Control Algorithms along Relative Equilibria of Underactuated Lagrangian Systems on Lie Groups

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Abstract

We present novel algorithms to control underactuated mechanical systems. For a class of invariant systems on Lie groups, we design iterative small-amplitude control forces to accelerate along, decelerate along, and stabilize relative equilibria. The technical approach is based upon a perturbation analysis and the design of inversion primitives and composition methods. We illustrate the algorithms on an underactuated planar rigid body and on a satellite with two thrusters.

I. Introduction

In this paper we study control of underactuated mechanical systems on Lie groups. We focus on the particular class of motions called relative equilibria. A relative equilibrium is a motion for which the body-fixed velocity is constant while no control forces are applied; thus when referring to a relative equilibrium a specific body-fixed velocity is implied. Accelerating/decelerating along a relative equilibrium means increasing/decreasing the velocity in the direction of a relative equilibrium while the configuration behaves accordingly. We concentrate on the construction of small-amplitude control forces that, when used iteratively, result in a given acceleration/deceleration along a relative equilibrium; stabilization is achieved as zero acceleration. Perturbation analysis

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and Lie group theory play a crucial role in the analysis. Example systems to which the theory applies are a hovercraft, modeled as an underactuated planar rigid body, and a satellite with two thrusters.

The motivation for studying underactuated mechanical systems is twofold. First, control algorithms for underactuated systems enable more general control designs than those in fully actuated systems, e.g., less costly designs or lighter designs. Second, control algorithms for underactuated systems are applicable in the situation of an actuator failure and, therefore, they improve robustness of the control system; this robustness is crucial in case the vehicle is in a hazardous environment or is hardly accessible (e.g., a satellite).

A vast literature is available on mechanical control systems. Extensive research has focused on underactuated mechanical systems, especially in the context of controlled Lagrangians and Hamiltonians, e.g., see [1], [2] and subsequent works. Somehow less research is available for controlling systems along relative equilibria; a related spin-up problem is considered in [3], the theory of kinematic reductions is exposed in [4]. Since this document builds directly upon the work in [5] we refer the reader to that document for a literature survey relevant for control algorithms for underactuated Lagrangian systems on Lie groups. A generalization of the theory in [5] to a larger class of mechanical systems can be found in [6]. An advantage of our approach compared with implicit methods such as, e.g., the RRT search heuristic presented in [7], is that the controls are given by closed-form expressions. Therefore, only limited computational power is required—this is an appealing property when the controls are to be calculated on-board and weight and reliability are important design parameters.

As main contribution of this paper, we propose algorithms to compute small amplitude control forces that speed up, slow down, or stabilize, an underactuated system along a relative equilibrium. The resulting algorithm amounts to a repeated invocation of a motion primitive which, in turn, is composed of two control primitives in succession; these are denoted "inversion primitives" as they amount to local inversion algorithms for the "controls to state" maps. The main advantage of the proposed approach is its applicability to systems that are not linearly controllable; the main limitation is that part of the results are applicable only to n-dimensional systems with $(n-1)$ controls. We mention that algorithms to control motion along relative equilibria are not presented in [5] which focused on control algorithms at velocities near zero.

This paper is organized as follows. First, we review the mathematical model of simple mechanical control systems on Lie groups, as described in [4], and perform perturbation analysis for small amplitude forcing and initial velocity close to a relative equilibrium. Based on this analysis we construct two inversion primitives and combine them into a single motion primitive. After an application of the motion primitive the system has accelerated or decelerated along a relative equilibrium. Using this motion primitive iteratively we design an algorithm which gives a control that results in a given acceleration/deceleration along a relative equilibrium. We illustrate the approach by applying the algorithm numerically to an underactuated planar rigid body and the satellite with two thrusters, and we end the note by summarizing the results in a conclusion.

II. Mathematical model and perturbation analysis near a relative equilibrium

A simple mechanical control system on a Lie group is a mechanical system which has as configuration manifold an n dimensional Lie group G, with Lie algebra $\mathfrak{g}$, and Lagrangian equal to the kinetic energy which is defined by an inertia tensor $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$. We assume that G is a matrix Lie group with identity element id and adjoint map $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ associated to each $g \in G$. Such a system has dynamics given by

$$\dot{g} = g \cdot \xi, \tag{1}$$

$$\mathbb{I}\dot{\xi} = \mathrm{ad}^*_\xi \mathbb{I}\xi + \sum_{i=1}^{m} f_i u_i(t), \tag{2}$$

where $g \in G$ is the configuration, $\xi \in \mathfrak{g}$ is the body-fixed velocity, $\mathrm{ad}_\xi : \mathfrak{g} \to \mathfrak{g}$ is the adjoint operator and $\mathrm{ad}^*_\xi : \mathfrak{g}^* \to \mathfrak{g}^*$ its dual, $f_i \in \mathfrak{g}^*$ defines the ith body-fixed force, and $u : \mathbb{R} \to \mathbb{R}^m$ is bounded and measurable and gives the resultant force on the system according to $\sum_{i=1}^m f_i u_i(t)$. In what follows, $\Sigma = (G, \mathbb{I}, \{f_1, \ldots, f_m\})$ denotes this mechanical control system.

