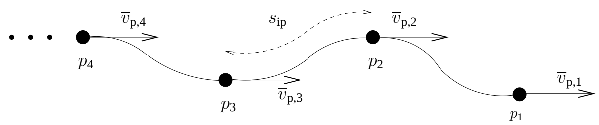

Fig. 3. A daisy-chain formation

Given $\{p_{k-1}, p_k, \overline{v}_{\text{p},k-1}, \overline{v}_{\text{p},k}\}$ in a daisy-chain formation with inter-pursuer separation s_{ip}, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be curves which are tangent to $\mathcal{B}_1(\eta(t))$ for every $t \in]0, s_{\text{ip}}[$, where for $a \in \mathbb{R}^2$, $\mathcal{B}_1(a) \subset \mathbb{R}^2$ denotes a circle of radius 1, centered at a. Here, η is a curve described in Definition III.1. Then, the evader is said to *move between* p_{k-1} *and* p_k if there is an evader trajectory from $\mathcal{C}_1$ to $\mathcal{C}_2$ (or vice-versa).

Given the pursuers' minimum turning radius R, for the evader/pursuers speed ratio $\gamma \le 1 - \frac{1}{R}$, we define *critical inter-pursuer separation* $s^*_{\text{ip}}(\gamma, R)$ as follows: if $r_1^2 := (1 + R^2) + 2R\sqrt{1-\gamma^2}$, $r_2^2 := (1 + R^2) - 2R\sqrt{1-\gamma^2}$, and

$$\Delta\theta := \arctan\left(\frac{1 - R\sqrt{1-\gamma^2}}{\gamma R}\right) - \arctan\left(\frac{1 + R\sqrt{1-\gamma^2}}{\gamma R}\right) + \frac{2\sqrt{1-\gamma^2}}{\gamma},$$

then

$$s^*_{\text{ip}}(\gamma, R) := R\Big(\Delta\theta + \arcsin\frac{\gamma}{r_1} + \arcsin\frac{\gamma}{r_2}\Big). \qquad (3)$$

This $s^*_{\text{ip}}(\gamma, R)$ has the following property, whose proof is presented in Section IV.

Lemma III.3 (Property of $s^*_{\text{ip}}(\gamma, R)$) *Given* $\{p_{k-1}, p_k, \overline{v}_{\text{p},k-1}, \overline{v}_{\text{p},k}\}$ *in a daisy-chain formation and the evader/pursuers speed ratio satisfying* $\gamma \le 1 - \frac{1}{R}$, *to prevent an evader from moving between* p_{k-1} *and* p_k *without being captured, the inter-pursuer separation must not exceed* $s^*_{\text{ip}}(\gamma, R)$.

Given a point p and a unit vector $x \in \mathbb{R}^2$, we define the *region of confinement* $\mathcal{G}_{\text{c}}(p, x) \subset \mathbb{R}^2$ for the CONFINE strategy as follows: choose $a, b, c \in \mathbb{R}^2$ such that

(i) x rotated counter-clockwise about p by $\frac{\pi}{2}$ becomes parallel to $(a - p)$, $(p - c) \parallel x$ and,

(ii) a, b, c, p are vertices of a rectangle such that $\|a - b\| = \|p - c\| = \frac{2\pi\gamma R}{1-\gamma}$ and $\|a - p\| = \|b - c\| = R + 1$, as shown in Figure 4.

Let $\text{Rect}\,(a, b, c, p) \subset \mathbb{R}^2$ denote the rectangular region formed by the four points a, b, c and p. Then, $\mathcal{G}_{\text{c}}(p, x) \subset \mathbb{R}^2$ is defined as interior of the union of $\mathcal{B}_{R+1}(\frac{a+p}{2})$, $\mathcal{B}_{R+1}(\frac{b+c}{2})$ and $\text{Rect}\,(a, b, c, p)$.

We define a point $a \in \mathbb{R}^2$ to be *aligned* with the k^{th} pursuer if $\overline{v}_{\text{p},k} \parallel (a - p_k)$.

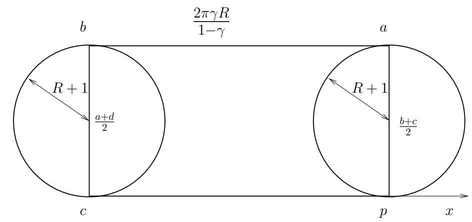

Fig. 4. Defining the region of confinement, $\mathcal{G}_{\text{c}}$ for the CONFINE strategy

We now describe the CONFINE strategy. Let the evader be located at $e(0)$, as shown in Figure 5. Let $\{p_1, \dots, p_N, \overline{v}_{\text{p},1}, \dots, \overline{v}_{\text{p},N}\}$ be a daisy-chain formation with $s_{\text{ip}} = s^*_{\text{ip}}(\gamma, R)$ and the first pursuer at $p_1(0)$. Let the angle between $\overline{v}_{\text{p},1}(0)$ and $(e(0) - p_1(0))$ be ϕ_0 and $\|p_1(0) - e(0)\| = L_0$. Due to Proposition III.2, it suffices to specify the strategy for p_1. The CONFINE strategy is as follows.

