Notes on averaging over acyclic digraphs and discrete coverage control

Chunkai Gao [a], Jorge Cortés [b], Francesco Bullo [a,*]

[a] *Department of Mechanical Engineering, University of California, Santa Barbara, CA 93106, USA*

[b] *Department of Mechanical and Aerospace Engineering, University of California, San Diego, CA 92093, USA*

Abstract

In this paper we study averaging algorithms and coverage control laws in a unified light. First, we characterize the convergence properties of averaging algorithms over acyclic digraphs with fixed and controlled-switching topology. Second, we introduce and study novel discrete coverage control laws, that are useful in practical implementations of coverage strategies. We characterize the close relationship of the novel discrete control laws with continuous coverage control laws and with averaging algorithms over a class of acyclic digraphs, that we term discrete Voronoi graphs. These results provide a unified framework to model a vast class of distributed optimization problems.

Key words: Averaging algorithm, Consensus, Switching topology, Coverage control, Nonsmooth analysis

1 Introduction

Consensus and coverage control are two distinct problems within the recent literature on multiagent coordination and cooperative robotics. Roughly speaking, in consensus problems, the objective is to analyze and design scalable distributed control laws to drive the groups of agents to agree upon certain quantities of interest. On the other hand, in coverage control problems, the objective is to deploy the agents to get optimal sensing performance of an environment of interest.

In the literature, many researchers have used averaging algorithms to solve consensus problems. The spirit of averaging algorithms is to let the state of each agent evolve according to the (weighted) average of the state of its neighbors. Averaging algorithms have been studied both in continuous time (Olfati-Saber and Murray, 2004; Ren and Beard, 2005) and in discrete time (Tsitsiklis et al., 1986; Jadbabaie et al., 2003; Moreau, 2005). In (Olfati-Saber and Murray, 2004), averaging algorithms are investigated via graph Laplacians under a variety of assumptions, including fixed and switching communication topologies, time delays, and directed and undirected information flow. The work (Moreau, 2005) adopts a set-valued Lyapunov approach to analyze the convergence properties of averaging algorithms. The works (Ren et al., 2007; Olfati-Saber et al., 2007) survey the results available for consensus problems using averaging algorithms. In the context of coverage control, (Cortés et al., 2004) proposes gradient descent algorithms for optimal coverage, and (Cortés et al., 2005) presents coverage control algorithms for groups of mobile sensors with limited-range interactions.

The contributions of this paper are (i) the characterization of the convergence properties of averaging algorithms over acyclic digraphs with fixed and controlled-switching topologies, and (ii) the synthesis of a unified perspective on averaging, consensus, and discrete and continuous coverage based on distributed optimization. Regarding (i), we provide a novel matrix representation of the disagreement function associated with a directed graph. Secondly, we prove that averaging over a fixed acyclic graph drives the agents to an equilibrium determined by the so-called "sinks" of the graph. Finally, we show that averaging over controlled-switching acyclic digraphs also makes the agents converge to the set of equilibria under suitable state-dependent switching signals. Regarding (ii), we present multi-center locational optimization functions in continuous and discrete settings, and introduce distributed coverage control algorithms that optimize them. Our analysis of the novel discrete coverage control law is relevant in that it provides a theoretically meaningful justification for discretized

* Corresponding author.

Email addresses: ckgao@engineering.ucsb.edu (Chunkai Gao), cortes@ucsd.edu (Jorge Cortés), bullo@engineering.ucsb.edu (Francesco Bullo).

implementations of the continuous coverage control law in (Cortés et al., 2004) that do not require the computation of integrals; this is helpful in practical implementations with finite resolution. Finally, we show how discrete coverage control laws over discrete Voronoi graphs can be casted and analyzed as averaging algorithms over a set of controlled-switching acyclic digraphs.

The paper is organized as follows. Section 2 reviews known results on averaging algorithms and contains convergence results over acyclic digraphs. Section 3 presents locational optimization functions in both continuous and discrete settings, and discusses coverage control laws. Finally, we gather our conclusions in Section 4.

2 Averaging algorithms over digraphs

We let $\mathbb{N}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{R}_{>0}$ denote the sets of, respectively, natural numbers, non-negative reals, and positive reals. The *quadratic form* associated with a symmetric matrix $B \in \mathbb{R}^{n\times n}$ is the function $x \mapsto x^T Bx$. The scalars $\lambda_1, \dots, \lambda_k$ are *convex combination coefficients* if $\lambda_i \geq 0$, for $i \in \{1, \dots, k\}$, and $\sum_{i=1}^k \lambda_i = 1$.

2.1 Digraphs and disagreement functions

A *weighted directed graph*, in short *digraph*, $\mathcal{G} = (\mathcal{U}, \mathcal{E}, \mathcal{A})$ of order n consists of a *vertex set* $\mathcal{U}$ with n elements, an *edge set* $\mathcal{E} \in 2^{\mathcal{U}\times\mathcal{U}}$ (recall that $2^{\mathcal{U}}$ is the collection of subsets of $\mathcal{U}$), and a *weighted adjacency matrix* $\mathcal{A}$ (sometimes denoted $\mathcal{A}(\mathcal{G})$) with nonnegative entries a_{ij}, $i, j \in \{1, \dots, n\}$. For simplicity, we take $\mathcal{U} = \{1, \dots, n\}$. For $i, j \in \{1, \dots, n\}$, the entry a_{ij} is positive if and only if the pair (i, j) is an edge of $\mathcal{G}$, i.e., $a_{ij} > 0 \iff (i, j) \in \mathcal{E}$. We assume no self loops, so that $a_{ii} = 0$ for all $i \in \{1, \dots, n\}$. The *out-degree* and the *in-degree* of node i are $d_{\text{out}}(i) = \sum_{j=1}^n a_{ij}$ and $d_{\text{in}}(i) = \sum_{j=1}^n a_{ji}$, respectively. The out-degree matrix $D_{\text{out}}(\mathcal{G})$ and the in-degree matrix $D_{\text{in}}(\mathcal{G})$ are the diagonal matrices defined by $(D_{\text{out}}(\mathcal{G}))_{ii} = d_{\text{out}}(i)$ and $(D_{\text{in}}(\mathcal{G}))_{ii} = d_{\text{in}}(i)$, respectively. The digraph $\mathcal{G}$ is *balanced* if $D_{\text{out}}(\mathcal{G}) = D_{\text{in}}(\mathcal{G})$. The *graph Laplacian* of the digraph $\mathcal{G}$ is

$$L(\mathcal{G}) = D_{\text{out}}(\mathcal{G}) - \mathcal{A}(\mathcal{G}).$$

A *directed path* in a digraph is an ordered sequence of vertices such that any two consecutive vertices in the sequence are an edge of the digraph. A *cycle* is a non-trivial directed path that starts and ends at the same vertex. A digraph is *acyclic* if it contains no directed cycles. A node of a digraph is *globally reachable* if it can be reached from any other node by traversing a directed path. A digraph is *strongly connected* if every node is globally reachable.