We define the symmetric product $\langle \cdot : \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by

$$\langle \xi : \eta \rangle := -\mathbb{I}^{-1}(\mathrm{ad}^*_\xi \mathbb{I}\eta + \mathrm{ad}^*_\eta \mathbb{I}\xi).$$

Defining $b_i := \mathbb{I}^{-1} f_i$, $i \in \{1, \ldots, m\}$, the dynamic equation (2) can be written as

$$\dot{\xi} = -\tfrac{1}{2}\langle \xi : \xi \rangle + \sum_{i=1}^{m} b_i u_i(t). \tag{3}$$

Remark 1 (Simplifying convention): It is well known that $\mathfrak{g}$ is an n-dimensional vector space. We make no distinction between $\mathfrak{g}$ and $\mathbb{R}^n$ in order to express a vector in $\mathfrak{g}$ as a column vector in $\mathbb{R}^n$ and in order to represent a linear map on $\mathfrak{g}$ as a matrix. Although we make this choice of notation, we shall be careful not to assume that the Lie algebra operation is commutative. •

A *relative equilibrium* for Σ is a curve $t \mapsto g_0 \exp(t\xi_{\text{re}}) \in G$, for $g_0 \in G$ and $\xi_{\text{re}} \in \mathbb{R}^n$, that is a solution to the dynamics (1), (2) for zero input u. It is easy to see that $t \mapsto g_0 \exp(t\xi_{\text{re}})$ is a relative equilibrium if and only if $\langle \xi_{\text{re}} : \xi_{\text{re}} \rangle = 0$. It is convenient to call relative equilibrium both the curve $t \mapsto g_0 \exp(t\xi_{\text{re}})$ and the vector ξ_{re}. Given a relative equilibrium ξ_{re}, we define the linear map $A_{\text{re}} : \mathbb{R}^n \to \mathbb{R}^n$ by $A_{\text{re}}\eta := -\langle \xi_{\text{re}} : \eta \rangle$.

Remark 2 (Time scaling): Let $\lambda > 0$ and $T > 0$ and define $\tau = t/\lambda$. If $(g(t), \xi(t))$ is a solution for $t \in [0, T]$ to (1)-(2) with control $u(t)$, then $(g(\tau/\lambda), \xi(\tau/\lambda)/\lambda)$ is a solution for $\tau \in [0, \lambda T]$ with control $u(\tau/\lambda)/\lambda^2$. In the following we choose $T = 2\pi$ for simplicity. •

We are interested in control signals $u \in C^0([0, 2\pi], \mathbb{R}^m)$ of the form

$$u(t) = \epsilon u^1(t) + \epsilon^2 u^2(t), \qquad 0 < \epsilon \ll 1,$$

where $u^i \in C^0([0, 2\pi], \mathbb{R}^m)$. Accordingly, we define $b^j(t) := \sum_{i=1}^{m} b_i u_i^j(t)$, $j \in \{1, 2\}$. In the perturbation analysis it will be convenient to define, for $f \in C^0([0, 2\pi], \mathbb{R}^n)$ and $\sigma \in \mathbb{R}$,

$$\overline{f}^{\sigma}(t) := \int_0^t e^{\sigma A_{\text{re}}(t-s)} f(s) \mathrm{d}s, \qquad \overline{f}(t) := \overline{f}^{0}(t).$$

In what follows, s and τ will be used as integration variables only.

Proposition 3 (Perturbation analysis): Let Σ be a mechanical control system on a Lie group with a relative equilibrium ξ_{re} and corresponding matrix A_{re}. For $0 < \epsilon \ll 1$ and $\sigma > 0$, let $[0, 2\pi] \ni t \mapsto (g(t), \xi(t))$ be the solution to (1) and (3) with $t \mapsto \sum_i^m b_i u_i(t) = \epsilon b^1(t) + \epsilon^2 b^2(t)$ and from initial velocity $\xi(0) = \sigma\xi_{\text{re}} + \epsilon^2 \xi_0^2$, for $\xi_0^2 = \mathcal{O}(1)$, and initial configuration $g(0) = \text{id}$. Let $h(t) := g(t) \cdot \exp(-t\sigma\xi_{\text{re}})$ and let $x(t) := \log(h(t))$ be the exponential coordinates of h. Then, for $t \in [0, 2\pi]$, it holds that $\xi(t, \epsilon) = \xi^0(t) + \epsilon\xi^1(t) + \epsilon^2\xi^2(t) + \mathcal{O}(\epsilon^3)$ with