(i) **Pre-Align phase:** This phase is needed if $e(0)$ is not aligned with $p_1(0)$ or if $L_0 < (1+\gamma)l_{\text{st}}$, where $l_{\text{st}} \triangleq \frac{2\pi\gamma R}{1-\gamma}$. Let l_{p} be the minimum of the roots of the quadratic equation:

$$(\gamma(l_{\text{p}} + 2\pi R) + (1+\gamma)l_{\text{st}} + R)^2 = (l_{\text{p}} - L_0\cos\psi_0)^2 + (R - L_0\sin\psi_0)^2. \qquad (4)$$

p_1 moves on a straight line path of length l_{p} and then moves on a circle of radius R and center on the side not containing $e(0)$ of the line along $\overline{v}_{\text{p},1}(l_{\text{p}})$. If $\phi_0 = 0$ or π, then the center of the circle of radius R can be chosen to be on either side of the line along $\overline{v}_{\text{p},1}(l_{\text{p}})$. Lemma IV.2 shows that this phase terminates in finite time with the evader aligned with p_1 and at a distance greater than $(1+\gamma)l_{\text{st}}$.

(ii) **Align phase:** p_1 moves with the following control law:

$$u_{\text{p},1}(\theta_{\text{e}}, e, \theta_{\text{p},1}, p_1) = \frac{R\gamma}{\|p_1 - e\|}\sin(\theta_{\text{e}} - \theta_{\text{p},1}),$$

until $\|p_1 - e\| = (1+\gamma)l_{\text{st}}$. Lemma IV.3 shows that this happens in finite time.

(iii) **Chase phase:** p_1 moves on a straight line path until either it captures the evader or $\overline{v}_{\text{p},1} \perp (e - p_1)$, where the symbol $\perp$ denotes perpendicularity. This is shown in Figure 6. Let t_{chase} denote the time at the end of the Chase phase.

(iv) **Close phase:** First, p_1 moves on a circle of radius R and center O located on the evader side of the line

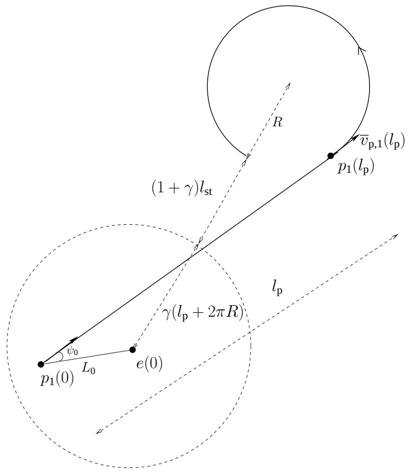

Fig. 5. The Pre-Align phase: the bold line shows the trajectory followed by p_1.

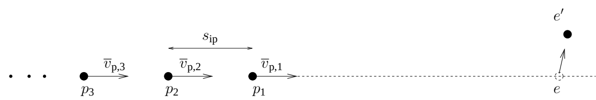

Fig. 6. The Chase phase

along $\overline{v}_{\mathrm{p},1}$, until it covers a distance πR. Second, p_1 moves on a straight line path of length l_{st}. Third, p_1 moves on a circle of radius R and center O' located on the evader side of the line along $\overline{v}_{\mathrm{p},1}$, until it covers a distance πR. This motion is shown in Figure 7.

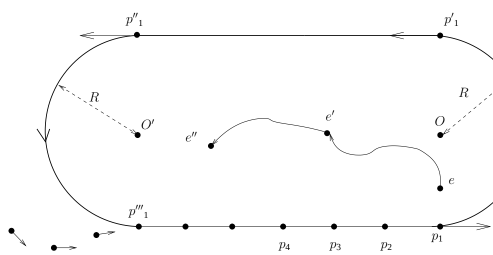

Fig. 7. The Close phase

(v) **Final phase:** p_1 moves on the boundary of $\mathcal{G}_\mathrm{c}(p_1(t_{\mathrm{chase}}), \overline{v}_{\mathrm{p},1}(t_{\mathrm{chase}}))$.