Next, we define reverse and mirror digraphs. Let $\tilde{\mathcal{E}}$ be the set of reverse edges of $\mathcal{G}$ obtained by reversing the order of all pairs in $\mathcal{E}$. The *reverse digraph* $\tilde{\mathcal{G}}$ of $\mathcal{G}$ is $(\mathcal{U}, \tilde{\mathcal{E}}, \tilde{\mathcal{A}})$, where $\tilde{\mathcal{A}} = \mathcal{A}^T$. The *mirror digraph* $\widehat{\mathcal{G}}$ of $\mathcal{G}$ is $(\mathcal{U}, \widehat{\mathcal{E}}, \widehat{\mathcal{A}})$, where $\widehat{\mathcal{E}} = \mathcal{E} \cup \tilde{\mathcal{E}}$ and $\widehat{\mathcal{A}} = (\mathcal{A} + \mathcal{A}^T)/2$. Note that $L(\tilde{\mathcal{G}}) = D_{\text{out}}(\tilde{\mathcal{G}}) - \mathcal{A}(\tilde{\mathcal{G}}) = D_{\text{in}}(\mathcal{G}) - \mathcal{A}(\mathcal{G})^T$.

Given a digraph $\mathcal{G}$ of order n, the *disagreement function* $\Phi_{\mathcal{G}} : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\Phi_{\mathcal{G}}(x) = \frac{1}{2} \sum_{i,j=1}^{n} a_{ij}(x_j - x_i)^2. \tag{1}$$

The following characterization of $\Phi_{\mathcal{G}}$ is novel.

Proposition 2.1 (Matrix form of disagreement) *Given a digraph $\mathcal{G}$ of order n, the disagreement function $\Phi_{\mathcal{G}} : \mathbb{R}^n \to \mathbb{R}$ is the quadratic form associated with the symmetric positive-semidefinite matrix*

$$P(\mathcal{G}) = \frac{1}{2}(D_{\text{out}}(\mathcal{G}) + D_{\text{in}}(\mathcal{G}) - \mathcal{A}(\mathcal{G}) - \mathcal{A}(\mathcal{G})^T).$$

Moreover, $P(\mathcal{G})$ is the graph Laplacian of the mirror graph $\widehat{\mathcal{G}}$, that is, $P(\mathcal{G}) = L(\widehat{\mathcal{G}}) = \frac{1}{2}\big(L(\mathcal{G}) + L(\tilde{\mathcal{G}})\big)$.

PROOF. For $x \in \mathbb{R}^n$, we compute

$$\begin{aligned}
x^T P(\mathcal{G})x &= \frac{1}{2}x^T(D_{\text{out}} + D_{\text{in}} - \mathcal{A} - \mathcal{A}^T)x \\
&= \frac{1}{2}\Big(\sum_{i,j=1}^{n} a_{ij}x_i^2 + \sum_{i,j=1}^{n} a_{ij}x_j^2 - 2\sum_{i,j=1}^{n} a_{ij}x_i x_j \Big) \\
&= \frac{1}{2}\Big(\sum_{i,j=1}^{n} a_{ij}(x_i^2 + x_j^2 - 2x_i x_j)\Big) = \Phi_{\mathcal{G}}(x).
\end{aligned}$$

Clearly, P is symmetric. Since $\Phi_{\mathcal{G}}(x) \geq 0$ for all $x \in \mathbb{R}^n$, we deduce $P(\mathcal{G})$ is positive semidefinite. Since

$$(D(\hat{\mathcal{G}}))_{ii} = \sum_{j=1}^{n} \hat{a}_{ij} = \sum_{j=1}^{n} \frac{1}{2}(a_{ij} + a_{ji}),$$

we have $D(\hat{\mathcal{G}}) = \frac{1}{2}(D_{\text{out}}(\mathcal{G}) + D_{\text{in}}(\mathcal{G}))$. Hence, from the definitions of reverse and mirror graphs, we have

$$\begin{aligned}
&L(\hat{\mathcal{G}}) = D(\hat{\mathcal{G}}) - \mathcal{A}(\hat{\mathcal{G}}) \\
&= \frac{1}{2}(D_{\text{out}}(\mathcal{G}) + D_{\text{in}}(\mathcal{G})) - \frac{1}{2}(\mathcal{A}(\mathcal{G}) + \mathcal{A}(\mathcal{G})^T) = P(\mathcal{G}). \qquad \square
\end{aligned}$$

Remark 2.2 *In general, $P(\mathcal{G}) \neq L(\mathcal{G})$ and, therefore, $\Phi_{\mathcal{G}}(x) \neq x^T L(\mathcal{G})x$. However, if the digraph $\mathcal{G}$ is balanced, then $D_{\text{out}}(\mathcal{G}) = D_{\text{in}}(\mathcal{G})$ and, in turn, $\Phi_{\mathcal{G}}(x) = x^T L(\mathcal{G})x$. This is the usual result for undirected graphs, e.g., (Olfati-Saber and Murray, 2004).* •

2.2 Averaging plus connectivity achieves consensus

To each node $i \in \mathcal{U}$ of a digraph $\mathcal{G}$, we associate a state $x_i \in \mathbb{R}$, that obeys a first-order dynamics of the form $\dot{x}_i = u_i$. The control u_i depends only on the state of the node i and of its neighbors in $\mathcal{G}$. Given a choice of controls u_i, $i \in \{1, \dots, n\}$, the closed-loop system asymptotically achieves consensus if each closed loop trajectory $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfies $x(t) \to \{\alpha(1, \dots, 1) \mid \alpha \in \mathbb{R}\}$ when $t \to +\infty$. In other words, all states become equal asymptotically. If α is the average of the initial state of the n nodes, then we say the nodes have reached *average-consensus*. The *averaging protocol* (Blondel et al., 2005; Ren et al., 2007; Olfati-Saber et al., 2007) is the linear control law defined by

$$u_i = \sum_{j=1}^{n} a_{ij}(x_j - x_i). \tag{2}$$

With this control law, the closed-loop system is

$$\dot{x}(t) = -L(\mathcal{G})x(t). \tag{3}$$

In a multi-agent system, as agents evolve, the communication topology changes, and this motivates the consideration of switching signals. Consider a family of digraphs $\{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ with the same vertex set $\{1, \dots, n\}$. A *switching signal* is a map $\sigma : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \{1, \dots, m\}$. A switching signal of the form $\sigma : \mathbb{R}_{\geq 0} \to \{1, \dots, m\}$ is *time-dependent*. A time-dependent switching signal σ has *finite dwell time* $\delta \in \mathbb{R}_{>0}$ if σ is constant over intervals of duration at least δ. Consider the following switched dynamical system

$$\begin{aligned} \dot{x}(t) &= -L(\mathcal{G}_k)x(t), \\ k &= \sigma(t, x(t)). \end{aligned} \tag{4}$$