$$\begin{aligned}
\xi^0(t) &= \sigma\xi_{\text{re}}, \\
\xi^1(t) &= \overline{b^1}^{\sigma}(t), \\
\xi^2(t) &= e^{\sigma A_{\text{re}} t}\xi_0^2 - \tfrac{1}{2}\overline{\langle \overline{b^1}^{\sigma} : \overline{b^1}^{\sigma} \rangle}^{\sigma}(t) + \overline{b^2}^{\sigma}(t),
\end{aligned}$$

and $x(t,\epsilon) = \epsilon x^1(t) + \epsilon^2 x^2(t) + \mathcal{O}(\epsilon^3)$ with

$$\begin{aligned} x^1(t) &= \overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(\overline{b^1}^{\sigma}(s))}(t), \\ x^2(t) &= \overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(e^{\sigma A_{\mathrm{re}} s}\xi_0^2)}(t) - \tfrac{1}{2}\overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(\overline{\langle \overline{b^1}^{\sigma} : \overline{b^1}^{\sigma}\rangle}^{\sigma}(s))}(t) \\ &\quad + \overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(\overline{b^2}^{\sigma}(s))}(t) - \tfrac{1}{2}\overline{[\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(\overline{b^1}^{\sigma}(s)), \overline{\mathrm{Ad}_{\exp(\tau\sigma\xi_{\mathrm{re}})}(\overline{b^1}^{\sigma}(\tau))}(s)]}(t). \end{aligned}$$

Proof: Since the input is analytic in ϵ so is the solution $\xi(t) = \sum_{j=0}^{+\infty} \epsilon^j \xi^j(t)$. Inserting the expansions for ξ into equation (3) and collecting terms of same order we compute

$$\begin{aligned} \dot{\xi}^0 &= -\tfrac{1}{2}\langle \xi^0 : \xi^0\rangle, \qquad \dot{\xi}^1 = -\langle \xi^0 : \xi^1\rangle + b^1(t), \\ \dot{\xi}^2 &= -\langle \xi^0 : \xi^2\rangle - \tfrac{1}{2}\langle \xi^1 : \xi^1\rangle + b^2(t). \end{aligned}$$

Inserting the initial condition then gives

$$\begin{aligned} \xi^0(t) &= \sigma\xi_{\mathrm{re}}, \qquad \xi^1(t) = \overline{b^1}^{\sigma}(t), \\ \xi^2(t) &= e^{\sigma A_{\mathrm{re}} t}\xi_0^2 - \tfrac{1}{2}\overline{\langle \xi^1 : \xi^1\rangle}^{\sigma}(t) + \overline{b^2}^{\sigma}(t) \\ &= e^{\sigma A_{\mathrm{re}} t}\xi_0^2 - \tfrac{1}{2}\overline{\langle \overline{b^1}^{\sigma} : \overline{b^1}^{\sigma}\rangle}^{\sigma}(t) + \overline{b^2}^{\sigma}(t). \end{aligned}$$

Since g is a solution to the kinematic equation (1), it follows that

$$\begin{aligned} \dot{h} &= \dot{g} \cdot \exp(-t\sigma\xi_{\mathrm{re}}) - g \cdot \exp(-t\sigma\xi_{\mathrm{re}}) \cdot \sigma\xi_{\mathrm{re}} = g \cdot \xi \cdot \exp(-t\sigma\xi_{\mathrm{re}}) - h \cdot \sigma\xi_{\mathrm{re}} \\ &= h \cdot (\exp(t\sigma\xi_{\mathrm{re}}) \cdot \xi \cdot \exp(-t\sigma\xi_{\mathrm{re}}) - \sigma\xi_{\mathrm{re}}) = h \cdot (\mathrm{Ad}_{\exp(t\sigma\xi_{\mathrm{re}})}(\xi) - \sigma\xi_{\mathrm{re}}) \\ &= h \cdot (\mathrm{Ad}_{\exp(t\sigma\xi_{\mathrm{re}})}(\sigma\xi_{\mathrm{re}} + \epsilon\xi^1 + \epsilon^2\xi^2 + \mathcal{O}(\epsilon^3)) - \sigma\xi_{\mathrm{re}}) = h \cdot \mathrm{Ad}_{\exp(t\sigma\xi_{\mathrm{re}})}(\epsilon\xi^1 + \epsilon^2\xi^2 + \mathcal{O}(\epsilon^3)). \end{aligned}$$