For every $k \in \{2, \ldots, N\}$, p_k moves on curve η described in Definition III.1 for time $t \leq s_{\mathrm{ip}}$. For $t > s_{\mathrm{ip}}$, $u_{\mathrm{p},k}(t) = u_{\mathrm{p},k-1}(t - s_{\mathrm{ip}})$.

This strategy gives us the following result.

Theorem III.4 (Confinement) *Consider a cooperative homicidal chauffeur game with parameters $N \in \mathbb{N}$, $R > \frac{2}{\pi-2}$, and $\gamma \in\ ]0,1[$. Let t_{chase} denote the time at the end of the Chase phase. The* CONFINE *strategy guarantees confinement of the evader inside the region $\mathcal{G}_\mathrm{c}(p_1(t_{\mathrm{chase}}), \overline{v}_{\mathrm{p},1}(t_{\mathrm{chase}}))$ at time t_{chase} if*

$$N \geq N_{\min} := \left\lceil \frac{2\pi R(1+\gamma)}{s^*_{\mathrm{ip}}(\gamma, R)(1-\gamma)} \right\rceil,$$

and if

$$\gamma \in [\gamma_{\min}(R),\ \gamma_{\max}(R)],$$

where $\gamma_{\max}(R)$ is the smallest root in the interval $]0,1[$ of

$$(1-\gamma^2)\left(2 - \frac{2\pi\gamma^2}{(1-\gamma)}\sqrt{\frac{1+\gamma}{1-\gamma}} + \frac{1}{R}\right)^2 = \pi^2\gamma^2. \qquad (5)$$

Moreover, if $\gamma > \gamma_{\max}(R)$ or if $N < N_{\min}$, then the evader can escape the region $\mathcal{G}_\mathrm{c}(p_1(t_{\mathrm{chase}}), \overline{v}_{\mathrm{p},1}(t_{\mathrm{chase}}))$.

Remark III.5 Because

$$\lim_{R\to+\infty} \gamma_{\min}(R) = 0^+,\ \lim_{R\to+\infty} \gamma_{\max}(R) \approx .293,$$

for sufficiently large values of R, the interval $[\gamma_{\min}(R),\ \gamma_{\max}(R)]$ is non-empty.

Remark III.6 (From confinement to capture) If a sufficiently large number of pursuers is available, then capture can be achieved in a final maneuver as follows: $N_{\min}$ pursuers confine the evader, while some additional pursuers arrange themselves in a line as shown in Figure 8 and move simultaneously through $\mathcal{G}_\mathrm{c}(p_1(t_{\mathrm{chase}}), \overline{v}_{\mathrm{p},1}(t_{\mathrm{chase}}))$.

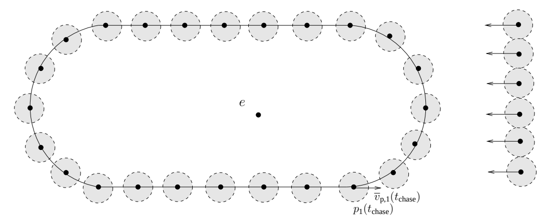

Fig. 8. Achieving capture with a sufficiently large number of pursuers

We now define a class of confinement strategies as follows.

Definition III.7 (Daisy chain-based strategy) *A confinement strategy is termed as* daisy chain-based *if, using that strategy,*

(i) *the evader is confined to a region $\mathcal{G}$ at time t^*,*
(ii) *the set $\{p_1(t^*), \ldots, p_N(t^*), \overline{v}_{\mathrm{p},1}(t^*), \ldots, \overline{v}_{\mathrm{p},N}(t^*)\}$ is a daisy-chain formation with inter-pursuer separation $s_{\mathrm{ip}} \leq s^*_{\mathrm{ip}}(\gamma, R)$ and,*
(iii) *the pursuer trajectories $p_k : [t^*, +\infty[\ \to \mathbb{R}^2$, $k \in \{2, \ldots, N\}$ satisfy*

$$u_{\mathrm{p},k}(t) = u_{\mathrm{p},k-1}(t - s_{\mathrm{ip}}), \quad \textit{for } t \geq t^*.$$

Clearly, the CONFINE strategy is a daisy chain-based strategy. We now have the following result.

Theorem III.8 (Minimum number of pursuers) *Let $R > \frac{2}{\pi-2}$ and the evader/pursuers speed ratio $\gamma \geq \gamma_{\min}(R)$, where $\gamma_{\min}(R)$ is the solution to equation* (2)*. Then, to achieve confinement using any daisy chain-based strategy, the number of pursuers must be at least $N^* = \lceil \frac{2\pi R}{s^*_{\mathrm{ip}}(\gamma,R)} \rceil$.*

IV. Proofs of the Main Results

In this section, we prove the main results from Section III. We begin with certain intermediate results which would be used to prove the main results. We first define the following terminology.