Note that the notion of solution for this system might not be well-defined for arbitrary switching signals. The properties of the linear system (3) and the system (4) under time-dependent switching signals have been investigated in (Olfati-Saber and Murray, 2004; Ren and Beard, 2005; Moreau, 2005; Ren et al., 2007) and are summarized as follows.

Theorem 2.3 (Averaging over digraphs) *Let $\mathcal{G}$ be a digraph. The following statements hold:*

(i) System (3) *asymptotically achieves consensus if and only if $\mathcal{G}$ has a globally reachable node;*
(ii) If $\mathcal{G}$ is strongly connected, then system (3) *asymptotically achieves average-consensus if and only if $\mathcal{G}$ is balanced.*

Next, let $\{\mathcal{G}_1, \dots, \mathcal{G}_m\}$ be digraphs with the same vertex set $\{1, \dots, n\}$, and let $\sigma : \mathbb{R}_{\geq 0} \to \{1, \dots, m\}$ have finite dwell time. The following statements hold:

(iii) System (4) *asymptotically achieves consensus if there exists an infinite sequence of uniformly upper bounded, nonoverlapping time intervals such that the union of the switching graphs across each interval has a globally reachable node;*
(iv) If each $\mathcal{G}_i$, $i \in \{1, \dots, m\}$, is strongly connected and balanced, then for any arbitrary piecewise constant function σ, the system (4) *globally asymptotically solves the* average-*consensus problem.*

2.3 Averaging protocol over a fixed acyclic digraph

Here we characterize the convergence properties of the averaging protocol (3) under different connectivity properties than the ones stated in Theorem 2.3(i)-(ii), namely assuming that the given digraph is acyclic. Acyclic digraphs are widely studied in the computer science and distributed signal processing literature, and are used to describe the interactions of agents in leader-following formation problems, e.g., (Tanner et al., 2004; Fax and Murray, 2004).

We start by reviewing some basic facts. Given an acyclic digraph $\mathcal{G}$, every vertex of in-degree 0 is named *source*, and every vertex of out-degree 0 is named *sink*. We associate a nonnegative number to each vertex, called *depth*, in the following way. First, we define the depth of the sinks of $\mathcal{G}$ to be 0; note that any acyclic digraph has at least one sink. Next, we consider the acyclic digraph that results from erasing the 0-depth vertices from $\mathcal{G}$ and the in-edges towards them; the depth of the sinks of this new acyclic digraph are defined to be 1. The higher depth vertices are defined recursively. The depth of the digraph is the maximum depth of its vertices. For $n, d \in \mathbb{N}$, let $\mathcal{S}_{n,d}$ be the set of acyclic digraphs with vertex set $\{1, \dots, n\}$ and depth d. Next, we relabel the n vertices of the acyclic digraph $\mathcal{G}$ with depth d in the following way: (1) label the sinks from 1 to n_0, where n_0 is the number of sinks; (2) label the vertices of depth k from $\sum_{j=0}^{k-1} n_j + 1$ to $\sum_{j=0}^{k-1} n_j + n_k$, where n_k is the number of vertices of depth k, for $k \in \{1, \dots, d\}$. Vertices with the same depth may be labeled in arbitrary order. With this labeling, the adjacency matrix $\mathcal{A}(\mathcal{G})$ is lower-diagonal with vanishing diagonal entries, and the Laplacian $L(\mathcal{G})$ takes the form

$$L(\mathcal{G}) = \begin{bmatrix} 0 & 0 & \dots & 0 \\ -a_{21} & \sum_{j=1}^{1} a_{2j} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ -a_{n1} & -a_{n2} & \dots & \sum_{j=1}^{n-1} a_{nj} \end{bmatrix},$$

or, alternatively,

$$L(\mathcal{G}) = \begin{bmatrix} 0_{n_0 \times n_0} & 0_{n_0 \times (n-n_0)} \\ L_{21} & L_{22} \end{bmatrix},$$

where $0_{k \times h}$ is the $k \times h$ matrix with zero entries, $L_{21} \in \mathbb{R}^{(n-n_0) \times n_0}$ and $L_{22} \in \mathbb{R}^{n-n_0 \times n-n_0}$. All eigenvalues of L are non-negative and the zero eigenvalues are simple.

Proposition 2.4 (Averaging over an acyclic digraph) *Let $\mathcal{G}$ be an acyclic digraph of order n with n_0 sinks and assume its vertices are labeled according to their depth. The following statements hold:*

(i) The equilibrium set of (3) *is the vector subspace*

$$\ker L(\mathcal{G}) = \{(x_{\rm s}, x_{\rm e}) \in \mathbb{R}^{n_0}\times\mathbb{R}^{n-n_0} \mid x_{\rm e} = -L_{22}^{-1}L_{21}x_{\rm s}\}.$$

(ii) Each trajectory $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of (3) *exponentially converges to the equilibrium x^* defined by*

$$x_i^* = \begin{cases} x_i(0), & i \in \{1,\dots,n_0\}, \\ \dfrac{\sum_{j=1}^{i-1} a_{ij}x_j^*}{\sum_{j=1}^{i-1} a_{ij}}, & i \in \{n_0+1,\dots,n\}. \end{cases}$$

(iii) If $\mathcal{G}$ has unit depth, then $\Phi_{\mathcal{G}}$ is monotonically non-increasing along any trajectory of (3)*.*

PROOF. Statement (i) is obvious. Statement (ii) follows from the fact that $-L_{22}$ is Hurwitz and from

$$0 = \sum_{j=1}^{i-1} a_{ij}(x_j^* - x_i^*) = \sum_{j=1}^{i-1} a_{ij}x_j^* - \Big(\sum_{j=1}^{i-1} a_{ij}\Big)x_i^*.$$