If we define $\zeta(t) := \mathrm{Ad}_{\exp(t\sigma\xi_{\mathrm{re}})}(\epsilon\xi^1 + \epsilon^2\xi^2 + \mathcal{O}(\epsilon^3))$, then we have, according to [8], that

$$x(t) = \overline{\zeta}(t) - \tfrac{1}{2}\overline{[\zeta, \overline{\zeta}]}(t) + \mathcal{O}(\epsilon^3). \tag{4}$$

Using $x = \epsilon x^1 + \epsilon^2 x^2 + \mathcal{O}(\epsilon^3)$ we achieve the result on x^1 and x^2 by inserting the expression for ζ into equation (4). ■

III. DESIGN: LOCAL INVERSION PRIMITIVES

In this section we construct two open-loop control primitives which act as inversion primitives. Later these will be combined into a single motion primitive which, in turn, will be used iteratively in a control algorithm.

For a mechanical control system $\Sigma = (G, \mathbb{I}, \{f_1, \dots, f_m\})$ with relative equilibrium ξ_{re} and corresponding matrix A_{re}, we present the following assumptions. First, we make the standing assumption that $\xi_{\text{re}} \notin \operatorname{span}\{b_1, \dots, b_m\}$, otherwise the theory of kinematic reductions [4] is readily applicable and the control problems we consider below are trivial.

Assumption 1 (Lack of linear controllability): The subspace $\operatorname{span}\{b_1, \dots, b_m\}$ is invariant under the linear map A_{re}, that is, $\langle \xi_{\text{re}} : b_i \rangle \in \operatorname{span}\{b_1, \dots, b_m\}$, for $i \in \{1, \dots, m\}$.

Assumption 2 (Nonlinear controllability): The subspace $\operatorname{span}\{b_i, \langle b_i : b_j \rangle \mid i, j \in \{1, \dots, m\}\}$ is full rank and $\langle b_i : b_i \rangle \in \operatorname{span}\{b_1, \dots, b_m\}$, for $i \in \{1, \dots, m\}$.

Assumption 3: $\langle \xi_{\text{re}} : \langle b_j : b_k \rangle \rangle \in \operatorname{span}\{b_1, \dots, b_m\}$, for $j, k \in \{1, \dots, m\}$ and $j \neq k$.

Assumption 4: The subspace $\operatorname{span}\{b_1, \dots, b_m\}$ is invariant under the linear map $\operatorname{ad}_{\xi_{\text{re}}}$.

Assumption 2 is the same controllability assumption adopted in [5]. If we define the matrix $B := [b_1, \dots, b_m] \in \mathbb{R}^{n \times m}$, then Assumption 1 is equivalent to the existence of a matrix $Q \in \mathbb{R}^{m \times m}$ such that $A_{\text{re}} B = BQ$, and in turn $e^{A_{\text{re}}} B = B e^{Q}$. Similarly, Assumption 4 is equivalent to the existence of a matrix $M \in \mathbb{R}^{m \times m}$ such that $\operatorname{ad}_{\xi_{\text{re}}} B = BM$.

Given $Q \in \mathbb{R}^{m \times m}$, define $F_Q : C^0([0, 2\pi], \mathbb{R}^m) \to \{f \in C^1([0, 2\pi], \mathbb{R}^m) \mid f(0) = 0\}$ by

$$F_Q[u](t) := \int_0^t e^{Q(t-s)} u(s) \mathrm{d}s.$$

Lemma 4 (Transformation of controls): The map F_Q is invertible and its inverse is given as follows: if $w = F_Q[u]$, then $u(t) = -Qw(t) + \dot{w}(t)$. Additionally, as in Assumption 1, let A_{re}, B and Q satisfy $A_{\text{re}} B = BQ$. If $u \in C^0([0, 2\pi], \mathbb{R}^m)$ and $w = F_{\sigma Q}[u]$, $\sigma \in \mathbb{R}$, then

$$\overline{Bu}^{\sigma}(t) = Bw(t).$$

Proof: One-to-one correspondence between u and w is readily checked. We compute $\overline{Bu}^{\sigma}(t) = \int_0^t e^{\sigma A_{\text{re}}(t-s)} Bu(s) \mathrm{d}s = B \int_0^t e^{\sigma Q(t-s)} u(s) \mathrm{d}s = Bw(t)$. ■

Definition 5 (Convenient forcing frequencies): Take $r = \lceil \frac{n}{m} \rceil$. For $(i, h) \in \{1, \dots, m\} \times \{1, \dots, r\}$, select numbers α_{ih} in the set $\{0, \dots, rm + \frac{1}{2} m(m-1)\}$ as follows:

$\mathcal{V} := \emptyset$; $\mathcal{I} := \{1, \dots, rm + \frac{1}{2} m(m-1)\}$
for $h \in \{1, \dots, r\}$ and **for** $i \in \{1, \dots, m\}$ **do**
  $\omega := \min(\mathcal{I})$; $v := \int_0^{2\pi} \operatorname{Ad}_{\exp(s\sigma\xi_{\text{re}})} b_i \sin(\omega s) \mathrm{d}s$
  if $v \in \operatorname{span}(\mathcal{V})$ **then** $\alpha_{ih} := 0$ **else** $\alpha_{ih} := \omega$; $\mathcal{I} := \mathcal{I} \setminus \{\omega\}$; $\mathcal{V} := \mathcal{V} \cup \{v\}$ **end if**
end for

Define the $n \times rm$ matrix

$$\mathcal{A}_{\sigma,\alpha} := \int_0^{2\pi} \mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(B[\mathrm{diag}(\sin(\alpha_{11}s), \dots, \sin(\alpha_{m1}s)), \dots, \mathrm{diag}(\sin(\alpha_{1r}s), \dots, \sin(\alpha_{mr}s))])\mathrm{d}s.$$

Next, for $(i,j) \in \{1,\dots,m\}^2$, select numbers β_{ij} as follows: for $i < j$ take $\beta_{ij} \in \{1,\dots,rm + \frac{1}{2}m(m-1)\} \setminus \{\alpha_{kh}\}_{(k,h)\in\{1,\dots,m\}\times\{1,\dots,r\}}$ all having distinct values, for $i > j$ take $\beta_{ij} = \beta_{ji}$, and for $i = j$ take $\beta_{ij} = 0$.

Remark 6: In other words, the numbers α_{ij} are selected sequentially in such a way as to maximize the rank of $\mathcal{A}_{\sigma,\alpha}$. Note that, for $i,j,k,l \in \{1,\dots,m\}$ and $h \in \{1,\dots,r\}$, we have: (i) all nonzero α_{ih} are distinct, (ii) all nonzero α_{ih} are distinct from all nonzero β_{jk}, and (iii) $\beta_{ij} = \beta_{kl}$ if and only if $(i,j) = (k,l)$ or $(i,j) = (l,k)$. •

Remark 7: The computations required by Definition 5 include checking that a vector belongs to a subspace. In practical numerical implementations it is sufficient to verify this condition up to a specified tolerance. It is convenient to choose this tolerance comparable with the accuracy of the control algorithms. •

For $Z \in \mathbb{R}^{m\times m}$ define $\lambda : \mathbb{R}^{m\times m} \to \mathbb{R}^{m\times m}$ by

$$\lambda_{jk}(Z) := \begin{cases} \mathrm{sign}(Z_{jk})\sqrt{|Z_{jk}|}, & j < k, \\ 0, & j = k, \\ \frac{1}{\pi}\sqrt{|Z_{kj}|}, & j > k. \end{cases}$$

We are now able to obtain the following result.

Proposition 8 (`speed_inversion` inversion primitive): Let Σ be a mechanical control system on a Lie group with a relative equilibrium ξ_{re} and corresponding matrix A_{re} and satisfying Assumptions 1, 2 and 3. Let $Q \in \mathbb{R}^{m\times m}$ satisfy $A_{\mathrm{re}}B = BQ$. Let $\eta \in \mathbb{R}^n$, $\sigma \in \mathbb{R}$, and compute $z \in \mathbb{R}^m$ and $Z \in \mathbb{R}^{m\times m}$ as the pseudoinverse solution to

$$\eta = \sum_{i=1}^{m} z_i b_i - \sum_{j=1}^{m-1}\sum_{k=j+1}^{m} Z_{jk}\langle b_j : b_k\rangle, \qquad Z_{jk} = 0 \text{ for } j \geq k.$$

Given r, α, $\mathcal{A}_{\sigma,\alpha}$, and β as in Definition 5, let

$$y_j(t) := \sum_{k=1}^{m} \lambda_{jk}(Z)\sin(\beta_{jk}t), \quad j \in \{1,\dots,m\},$$

and let $\gamma = (\gamma_{11},\dots,\gamma_{m1},\dots,\gamma_{1r},\dots,\gamma_{mr})^T$ be the unique solution to

$$\begin{aligned} \mathcal{A}_{\sigma,\alpha}\gamma &= -\overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(By(s))}(2\pi), \\ \gamma_{ih} &= 0 \ \text{ if } \ \alpha_{ih} = 0 \ \text{ for } \ (i,h) \in \{1,\dots,m\}\times\{1,\dots,r\}. \end{aligned} \tag{5}$$