We say that a daisy-chain with separation s_{ip} is *closed* if for some p_k $(k \neq 1)$, there exists a $t_k \leq s_{\text{ip}}$ and a solution $\eta : [0, t_k] \to \mathbb{R}^2$ of equation (1) satisfying

$$\begin{aligned} \eta(0) &= p_1, & \dot{\eta}(0) &= \overline{v}_{\text{p},1}, \\ \eta(t_k) &= p_k, & \dot{\eta}(t_k) &= \overline{v}_{\text{p},k}, \end{aligned}$$

such that the evader cannot move between p_1 and p_k without being captured. We first prove Lemma III.3.

Proof of Lemma III.3: We show that the separation $s_{\text{ip}} \leq s^*_{\text{ip}}(\gamma, R)$ is necessary so that there does not exist any evader escape trajectory from arcs UW to PQ, without entering any capture ball when pursuers p_{k-1} and p_k are placed on a circle of radius R as shown in Figure 9. Let $\alpha_{\text{e}}(t) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ denote the angle made by the evader's velocity in the ground frame with eN, such that $(e - O) \parallel (N - e)$. Consider evader motion in a reference frame attached to the center O of the circle of radius R through pursuers p_{k-1} and p_k and rotating with angular speed $\frac{1}{R}$ in the direction of pursuer motion. Let $r(t), \theta(t)$ denote the evader's polar coordinates. $\theta(t)$ is measured with respect to Op_{k-1}. The equations of motion for the evader are,

$$\begin{aligned} \dot{r}(t) &= \gamma \cos \alpha_{\text{e}}(t), \\ \dot{\theta}(t) &= \frac{\gamma \sin \alpha_{\text{e}}(t)}{r(t)} + \frac{1}{R}. \end{aligned}$$

The optimal evader motion in this case is given by $\alpha^*_{\text{e}}(t) = -\arcsin \frac{\gamma R}{r(t)}$ [9]. This is well-defined since $\gamma \leq 1 - \frac{1}{R}$. Substituting in the differential equation for $r(t)$, we have

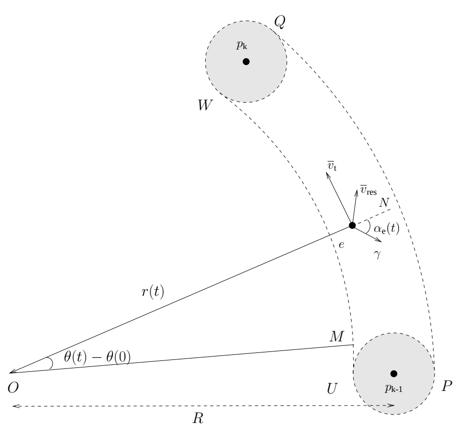

Fig. 9. Illustrating proof of Lemma IV.1. The shaded regions are the capture discs of p_{k-1} and p_k.

$$\dot{r}(t) = \gamma \sqrt{1 - \frac{\gamma^2 R^2}{r^2(t)}},$$

and, integrating,

$$r^2 = (\gamma t + c)^2 + \gamma^2 R^2, \tag{6}$$

where $c = \sqrt{(R-1)^2 - \gamma^2 R^2}$, which can be verified to be well-defined. Upon solving for t,

$$t = \frac{\sqrt{r^2 - \gamma^2 R^2} - \sqrt{(R-1)^2 - \gamma^2 R^2}}{\gamma}. \tag{7}$$

Substituting the expression for r in the equation for $\theta(t)$,

$$\dot{\theta}(t) = -\frac{\gamma^2 R}{r^2} + \frac{1}{R}.$$

Integrating, we obtain

$$\begin{aligned} \theta(t) &= -\int \frac{\gamma^2 R \mathrm{d}t}{(\gamma t + c)^2 + \gamma^2 R^2} + \frac{t}{R} \\ &= -\arctan\left(\frac{\gamma t + \sqrt{(R-1)^2 - \gamma^2 R^2}}{\gamma R}\right) + \frac{t}{\gamma R}, \end{aligned} \tag{8}$$

with

$$\theta(0) = -\arctan\left(\frac{\sqrt{(R-1)^2 - \gamma^2 R^2}}{\gamma R}\right).$$