Regarding statement (iii), when $\mathcal{G}$ has unit depth, the adjacency and out-degree matrices are, respectively,

$$\begin{bmatrix} 0_{n_0\times n_0} & 0_{n_0\times(n-n_0)} \\ -L_{21} & 0_{(n-n_0)\times(n-n_0)} \end{bmatrix}, \begin{bmatrix} 0_{n_0\times n_0} & 0_{n_0\times(n-n_0)} \\ 0_{(n-n_0)\times n_0} & L_{22} \end{bmatrix}.$$

Therefore, for an appropriate $\tilde{L}_{11} \in \mathbb{R}^{n_0\times n_0}$, we have

$$L(\tilde{\mathcal{G}}) = \begin{bmatrix} \tilde{L}_{11} & L_{21}^T \\ 0_{(n-n_0)\times n_0} & 0_{(n-n_0)\times(n-n_0)} \end{bmatrix}.$$

By Proposition 2.1, we have $P(\mathcal{G}) = \frac{1}{2}(L(\mathcal{G}) + L(\tilde{\mathcal{G}}))$. Hence, given a trajectory $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of (3), we have

$$\begin{aligned} \frac{d}{dt}\big(\Phi_{\mathcal{G}}(x(t))\big) &= -x(t)^T(L(\mathcal{G})^TP(\mathcal{G})+P(\mathcal{G})L(\mathcal{G}))x(t) \\ &= -x(t)^TL(\mathcal{G})^TL(\mathcal{G})x(t) - x(t)^TL(\mathcal{G})^TL(\tilde{\mathcal{G}})x(t) \\ &= -x(t)^TL(\mathcal{G})^TL(\mathcal{G})x(t) \leq 0, \end{aligned}$$

where the last equality relies upon the fact that $L(\mathcal{G})^TL(\tilde{\mathcal{G}}) = L(\tilde{\mathcal{G}})^TL(\mathcal{G}) = 0_{n\times n}$. □

Remark 2.5 *If the digraph has a single sink, then the convergence statement in part (ii) of Proposition 2.4 is equivalent to part (i) of Theorem 2.3. Note also that statement (iii) is not true for digraphs with depth larger than 1. The digraph in Figure 1 is a counterexample.* •

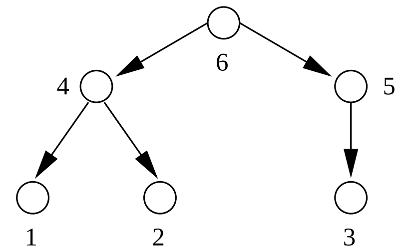

Fig. 1. For this digraph of depth 2, the Lie derivative of the disagreement (1) along the protocol (3) is sign indefinite.

2.4 Averaging protocol over switching acyclic digraphs

Given a family of digraphs $\Gamma = \{\mathcal{G}_1,\dots,\mathcal{G}_m\}$ with common vertex set $\{1,\dots,n\}$, the *minimal disagreement function* $\Phi_\Gamma : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\Phi_\Gamma(x) = \min_{k\in\{1,\dots,m\}} \Phi_{\mathcal{G}_k}(x). \tag{5}$$

Let $I(x) = \operatorname{argmin}\{\Phi_{\mathcal{G}_k}(x) \mid k \in \{1,\dots,m\}\}$. We consider state-dependent switching signals $\sigma : \mathbb{R}^n \to \{1,\dots,m\}$ with the property that $\sigma(x) \in I(x)$, that is, at each $x \in \mathbb{R}^n$, $\sigma(x)$ corresponds to the index of a graph with minimal disagreement. Clearly, for any such σ, one has $\Phi_\Gamma(x) = \Phi_{\mathcal{G}_{\sigma(x)}}(x)$.

Proposition 2.6 (Averaging over acyclic digraphs) *Let $\Gamma = \{\mathcal{G}_1,\dots,\mathcal{G}_m\}$ be a set of acyclic digraphs with vertices $\{1,\dots,n\}$ and depth d. Assume that $\cup_{k\in\{1,\dots,m\}}\mathcal{G}_k \in \mathcal{S}_{n,1}$ and that $\sigma : \mathbb{R}^n \to \{1,\dots,m\}$ satisfies $\sigma(x) \in I(x)$. Consider the discontinuous system*

$$\dot{x}(t) = Y(x(t)) = -L(\mathcal{G}_k)x(t), \quad \textit{for} \quad k = \sigma(x(t)), \tag{6}$$

whose solutions are understood in the Filippov sense (Filippov, 1988). The following statements hold:

(i) The point $x^* \in \mathbb{R}^n$ is an equilibrium for (6) *if and only if there exist convex combination coefficients $\{\lambda_i\}_{i\in I(x^*)}$ such that*

$$x^* \in \ker\Big(\textstyle\sum_{i\in I(x^*)} \lambda_i L(\mathcal{G}_i)\Big).$$

(ii) Each trajectory $x : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ of (6) *converges to the set of equilibria.*

(iii) The minimum disagreement function Φ_Γ is monotonically non-increasing along any trajectory of (6)*.*

PROOF. Statement (i) follows directly from the notion of equilibrium in discontinuous systems (Filippov, 1988). Given the assumptions on $\{\mathcal{G}_1,\dots,\mathcal{G}_m\}$ for any $i, j \in \{1,\dots,m\}$, we have

$$L(\mathcal{G}_i)^TL(\tilde{\mathcal{G}}_j) = 0_{n\times n}. \tag{7}$$

Next, we study the smoothness of Φ_Γ. Because $-\Phi_\Gamma$ is the maximum of smooth functions, Φ_Γ is (Clarke, 1983) locally Lipschitz and has generalized gradient

$$\partial\Phi_\Gamma(x) = \mathrm{co}\{2P(\mathcal{G}_i)x \mid i \in I(x)\}.$$

Given a locally Lipschitz function $f : \mathbb{R}^n \to \mathbb{R}$ and a vector field X, let $\widetilde{\mathcal{L}}_X f(x)$ denote its set-valued Lie derivative with respect to X (see (Cortés and Bullo, 2005). Let $a \in \widetilde{\mathcal{L}}_Y \Phi_\Gamma(x)$. By definition, there exist convex combination coefficients $\{\lambda_i\}_{i\in I(x)}$ such that $a = -\zeta^T(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)x)$, for all $\zeta \in \partial\Phi_\Gamma(x)$. In particular, for $\zeta = \sum_{i\in I(x)} 2\lambda_i P(\mathcal{G}_i)x \in \partial\Phi_\Gamma(x)$, we have