Additionally, if we take

$$w_j^1(t) = y_j(t) + \sum_{l=1}^{r} \gamma_{jl} \sin(\alpha_{jl} t), \quad j \in \{1, \ldots, m\},$$
$$u^1(t) = F_{\sigma Q}^{-1}[w^1](t), \qquad u^2(t) = \tfrac{1}{2\pi} e^{\sigma Q(t-2\pi)}(\chi + z),$$

where $\chi \in \mathbb{R}^m$ is the unique solution to

$$B\chi = \sum_{j=1}^{m-1} \sum_{k=j+1}^{m} \int_0^{2\pi} (e^{\sigma A_{\mathrm{re}}(2\pi - s)} - I) w_j^1(s) w_k^1(s) \mathrm{d}s \, \langle b_j : b_k \rangle + \tfrac{1}{2} \sum_{i=1}^{m} \int_0^{2\pi} e^{\sigma A_{\mathrm{re}}(2\pi - s)} (w_i^1(s))^2 \mathrm{d}s \, \langle b_i : b_i \rangle, \quad (6)$$

then $b^1(t) = Bu^1(t)$ and $b^2(t) = Bu^2(t)$ satisfy

$$-\tfrac{1}{2} \overline{\langle \overline{b^1}^{\sigma} : \overline{b^1}^{\sigma} \rangle}^{\sigma}(2\pi) + \overline{b^2}^{\sigma}(2\pi) = \eta, \quad (7)$$
$$\overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(\overline{b^1}^{\sigma}(s))}(2\pi) = 0. \quad (8)$$

We call this inversion primitive $\texttt{speed\_inversion}(\sigma, \eta) = (b^1(t), b^2(t))$.

Proof: Existence and uniqueness of the solution to (6) is a consequence of Assumptions 3 and 2. Regarding existence and uniqueness of the solution to (5), Definition 5 ensures that

$$\overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(By(s))}(2\pi) \in \mathrm{Image}(\mathcal{A}_{\sigma,\alpha}).$$

Since every nonzero column in $\mathcal{A}_{\sigma,\alpha}$ contributes to the rank of $\mathcal{A}_{\sigma,\alpha}$, the entries of γ corresponding to these will be unique. The remaining γ-values are defined to be 0.

Regarding the proof of equation (8), direct calculations show that

$$\overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(\overline{b^1}^{\sigma}(s))}(2\pi) = \overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(Bw^1(s))}(2\pi) = \mathcal{A}_{\sigma,\alpha}\gamma + \overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(By(s))}(2\pi) = 0.$$

Regarding the proof of equation (7), from Lemma 4 we compute

$$\langle \overline{b}^{\sigma} : \overline{b}^{\sigma} \rangle(t) = \langle \sum_{j=1}^{m} w_j^1(t) b_j : \sum_{k=1}^{m} w_k^1(t) b_k \rangle$$
$$= 2 \sum_{j=1}^{m-1} \sum_{k=j+1}^{m} w_j^1(t) w_k^1(t) \langle b_j : b_k \rangle + \sum_{i=1}^{m} (w_i^1(t))^2 \langle b_i : b_i \rangle.$$

Since all nonzero α-values are distinct and are distinct from the β-values we have for $j < k$

$$\begin{aligned}\int_0^{2\pi} w_j^1(t) w_k^1(t)\mathrm{d}t &= \sum_{l,q=1}^{m} \lambda_{jl}(Z)\lambda_{kq}(Z)\int_0^{2\pi} \sin(\beta_{jl}t)\sin(\beta_{kq}t)\mathrm{d}t \\ &= \sum_{l,q=1}^{m} \lambda_{jl}(Z)\lambda_{kq}(Z)\delta_{\beta_{kq}}^{\beta_{jl}}\pi = \lambda_{jk}(Z)\lambda_{kj}(Z)\pi = Z_{jk}.\end{aligned}$$