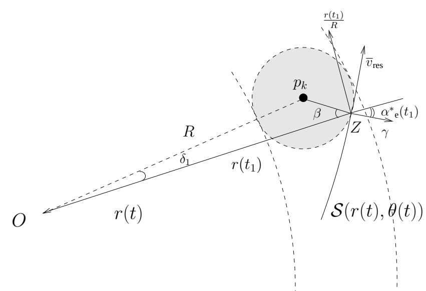

Fig. 10. Computing an expression for $s^*_{\text{ip}}(\gamma, R)$

We now determine the analytic expression for $s^*_{\text{ip}}(\gamma, R)$. Let $\mathcal{S}(r(t), \theta(t))$ denote the trajectory defined by equations (6) and (8). Let it touch the capture ball of p_k at Z, as shown in Figure 10. Let $\delta_1 \triangleq \angle p_k OZ$ and $\beta \triangleq \angle p_k ZO$. Due to tangency of $\mathcal{S}(r(t), \theta(t))$ and capture ball of p_k, angle between $\overline{v}_{\text{res}}$ and line OZ is $\frac{\pi}{2} - \beta$. From geometry,

$$\tan \beta = \frac{\gamma \cos \alpha^*_{\text{e}}(t_1)}{\frac{r(t_1)}{R} - \gamma \sin \alpha^*_{\text{e}}(t_1)},$$

where t_1 is the value of parameter t such that $\mathcal{S}(r(t_1), \theta(t_1)) = Z$ and $\alpha^*_{\text{e}}(t_1)$ is the optimal choice of evader motion at time t_1. On simplifying,

$$\sin \beta = \frac{\gamma R}{r(t_1)}, \quad \cos \beta = \sqrt{1 - \frac{\gamma^2 R^2}{r^2(t_1)}}.$$

Again, from geometry,

$$\sin \delta_1 = \frac{\sin \beta}{R} = \frac{\gamma}{R} \implies \cos \delta_1 = \sqrt{1 - \frac{\gamma^2}{r^2(t_1)}}.$$

Also, at the point of tangency,

$$r(t_1) = R \cos \delta_1 + \cos \beta,$$

Thus, $r(t_1) := r_1$ is a root of

$$x^4 - 2(1+R^2)x^2 + (R^2-1)^2 + 4\gamma^2 R^2 = 0.$$

The other positive root $\triangleq r(t_2) := r_2$ corresponds to Z_2, which is the point of tangency of $\mathcal{S}$ with the capture ball of p_{k-1}. Thus, we obtain

$$\{r_1^2, r_2^2\} = (1+R^2) \pm 2R\sqrt{1-\gamma^2}.$$

Substituting in equation (8),

$$\begin{aligned}\Delta\theta := \theta(t_1) - \theta(t_2) &= \arctan\left(\frac{\sqrt{r^2(t_2) - \gamma^2 R^2}}{\gamma R}\right) \\ &\quad - \arctan\left(\frac{\sqrt{r^2(t_1) - \gamma^2 R^2}}{\gamma R}\right) \\ &\quad + \frac{\sqrt{r^2(t_1) - \gamma^2 R^2} - \sqrt{r^2(t_2) - \gamma^2 R^2}}{\gamma R}.\end{aligned}$$

The angles δ_1 and analogously δ_2 (which is the angle between lines Op_{k-1} and OZ_2), are given by

$$\delta_1 = \arcsin\frac{\gamma}{r(t_1)}, \delta_2 = \arcsin\frac{\gamma}{r(t_2)}.$$

Thus, we get equation (3)

$$s^*_{\text{ip}}(\gamma, R) = R(\theta(t_1) - \theta(t_2) + \delta_1 + \delta_2).$$

For any value of inter-pursuer separation exceeding $s^*_{\text{ip}}(\gamma, R)$, there exists a trajectory between two consecutive pursuers p_k and p_{k-1} placed on a circle of radius R, such that the evader does not enter the capture balls of those two pursuers. This completes the proof. ■

We have the following result for the CONFINE strategy.

Lemma IV.1 (A CONFINE strategy property) *For the* CONFINE *strategy, the evader cannot move without being captured between pursuers p_k and p_{k-1}, for all $k \in \{2, \ldots, N\}$ and between any two consecutive pursuers if the daisy-chain is closed.*

Proof: We show that the inter-pursuer spacing, $s_{\text{ip}} = s^*_{\text{ip}}(\gamma, R)$ is actually necessary in the CONFINE strategy to prevent the evader from escaping between any two pursuers p_{k-1} and p_k and between any two pursuers when the daisy-chain is closed. It is known from [9] that when p_{k-1} is aligned with p_k, the inter-pursuer separation $s_{\text{ip}} = \frac{2}{\gamma}$ necessary and sufficient so that the evader cannot move between them without being captured.