$$\begin{aligned} a &= \Big(-\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)x\Big)^T \Big(\sum_{i\in I(x)} 2\lambda_i P(\mathcal{G}_i)x\Big) \\ &= -x^T\Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)\Big)^T\Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)\Big)x \\ &\quad - x^T\Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)^T\Big)\Big(\sum_{i\in I(x)} \lambda_i L(\tilde{\mathcal{G}}_i)\Big)x \\ &= -x^T\Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)\Big)^T\Big(\sum_{i\in I(x)} \lambda_i L(\mathcal{G}_i)\Big)x \le 0, \end{aligned}$$

where the last equality follows from (7). Therefore, for $x \in \mathbb{R}^n$ and $a \in \widetilde{\mathcal{L}}_Y \Phi_\Gamma(x)$, we have $a \le 0$, that is, $\max \widetilde{\mathcal{L}}_Y \Phi_\Gamma(x) \le 0$. Statement (iii) is now obvious and statement (ii) is an immediate application of the LaSalle Invariance Principle for discontinuous systems as stated in (Bacciotti and Ceragioli, 1999). □

3 Discrete coverage control

In this section, we first review the multi-center optimization problem and the corresponding coverage control algorithm proposed in (Cortés et al., 2004). We then study the multi-center optimization problem in discrete space and derive a discrete coverage control law. Consider n robotic agents placed at locations $p_1, \dots, p_n \in \mathbb{R}^2$ moving according to $\dot{p}_i = u_i,\ i \in \{1, \dots, n\}$. We denote by P the vector $(p_1, \dots, p_n) \in (\mathbb{R}^2)^n$. Additionally, we define

$$\mathcal{S}_{\mathrm{coinc}} = \{(p_1, \dots, p_n) \in (\mathbb{R}^2)^n \mid p_i = p_j, \text{ for some } i \neq j\}.$$

Let $Q \subset \mathbb{R}^2$. For $P \notin \mathcal{S}_{\mathrm{coinc}}$, we let $\{V_i(P)\}_{i\in\{1,\dots,n\}}$ denote the Voronoi partition of Q generated by P, i.e.,

$$V_i(P) = \{q \in Q \mid \|q - p_i\| \le \|q - p_j\|, \text{ for all } j \neq i\}.$$

Voronoi partitions are discussed in (Cortés et al., 2004) and references therein. Figure 2 illustrates this notion.

3.1 Continuous and discrete multi-center functions

Let Q be a convex polygon in $\mathbb{R}^2$ including its interior and let $\phi : \mathbb{R}^2 \to \mathbb{R}_{\ge 0}$ be an integrable function whose support is Q. Analogously, let $\{q_1, \dots, q_N\} \subset Q$ be a

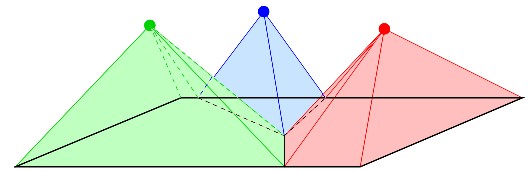

Fig. 2. Voronoi partition of a rectangle. The generators $p_1, \dots, p_n$ are elevated from the plane for intuition's sake.

point set and $\{\phi_1, \dots, \phi_N\}$ be nonnegative weights associated to them. Given a non-increasing function $f : \mathbb{R}_{\ge 0} \to \mathbb{R}$, we consider the *continuous* and *discrete multi-center functions* $\mathcal{H} : (\mathbb{R}^2)^n \to \mathbb{R}$ and $\mathcal{H}_{\mathrm{dscrt}} : (\mathbb{R}^2)^n \to \mathbb{R}$ defined by

$$\begin{aligned} \mathcal{H}(P) &= \int_Q \max_{i\in\{1,\dots,n\}} f(\|q - p_i\|)\phi(q)dq, \\ \mathcal{H}_{\mathrm{dscrt}}(P) &= \sum_{j=1}^N \max_{i\in\{1,\dots,n\}} \phi_j f(\|q_j - p_i\|). \end{aligned}$$

Remark 3.1 (Interpretation) *The function f plays the role of a performance function: given a sensor at location p_i, $f(\|q - p_i\|)$ is the quality of service provided by sensor for an event taking place at point q. Events take place inside the environment Q with likelihood ϕ. It is of interest to find local maxima for $\mathcal{H}$ and $\mathcal{H}_{\mathrm{dscrt}}$. These optimal sensor placement problems are studied in locational and geometric optimization, quantization theory, clustering analysis, and statistical pattern recognition; see (Cortés et al., 2004) and references therein.* •

Next, we provide an alternative expression of the multicenter functions using Voronoi partitions. Let $d(q) = \min_{j\in\{1,\dots,n\}} \|q - p_j\|$ and define

$$\mathcal{S}_{\mathrm{equid}} = \{(p_1, \dots, p_n) \in (\mathbb{R}^2)^n \mid \|q - p_i\| = \|q - p_k\| = d(q) \\ \text{for some } q \in \{q_1, \dots, q_N\} \text{ and for some } i \neq k\}.$$

In other words, if $P \notin \mathcal{S}_{\mathrm{equid}}$, then no point q_j is equidistant to two or more nearest robots. Note that $\mathcal{S}_{\mathrm{equid}}$ is a set of measure zero because it is the union of the solutions of a finite number of algebraic equations. For $P \notin \mathcal{S}_{\mathrm{coinc}}$, we may write

$$\begin{aligned} \mathcal{H}(P) &= \sum_{i=1}^n \int_{V_i(P)} f(\|q - p_i\|)\phi(q)dq, \\ \mathcal{H}_{\mathrm{dscrt}}(P) &= \sum_{i=1}^n \sum_{q_j\in V_i(P)} \frac{\phi_j}{\mathrm{card}(q_j, P)} f(\|q_j - p_i\|), \end{aligned}$$

where $\mathrm{card} : \mathbb{R}^2 \times (\mathbb{R}^2)^n \to \{1, \dots, n\}$ denotes the number of indices k for which $\|q_j - p_k\| = \min_{i\in\{1,\dots,n\}} \|q_j - p_i\|$. If q_j is a point in the interior of $V_i(P)$ for some i, then $\mathrm{card}(q_j, P) = 1$. The following result is discussed in (Cortés et al., 2005) for the continuous multi-center function; the result for the discrete function is novel.