By straightforward calculations we then obtain

$$\begin{aligned}&-\tfrac{1}{2}\overline{\langle \overline{b^1}^\sigma : \overline{b^1}^\sigma \rangle}^\sigma(2\pi) + \overline{b^2}^\sigma(2\pi) \\ &= -\tfrac{1}{2}\int_0^{2\pi} e^{\sigma A_{\mathrm{re}}(2\pi - s)}\langle \overline{b^1}^\sigma : \overline{b^1}^\sigma\rangle(s)\mathrm{d}s + B\int_0^{2\pi} e^{\sigma Q(2\pi-s)} u^2(s)\mathrm{d}s \\ &= -\sum_{j=1}^{m-1}\sum_{k=j+1}^{m}\Big(\int_0^{2\pi} w_j^1(s)w_k^1(s)\mathrm{d}s\langle b_j : b_k\rangle + \int_0^{2\pi}(e^{\sigma A_{\mathrm{re}}(2\pi-s)} - I)w_j^1(s)w_k^1(s)\mathrm{d}s\langle b_j : b_k\rangle\Big) \\ &\quad - \tfrac{1}{2}\sum_{j=1}^{m}\int_0^{2\pi} e^{\sigma A_{\mathrm{re}}(2\pi-s)}(w_j^1(s))^2\mathrm{d}s\langle b_j : b_j\rangle + \sum_{i=1}^{m}(\chi_i + z_i)b_i \\ &= -\sum_{j=1}^{m-1}\sum_{k=j+1}^{m} Z_{jk}\langle b_j : b_k\rangle + \sum_{i=1}^{m} z_i b_i = \eta.\end{aligned}$$

■

Remark 9: From the proof of Proposition 8 we see that Definition 5 ensures that

$$x^1(2\pi) = \overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(\overline{b^1}^\sigma(s))}(2\pi) = 0,$$

after an application of `speed_inversion`. Thus, using the controls given by `speed_inversion` the deviation in the configuration from the relative equilibrium is of order $\mathcal{O}(\epsilon^2)$. •

Proposition 10 (`configuration_inversion` *inversion primitive):* Let Σ be a mechanical control system on a Lie group with a relative equilibrium ξ_{re} and corresponding matrix A_{re} and satisfying Assumptions 1 and 4. Let $Q, M \in \mathbb{R}^{m\times m}$ satisfy $A_{\mathrm{re}}B = BQ$ and $\mathrm{ad}_{\xi_{\mathrm{re}}}B = BM$. If $\mu \in \mathbb{R}^m$, $\sigma \in \mathbb{R}$ and

$$\begin{aligned}u^1(t) &= 0, \\ u^2(t) &= F_{\sigma Q}^{-1}[w^2](t), \qquad w^2(t) = \tfrac{1}{\pi}e^{-\sigma M t}\mu\sin^2(t),\end{aligned}$$

then $b^1(t) = Bu^1(t)$ and $b^2(t) = Bu^2(t)$ satisfy

$$\begin{aligned}-\tfrac{1}{2}\overline{\langle \overline{b^1}^\sigma : \overline{b^1}^\sigma \rangle}^\sigma(2\pi) + \overline{b^2}^\sigma(2\pi) &= 0, \\ \overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(\overline{b^2}^\sigma(s))}(2\pi) &= B\mu.\end{aligned}$$

We denote this inversion primitive `configuration_inversion`$(\sigma, \mu) = (b^1(t), b^2(t)) = (0, b^2(t))$.

Proof: For $b^1(t) = 0$ we have, using Lemma 4 and $w^2(t) = \frac{1}{\pi} e^{-\sigma M t} \mu \sin^2(t)$, that

$$-\frac{1}{2}\overline{\langle \overline{b^1}^\sigma : \overline{b^1}^\sigma \rangle}^\sigma(2\pi) + \overline{b^2}^\sigma(2\pi) = \overline{b^2}^\sigma(2\pi) = Bw^2(2\pi) = 0.$$

Using Assumption 4 and Lemma 4 we compute

$$\begin{aligned}\overline{\mathrm{Ad}_{\exp(s\sigma\xi_{\mathrm{re}})}(\overline{b^2}^\sigma(s))}(2\pi) &= \overline{\exp(s\sigma\mathrm{ad}_{\xi_{\mathrm{re}}})(Bw^2(s))}(2\pi) = \overline{Be^{\sigma M s}w^2(s)}(2\pi) \\ &= \tfrac{1}{\pi}\overline{B\mu\sin^2(s)}(2\pi) = B\mu.\end{aligned}$$

■

IV. DESIGN: GLOBAL MOTION ALGORITHMS

In this section we combine the two inversion primitives constructed in the previous section into a single motion primitive used iteratively in a control algorithm to achieve speeding up or slowing down along a relative equilibrium.

The algorithm presented in this section requires the following additional assumption.

Assumption 5: The n dimensional system Σ has $n-1$ control forces, that is, $m = n-1$.