Using equation (3), it can be easily verified that for a given γ, $s^*_{\text{ip}}(\gamma, R)$ increases monotonically with R and as $R \to \infty$, $s^*_{\text{ip}}(\gamma, R) \to \frac{2}{\gamma}$. Thus, the necessary spacing for the CONFINE strategy is $s^*_{\text{ip}}(\gamma, R)$. ■

We now prove the following property of the Pre-Align phase.

Lemma IV.2 (Pre-Align phase) *The Pre-Align phase terminates in finite time with the evader aligned with p_1 at a distance greater than $(1+\gamma)l_{\text{st}}$.*

Proof: The total time taken by p_1 to cover a distance l_{p} followed by distance of $2\pi R$ is $(l_{\text{p}} + 2\pi R)$. In that time, the evader's reachability set is the dotted circle of radius $\gamma(l_{\text{p}} + 2\pi R)$, centered at $e(0)$, as shown in Figure 5. Thus, to compute l_{p}, we impose the condition that the minimum distance between the evader's reachability set and the circular portion of the path of p_1 must be $(1+\gamma)l_{\text{st}}$. Using elementary geometry, the equation (4) for l_{p} follows. ■

We now prove the following property of the Align phase.

Lemma IV.3 (Align phase) *The Align phase of* CONFINE *strategy terminates after finite time with the evader aligned with p_1 and $\|p_1 - e\| = (1+\gamma)l_{\text{st}}$.*

Proof: Consider the system as shown in Figure 2 with $k = 1$. Let α be the angle between the global X axis and the vector $e(t) - p_1(t)$ and $L = \|e(t) - p_1(t)\|$. In the reference frame of the pursuer, the equations of motion are as follows [2].

$$\begin{aligned}\dot{L} &= v_{\text{e}}\cos(\theta_{\text{e}} - \alpha) - v_{\text{p}}\cos(\theta_{\text{p},1} - \alpha), \\ \dot{\alpha} &= \frac{1}{L}[v_{\text{e}}\sin(\theta_{\text{e}} - \alpha) - v_{\text{p}}\sin(\theta_{\text{p},1} - \alpha)], \\ \dot{\theta}_{\text{p},1} &= \frac{v_{\text{p}} u_{\text{p},1}}{R}.\end{aligned}$$

Define $\phi \triangleq \alpha - \theta_{\text{p},1}$. Thus, we have,

$$\dot{\phi} = \frac{1}{R}\left(\frac{R}{L}(\gamma\sin(\theta_{\text{e}} - \theta_{\text{p},1} - \phi) + \sin\phi) - u_{\text{p},1}\right).$$

Once evader is aligned with the pursuer, i.e., $\phi = 0$, we would like to ensure that $\dot{\phi} = 0$, for all subsequent t. This is possible if

$$u_{\text{p},1} = \frac{R\gamma}{L}\sin(\theta_{\text{e}} - \theta_{\text{p},1}).$$

But the constraint on $\|u_{\text{p},1}\|$ implies that this is possible if $L(t) \geq R\gamma$. It can be easily verified that the quantity $(1+\gamma)l_{\text{st}}$ satisfies this condition. Further, observe that once $\phi = 0$ and $\dot{\phi} = 0$, $\dot{L} \leq -(1-\gamma)$. Thus, L is reduced to $(1+\gamma)l_{\text{st}}$ in finite time. ■

We now obtain an upper-bound for the distance between the first pursuer and evader at the end of the Chase phase.

Lemma IV.4 (Upper bound on d) *Let d denote the distance $\|p_1 - e\|$ at the end of the Chase phase. Then,*

$$d \leq \gamma\sqrt{\frac{1+\gamma}{1-\gamma}}l_{\text{st}},$$

or, equivalently, for $\mu \triangleq \frac{d-1}{R}$ and $l_{\text{st}} = \frac{2\pi\gamma R}{1-\gamma}$,

$$\mu \leq \frac{2\pi\gamma^2}{(1-\gamma)}\sqrt{\frac{1+\gamma}{1-\gamma}} - \frac{1}{R}. \tag{9}$$

Proof: We have the evader aligned with the pursuer at a distance $(1+\gamma)l_{\text{st}}$ as shown in Figure 11. Let the evader strike out at an angle ψ as shown. So at the end of the Chase phase, it follows from trigonometry that

$$d = \frac{(1+\gamma)l_{\text{st}}\sin\psi}{\left(\frac{1}{\gamma} - \cos\psi\right)}.$$

Thus, d is a maximum when $\cos\psi = \gamma$ and the result follows. ∎

Fig. 11. Illustrating proof of Lemma IV.4

We would like to point out that the right hand side of equation (9) is positive and monotonic increasing with γ for $\gamma \geq \gamma_{\text{min}}(R)$. We now establish an upper bound for the speed ratio γ so that the evader always remains on the same side of the line along velocity vector of p_1.