Proposition 3.2 (Derivatives of $\mathcal{H}$ and $\mathcal{H}_{\mathrm{dscrt}}$) *If f is locally Lipschitz, then $\mathcal{H}$ and $\mathcal{H}_{\mathrm{dscrt}}$ are locally Lipschitz on Q^n. If f is differentiable, then*

(i) $\mathcal{H}$ is differentiable on $Q^n \setminus \mathcal{S}_{\mathrm{coinc}}$ with

$$\frac{\partial \mathcal{H}}{\partial p_i}(P) = \int_{V_i(P)} \frac{\partial}{\partial p_i} f(\|q - p_i\|)\phi(q)dq,$$

(ii) $\mathcal{H}_{\mathrm{dscrt}}$ is locally Lipschitz and regular on Q^n with generalized gradient

$$\partial \mathcal{H}_{\mathrm{dscrt}}(P) = \sum_{j=1}^{N} \phi_j \operatorname{co}\Big\{ \frac{\partial}{\partial P} f(\|q_j - p_k\|) \mid k \in I(q_j, P) \Big\},$$

where $I(q_j, P)$ is the set of indices k for which $f(\|q_j - p_k\|) = \max_{i \in \{1,\dots,n\}} f(\|q_j - p_i\|)$. Additionally, if $P \notin \mathcal{S}_{\mathrm{coinc}} \cup \mathcal{S}_{\mathrm{equid}}$, then $\mathcal{H}_{\mathrm{dscrt}}$ is differentiable at P, and for each $i \in \{1,\dots,n\}$

$$\frac{\partial \mathcal{H}_{\mathrm{dscrt}}}{\partial p_i}(P) = \sum_{q_j \in V_i(P)} \phi_j \frac{\partial}{\partial p_i} f(\|q_j - p_i\|).$$

PROOF. We refer to (Cortés et al., 2005) for the proof of (i). For $j \in \{1,\dots,N\}$, we define

$$F_j(P) = \max_{i \in \{1,\dots,n\}} f(\|q_j - p_i\|).$$

and write $\mathcal{H}_{\mathrm{dscrt}} = \sum_{j=1}^{N} \phi_j F_j(P)$. By assumption, f is locally Lipschitz and, therefore, so are F_j, for $j \in \{1,\dots,N\}$, and $\mathcal{H}_{\mathrm{dscrt}}$. Additionally, if f is differentiable, then F_j is regular with generalized derivative

$$\partial F_j(P) = \operatorname{co}\Big\{ \frac{\partial}{\partial P} f(\|q_j - p_i\|) \mid i \in I(q_j, P) \Big\},$$

where $I(q_j, P)$ is the set of indexes k for which $f(\|q_j - p_k\|) = F_j(P)$. Since $\mathcal{H}_{\mathrm{dscrt}}$ is a finite sum of F_j with nonnegative weights ϕ_j, so $\mathcal{H}_{\mathrm{dscrt}}$ is also regular on Q^n. From the regularity of F_j, we obtain (Clarke, 1983) the generalized gradient of $\mathcal{H}_{\mathrm{dscrt}}$ as stated in (ii). □

For particular choices of f, the multi-center functions and their partial derivatives may simplify. For example, if $f(x) = -x^2$, the partial derivative of the multi-center function $\mathcal{H}$ reads, for $P \notin \mathcal{S}_{\mathrm{coinc}}$,

$$\frac{\partial \mathcal{H}}{\partial p_i}(P) = 2M_{V_i(P)}(C_{V_i(P)} - p_i),$$

where mass and the centroid of $W \subset Q$ are

$$M_W = \int_W \phi(q)\, dq, \quad C_W = \frac{1}{M_W} \int_W q\, \phi(q)\, dq.$$

Additionally, the critical points P^* of $\mathcal{H}$ have the property that $p_i^* = C_{V_i(P^*)}$, for $i \in \{1,\dots,n\}$; accordingly, they are called (Cortés et al., 2004) *centroidal Voronoi configurations*. Analogously, if $f(x) = -x^2$, the discrete function and its generalized gradient are

$$\mathcal{H}_{\mathrm{dscrt}}(P) = -\sum_{j=1}^{N} \max_{i \in \{1,\dots,n\}} \phi_j \|q_j - p_i\|^2,$$

$$\partial \mathcal{H}_{\mathrm{dscrt}}(P) = \sum_{j=1}^{N} \phi_j \operatorname{co}\Big\{ 2(q_j - p_k) \frac{\partial p_k}{\partial P} \mid k \in I(q_j, P) \Big\}.$$

For each $j \in \{1,\dots,N\}$, given nonnegative scalars λ_{ij}, $i \in I(q_j, P)$, define $(M_{\mathrm{dscrt}})_{V_i(P)}$ and $(C_{\mathrm{dscrt}})_{V_i(P)}$ by

$$(M_{\mathrm{dscrt}})_{V_i(P)} = \sum_{q_j \in V_i(P)} \lambda_{ij} \phi_j,$$

$$(C_{\mathrm{dscrt}})_{V_i(P)} = \begin{cases} p_i, & \text{if } (M_{\mathrm{dscrt}})_{V_i(P)} = 0, \\ \dfrac{\sum_{q_j \in V_i(P)} \lambda_{ij} \phi_j q_j}{(M_{\mathrm{dscrt}})_{V_i(P)}}, & \text{otherwise.} \end{cases}$$

With this notation, P^* is a critical point of $\partial H_{\mathrm{dscrt}}$, that is, $0 \in \partial H_{\mathrm{dscrt}}(P^*)$ if, for any $j \in \{1,\dots,N\}$, there exist convex combination coefficients λ_{ij}, for $i \in I(q_j, P^*)$, such that $p_i^* = (C_{\mathrm{dscrt}})_{V_i(P^*)}$, for each $i \in \{1,\dots,n\}$. We call such points P^* *discrete centroidal Voronoi configurations.*

3.2 Continuous and discrete coverage control

Based on the expressions obtained in the previous subsection, it is possible to design motion coordination algorithms for the robots $p_1,\dots,p_n$. We call *continuous* and *discrete coverage optimization* the problems of maximizing the multi-center function $\mathcal{H}$ and $\mathcal{H}_{\mathrm{dscrt}}$, respectively. The continuous problem is studied in (Cortés et al., 2004). We simply impose that the locations $p_1,\dots,p_n$ follow a gradient ascent law defined over the set $Q^n \setminus \mathcal{S}_{\mathrm{coinc}}$. The (continuous) *coverage control law* is

$$u_i = k_{\mathrm{prop}} \frac{\partial \mathcal{H}}{\partial p_i}(P), \quad i \in \{1,\dots,n\}, \tag{8}$$

where k_{prop} is a positive gain. Analogously, the *discrete coverage control law* is

$$u_i = k_{\mathrm{prop}} X_i(P), \quad i \in \{1,\dots,n\}, \tag{9}$$

where $X_i : Q^n \to \mathbb{R}^2$ is defined by

$$X_i(P) = \sum_{q_j \in V_i(P)} \frac{\phi_j}{\operatorname{card}(q_j, P)} \frac{\partial}{\partial p_i} f(\|q_j - p_i\|).$$

Note that X_i is discontinuous on Q^n, continuous on $Q^n \setminus \mathcal{S}_{\text{coinc}} \cup \mathcal{S}_{\text{equid}}$, and satisfies

$$X_i(P) = \frac{\partial \mathcal{H}_{\text{dscrt}}}{\partial p_i}(P).$$

Note that both laws are distributed in the sense that each robot only needs information about its Voronoi cell in order to compute its control.

To handle the discontinuity of the discrete coverage control law (9), we define the vector field $X = [X_1, \dots, X_n]^T$ and write

$$\dot{P} = k_{\text{prop}}\, X(P). \tag{10}$$

We understand the solution of this equation in the Filippov sense (Filippov, 1988). We finally state the convergence properties of the solution of (8) and (9).

Proposition 3.3 (Continuous and discrete coverage control) *For the closed-loop systems induced by equation (8) and by equation (9) starting at $P_0 \in Q^n \setminus \mathcal{S}_{\text{coinc}}$, the agents location converges asymptotically to the set of critical points of $\mathcal{H}$ and of $\mathcal{H}_{\text{dscrt}}$, respectively.*

PROOF. The convergence of the continuous coverage control law to the critical points of $\mathcal{H}$ follows from the fact that (8) defines a gradient dynamical system for $\mathcal{H}$. Regarding the discrete coverage control law, note that the Filippov set-valued map is given by $K[k_{\text{prop}}\, X](P) = k_{\text{prop}} \partial \mathcal{H}_{\text{dscrt}}(P)$. Therefore, (9) defines a nonsmooth gradient dynamical system for $\mathcal{H}_{\text{dscrt}}$, and convergence to the critical points of $\mathcal{H}_{\text{dscrt}}$ follows (e.g., see (Cortés and Bullo, 2005, Proposition 2.9)). □

3.3 The relationship between discrete coverage and averaging over switching acyclic digraphs

As above, let $\{p_1, \dots, p_n\}$ be the robot positions in a convex polygon Q, and let $\{q_1, \dots, q_N\} \subset Q$ be N fixed points with nonnegative weights $\{\phi_1, \dots, \phi_N\}$. Whenever a fixed point q_j belongs to the interior of the Voronoi cell $V_i(P)$, we say that q_j is assigned to the robot i. We describe this assignment relationship through a novel useful digraph. A *discrete Voronoi graph* $\mathcal{G}_{\text{dscrt-Vor}}$ is a digraph with $(n+N)$ vertices $\{p_1, \dots, p_n, q_1, \dots, q_N\}$, with N directed edges

$$\{(p_i, q_j) | \text{ for each } q_j, p_i \text{ is the robot, or one of the robots, that is closest to } q_j\},$$

and with corresponding edge weights ϕ_j, for all $j \in \{1, \dots, N\}$. We illustrate one such graph in Figure 3. As the robots $(p_1, \dots, p_n) \in (\mathbb{R}^2)^n$ move, the digraph $\mathcal{G}_{\text{dscrt-Vor}}$ changes; when we need to emphasize this dependence, we denote the digraph by $\mathcal{G}_{\text{dscrt-Vor}}(P)$. For convenience, we denote the digraph nodes by $Z = (z_1, \dots, z_{n+N}) = (p_1, \dots, p_n, q_1, \dots, q_N)$. Additionally, we denote the digraph weights by $a_{\alpha\beta}$, for

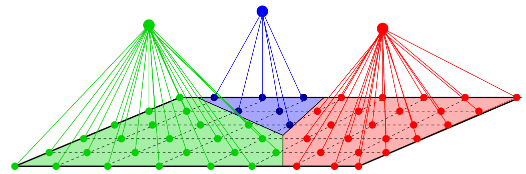

Fig. 3. The discrete Voronoi graph over 3 robots and 6×9 grid points in the rectangle of Figure 2. A top/down edge represents a grid point assigned to a robot.

$\alpha, \beta \in \{1, \dots, n+N\}$, with the convention that the only non-vanishing weights are $a_{\alpha\beta} = \phi_j$ precisely when $\beta = n + j$, for $j \in \{1, \dots, N\}$, and $\alpha \in \{1, \dots, n\}$ is the robot index in the directed edge (p_α, q_j).

Next, it is convenient to define a set of digraphs of which the discrete Voronoi graphs are examples. Let the set of *assignment functions* $H_{N,n}$ be the set of functions from $\{1, \dots, N\}$ to $\{1, \dots, n\}$. Roughly speaking, an assignment function $h : \{1, \dots, N\} \to \{1, \dots, n\}$ assigns to each point q_j, $j \in \{1, \dots, N\}$, the robot p_i, with $i = h(j) \in \{1, \dots, n\}$. Given an assignment function h, the *assignment digraph* $\mathcal{G}_h$ is the digraph with $(n+N)$ vertices $\{p_1, \dots, p_n, q_1, \dots, q_N\}$, with N directed edges $\{(p_{h(1)}, q_1), \dots, (p_{h(N)}, q_N)\}$, and corresponding edge weights $\phi_1, \dots, \phi_N$. By construction, we observe that $\mathcal{G}_{\text{dscrt-Vor}}(P) = \mathcal{G}_h$ for any assignment function h satisfying $h(j) \in \operatorname{argmin}\{\|q_j - p_i\| \mid i \in \{1, \dots, n\}\}$. We state a property of assignment digraphs without proof.

Lemma 3.4 *The digraphs $\mathcal{G}_h$, for $h \in H_{N,n}$, and the digraph $\cup_{h \in H_{N,n}} \mathcal{G}_h$ are acyclic and have unit depth.*

We now study appropriate disagreement functions for an assignment digraph $\mathcal{G}_h$. We define the function $\Phi_{\mathcal{G}_h} : (\mathbb{R}^2)^{n+N} \to \mathbb{R}$ by

$$\Phi_{\mathcal{G}_h}(Z)|_{Z=(p_1,\dots,p_n,q_1,\dots,q_N)} = \tfrac{1}{2} \sum_{j=1}^{N} \phi_j \|q_j - p_{h(j)}\|^2,$$

where we have used the fact that all weights $a_{\alpha\beta}$, $\alpha, \beta \in \{1, \dots, n+N\}$ of $\mathcal{G}_h$ vanish except for $a_{h(j),j} = \phi_j$, $j \in \{1, \dots, N\}$. We now state our main correspondence result whose proof is based on simple book-keeping and is therefore omitted.