Lemma IV.5 (Upper bound on γ) *For the* CONFINE *strategy to succeed, the evader/pursuers speed ratio γ must satisfy,*

$$\gamma \leq \frac{2-\mu}{\sqrt{\pi^2 + (2-\mu)^2}}, \text{ where } \mu = \frac{d-1}{R}. \qquad (10)$$

Proof: We first show that for any angle $\omega \in [0, \frac{\pi}{2}]$, as shown in Figure 12, equation (10) holds. Let p_1B' be a path taken by the pursuer. To guarantee escape against the pursuer strategy, the evader needs to cover a distance of at least eB as shown in Figure 12. Thus, we obtain

$$\frac{\pi R + (2R-d+1)\tan\omega}{1} \leq \frac{(2R-d+1)\sec\omega}{\gamma}$$
$$\implies \quad \gamma(\pi R\cos\omega + (2R-d+1)\sin\omega) \leq (2R-d+1).$$

Now,

$$\max_{\omega\in[0,\frac{\pi}{2}]}(\pi R\cos\omega + (2R-d+1)\sin\omega) = R\sqrt{\pi^2 + (2-\mu)^2}.$$

Thus, working backwards, we get that satisfying equation (10) satisfies $\gamma(\pi R\cos\omega + (2R-d+1)\sin\omega) \leq (2R-d+1)$ for all $\omega \in [0, \frac{\pi}{2}]$. Note that this upper bound is achieved for $\omega = \arctan(\frac{2-\mu}{\pi})$. It can be shown that for this choice of ω and for $\gamma = \frac{2-\mu}{\sqrt{\pi^2+(2-\mu)^2}}$, the evader trajectory does not enter the capture ball of p_1. We skip this detail of the proof due to space constraints.

Finally, we need to show that the same bound holds if $\omega \in [-\pi, 0[$. Let eC be a path taken by the evader, as shown in Figure 13. We need to ensure that for any point C, p_1 reaches C' sooner than e reaches C. Applying the cosine rule to triangle OeC, this is achieved if, for $\alpha \in [0, \pi]$,

$$\alpha R \leq \frac{\sqrt{(R+1)^2 + (R-d)^2 - 2(R+1)(R-d)\cos\alpha}}{\gamma}.$$

Upon simplifying, it suffices to show that, for $\alpha \in [0, \pi]$ and $d > 1$,

$$(1+\frac{1}{R})^2 + (1-\frac{d}{R})^2 - 2(1+\frac{1}{R})(1-\frac{d}{R})\cos\alpha$$
$$-\frac{(2-\frac{(d-1)}{R})^2}{\pi^2 + (2-\frac{(d-1)}{R})^2}\alpha^2 \geq 0.$$

It can be easily verified that it is indeed the case and the proof is complete. ∎

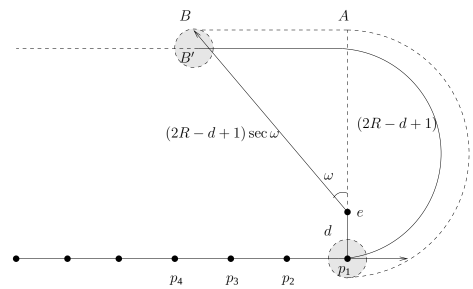

Fig. 12. Illustrating proof of Lemma IV.5. The shaded region about B' indicates capture region of p_1 when it reaches B'

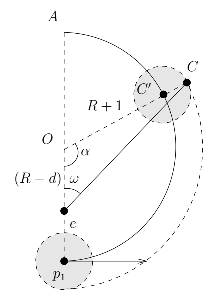

Fig. 13. Illustrating proof of Lemma IV.5. The shaded region about C' indicates capture region of p_1 when it reaches C'

We are now ready to prove Theorem III.4.

Proof of Theorem III.4:

The Align and Chase phases of the strategy ensure the property that immaterial of the orientations of the other following pursuers, a certain portion of the chain, equivalent to a length of *at least* l_{st}, will be a straight segment. For the Close phase to guarantee evader confinement, it suffices that the daisy chain is closed at the end of the Close phase. This final configuration is illustrated in Figure 14. Thus, by