Theorem 3.5 (Complete correspondence) *The following statements hold:*

(i) The discrete multi-center function $\mathcal{H}_{\text{dscrt}}$ with $f(x) = -x^2$, and the minimum disagreement function over the set of assignment digraphs $\mathcal{G}_h$ satisfy

$$-\frac{1}{2}\mathcal{H}_{\text{dscrt}}(P) = \min_{h \in H_{N,n}} \Phi_{\mathcal{G}_h}(p_1, \dots, p_n, q_1, \dots, q_N).$$

(ii) For $P \notin \mathcal{S}_{\text{coinc}} \cup \mathcal{S}_{\text{equid}}$, the discrete coverage law for $f(x) = -x^2$ and the averaging protocol over the

discrete Voronoi digraph satisfy, for $i \in \{1, \dots, n\}$,

$$\frac{1}{2}\frac{\partial \mathcal{H}_{\mathrm{dscrt}}}{\partial p_i}(P) = \sum_{\beta=1}^{n+N} a_{\alpha\beta}(z_\beta - z_\alpha),$$

where z_α and $a_{\alpha\beta}$, $\alpha, \beta \in \{1, \dots, n+N\}$, are nodes and weights of $\mathcal{G}_{\text{dscrt-Vor}}$.

(iii) $P^* \in Q^n$ is an equilibrium of the discrete coverage control system with $f(x) = -x^2$ if and only if $Z^* = (p_1^*, \dots, p_n^*, q_1, \dots, q_N)$ is an equilibrium of system (6) over the set of assignment digraphs $\mathcal{G}_h$.

(iv) Given any initial position of robots $P_0 \in Q^n$, the evolution of the discrete coverage control system (10) and the evolution of the averaging system (6) under the switching signal $\sigma : Q^n \to \{\mathcal{G}_h \mid h \in H_{N,n}\}$ defined by $\sigma(P) = \mathcal{G}_{\text{dscrt-Vor}}(Z)$ are identical in the Filippov sense. Therefore, the two systems converge to the same set of equilibrium placement of robots, as described in (iii).

Note that statement (ii) is about the two gradients being equal almost everywhere. It is possible to make an analogous statement about the corresponding generalized gradient being equal everywhere.

Remark 3.6 (Interpretation) *The theorem establishes a complete correspondence between discrete coverage control laws and averaging protocols over a certain class of acyclic graphs. One can argue that the coverage control problem and the consensus problem are special cases of a general class of distributed optimization problems.* •

4 Conclusions

We have studied averaging protocols over fixed and controlled-switching acyclic digraphs, and characterized their asymptotic convergence properties. We have also discussed continuous and discrete multi-center locational optimization functions, and distributed control laws that optimize them. Viewed separately, averaging, consensus, and coverage have numerous applications in distributed estimation and cooperative control. The results presented here connect these problems from a unified distributed optimization perspective.

Acknowledgments

The authors thank Dr Ali Jadbabaie for his initial motivating suggestion. This material is based upon work supported in part by ARO MURI Award W911NF-05-1-0219, by NSF CMS Award 0442041 and by NSF CAREER Award ECS-0546871.

References

A. Bacciotti and F. Ceragioli. Stability and stabilization of discontinuous systems and nonsmooth Lyapunov functions. *ESAIM. Control, Optimisation & Calculus of Variations*, 4:361–376, 1999.

V. D. Blondel, J. M. Hendrickx, A. Olshevsky, and J. N. Tsitsiklis. Convergence in multiagent coordination, consensus, and flocking. In *IEEE Conf. on Decision and Control and European Control Conference*, pages 2996–3000, Seville, Spain, December 2005.

F. H. Clarke. *Optimization and Nonsmooth Analysis*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley, 1983. ISBN 047187504X.

J. Cortés and F. Bullo. Coordination and geometric optimization via distributed dynamical systems. *SIAM Journal on Control and Optimization*, 44(5):1543–1574, 2005.

J. Cortés, S. Martínez, T. Karatas, and F. Bullo. Coverage control for mobile sensing networks. *IEEE Transactions on Robotics and Automation*, 20(2):243–255, 2004.

J. Cortés, S. Martínez, and F. Bullo. Spatially-distributed coverage optimization and control with limited-range interactions. *ESAIM. Control, Optimisation & Calculus of Variations*, 11:691–719, 2005.

J. A. Fax and R. M. Murray. Information flow and cooperative control of vehicle formations. *IEEE Transactions on Automatic Control*, 49(9):1465–1476, 2004.

A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*, volume 18 of *Mathematics and Its Applications*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1988.

A. Jadbabaie, J. Lin, and A. S. Morse. Coordination of groups of mobile autonomous agents using nearest neighbor rules. *IEEE Transactions on Automatic Control*, 48(6):988–1001, 2003.

L. Moreau. Stability of multiagent systems with time-dependent communication links. *IEEE Transactions on Automatic Control*, 50(2):169–182, 2005.

R. Olfati-Saber and R. M. Murray. Consensus problems in networks of agents with switching topology and time-delays. *IEEE Transactions on Automatic Control*, 49(9):1520–1533, 2004.

R. Olfati-Saber, J. A. Fax, and R. M. Murray. Consensus and cooperation in networked multi-agent systems. *IEEE Proceedings*, 95(1):215–233, 2007.

W. Ren and R. W. Beard. Consensus seeking in multi-agent systems under dynamically changing interaction topologies. *IEEE Transactions on Automatic Control*, 50(5):655–661, 2005.

W. Ren, R. W. Beard, and E. M. Atkins. Information consensus in multivehicle cooperative control: Collective group behavior through local interaction. *IEEE Control Systems Magazine*, 27(2):71–82, 2007.

H. G. Tanner, G. J. Pappas, and V. Kumar. Leader-to-formation stability. *IEEE Transactions on Robotics and Automation*, 20(3):443–455, 2004.

J. N. Tsitsiklis, D. P. Bertsekas, and M. Athans. Distributed asynchronous deterministic and stochastic gradient optimization algorithms. *IEEE Transactions on Automatic Control*, 31(9):803–812, 1986.