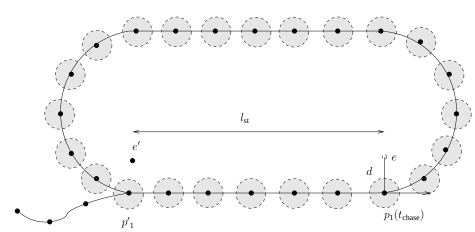

Fig. 14. Closing of the daisy-chain

the time the evader covers at most a distance l_{st} in the Close phase, p_1 must cover a total length of $2\pi R + l_{\text{st}}$, which gives us the following condition on l_{st}:

$$\frac{l_{\text{st}}}{\gamma} \geq 2\pi R + l_{\text{st}} \quad \Longrightarrow \quad l_{\text{st}} \geq \frac{2\pi\gamma R}{1-\gamma}.$$

So we let $l_{\text{st}} = \frac{2\pi\gamma R}{1-\gamma}$. This justifies our definition of the quantity l_{st} throughout the strategy. Using Lemma IV.5, we seek to determine the *minimum* over the set of all μ that satisfy equation (9) . This gives us a uniform upper-bound γ_{max}. Thus,

$$\gamma_{\text{max}} = \min_{\mu} \frac{2-\mu}{\sqrt{\pi^2 + (2-\mu)^2}}.$$

But the function on the right-hand side is monotonic decreasing with μ. The maximum value of μ is achieved when equality holds in equation (9), since the right-hand side of equation (9) is a monotonic increasing function of γ. Thus, the upper bound on speed ratio $\gamma_{\text{max}}(R)$ is obtained by solving equation (5) which upon simplification gives a polynomial equation in γ and $\gamma_{\text{max}}(R)$ is its smallest root in the interval $]0, 1[$.

A sufficient number of pursuers for the strategy CONFINE to succeed is obtained by ensuring that the daisy-chain is closed at the end of Close phase. The total distance covered by p_1 in the Close phase added to l_{st} gives the total length of the closed daisy-chain as $2(l_{\text{st}} + \pi R) = \frac{2\pi(1+\gamma)R}{(1-\gamma)}$. Thus, the number of pursuers needed is at least

$$N_{\text{min}} := \left\lceil \frac{2\pi(1+\gamma)R}{(1-\gamma)s^*_{\text{ip}}(\gamma, R)} \right\rceil.$$

Since the daisy-chain is closed at the end of Close phase and there is no escape trajectory between any two consecutive pursuers, it is clear that the evader is confined to $\mathcal{G}_{\text{c}}(p_1(t_{\text{chase}}), \overline{v}_{\text{p},1}(t_{\text{chase}}))$.

Moreover, if $\gamma > \gamma_{\text{max}}(R)$, then there exists an evader trajectory that leaves $\mathcal{G}_{\text{c}}(p_1(t_{\text{chase}}), \overline{v}_{\text{p},1}(t_{\text{chase}}))$ as seen in the proof of Lemma IV.5. Also, if the number of pursuers $N < N_{\text{min}}$, then the daisy chain is not closed at the end of the Close phase. Thus, there exists an initial condition and a corresponding evader strategy so that it can leave $\mathcal{G}_{\text{c}}(p_1(t_{\text{chase}}), \overline{v}_{\text{p},1}(t_{\text{chase}}))$ by moving between p_1 and p_N. This completes the proof. ■

Proof of Theorem III.8:

Let the evader be confined to a region $\mathcal{G}$ at time t^*. Due to part (iii) of Definition III.7, daisy chain formation is maintained at all times $t \geq t^*$ and the boundary of region $\mathcal{G}$ must be a continuous, closed curve with radius of curvature bounded below by R. To minimize the number of pursuers, the pursuers need to be placed on the smallest possible region having this property, which is a circle of radius R. From Lemma IV.1, we deduce that the inter-pursuer spacing $s_{\text{ip}} = s^*_{\text{ip}}(\gamma, R)$ is necessary to ensure that the evader cannot move between any two successive pursuers in the daisy-chain formation, without being captured. Thus, the number of pursuers must be at least $N^* = \lceil \frac{2\pi R}{s^*_{\text{ip}}(\gamma, R)} \rceil$. ■

V. CONCLUSIONS AND FUTURE DIRECTIONS

We addressed a cooperative homicidal chauffeur game in which a single pursuer is unable to capture an evader, given an arbitrary initial condition. We proposed a bio-inspired cooperative multiple pursuer strategy that guarantees confinement of an evader to a bounded region, from any initial condition. We characterized the required number of pursuers and the required value of the evader/pursuers speed ratio for which our CONFINE strategy is guaranteed to lead to confinement. We also characterized a class of confinement strategies and determined a minimum number of pursuers needed for any strategy in that class to achieve confinement.

Future research will consider how to determine alternate strategies possibly involving multiple maneuvers to achieve confinement. It would also be interesting to know the minimum number of pursuers required for any confinement strategy.

ACKNOWLEDGMENTS

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