Remark 11: Assumption 5 together with the standing assumption $\xi_{\mathrm{re}} \notin \mathrm{span}\{b_1, \ldots, b_m\}$ implies $\mathbb{R}^n = \mathrm{span}\{b_1, \ldots, b_m, \xi_{\mathrm{re}}\}$. Additionally, one can verify that Assumptions 5 and 1 together imply Assumption 3. Assumption 4, which is needed for Proposition 10, can be weakened to assuming that $\mathrm{span}\{b_1, \ldots, b_m, \xi_{\mathrm{re}}\}$ is invariant under $\mathrm{ad}_{\xi_{\mathrm{re}}}$, a condition which is automatically satisfied under Assumption 5 and the standing assumption; see [9]. •

Define the projection operators $\mathcal{P}_B : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{P}_{\xi_{\mathrm{re}}} : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\mathcal{P}_{\xi_{\mathrm{re}}}(\nu) := (\nu \cdot \xi_{\mathrm{re}})\xi_{\mathrm{re}}, \qquad \mathcal{P}_B := \mathrm{id} - \mathcal{P}_{\xi_{\mathrm{re}}}.$$

where $\cdot$ is the dot product in $\mathbb{R}^n$ defined by requiring $\{b_1, \ldots, b_m, \xi_{\mathrm{re}}\}$ to be an orthonormal basis. Notice that, under Assumption 4, these projection operators commute with $\mathrm{ad}_{\xi_{\mathrm{re}}}$. This allows us to construct the following motion primitive.

Proposition 12 (`change_speed` motion primitive): Let Σ be a mechanical control system on a Lie group with a relative equilibrium ξ_{re} and corresponding matrix A_{re} and satisfying Assumptions 1, 2, 4, and 5. For $0 < \epsilon \ll 1$, assume that

$$g(0) = g_0 \exp(\epsilon^2 \nu_{\mathrm{error}}),$$

$$\xi(0) = \sigma\xi_{\mathrm{re}} + \epsilon^2 \xi_{\mathrm{error}},$$

for some $g_0 \in G$, $\sigma \in \mathbb{R}$, $\nu_{\text{error}}, \xi_{\text{error}} \in \mathbb{R}^n$ with $\nu_{\text{error}} = \mathcal{O}(1)$ and $\xi_{\text{error}} = \mathcal{O}(1)$. If we take $\rho \in \mathbb{R}$ and

$$(b^1(t), b^2(t)) = \begin{cases} \texttt{speed\_inversion}(\sigma, \rho\xi_{\text{re}} - e^{2\pi\sigma A_{\text{re}}}\xi_{\text{error}}), & t \in [0, 2\pi], \\ \texttt{configuration\_inversion}(\sigma, \mu), & t \in [2\pi, 4\pi], \end{cases}$$

$$B\mu = -\text{Ad}_{\exp(-2\pi\sigma\xi_{\text{re}})}\mathcal{P}_B\Big(\nu_{\text{error}} + \frac{1}{\epsilon^2}\log\big(g(0)^{-1}g(2\pi)\exp(-2\pi\sigma\xi_{\text{re}})\big)\Big),$$

then we obtain

$$g(4\pi) = g_0^* \exp(\epsilon^2 \nu_{\text{error}}^*),$$

$$\xi(4\pi) = (\sigma + \epsilon^2\rho)\xi_{\text{re}} + \epsilon^2 \xi_{\text{error}}^*,$$

for some $\nu_{\text{error}}^*, \xi_{\text{error}}^* \in \mathbb{R}^n$ with $\mathcal{P}_{\xi_{\text{re}}}(\nu_{\text{error}}^*) = \mathcal{O}(1)$, $\mathcal{P}_B(\nu_{\text{error}}^*) = \mathcal{O}(\epsilon)$, $\xi_{\text{error}}^* = \mathcal{O}(\epsilon)$ and for

$$g_0^* = g_0 \exp\Big((4\pi\sigma + 2\pi\epsilon^2\rho)\xi_{\text{re}} + \epsilon^2\mathcal{P}_{\xi_{\text{re}}}(\nu_{\text{error}})\Big).$$

We denote this control map by $(g_0^*, \nu_{\text{error}}^*, \sigma + \epsilon^2\rho, \xi_{\text{error}}^*) = \texttt{change\_speed}(g_0, \nu_{\text{error}}, \sigma, \xi_{\text{error}}, \rho)$.

Proof: Using Propositions 3 and 8 we compute

$$\xi(2\pi) = \sigma\xi_{\text{re}} + \epsilon^2\left(e^{\sigma A_{\text{re}}2\pi}\xi_{\text{error}} + \rho\xi_{\text{re}} - e^{\sigma A_{\text{re}}2\pi}\xi_{\text{error}}\right) + \mathcal{O}(\epsilon^3) = (\sigma + \rho\epsilon^2)\xi_{\text{re}} + \mathcal{O}(\epsilon^3),$$

and from this, Propositions 3 and 10 we have $\xi(4\pi) = (\sigma + \rho\epsilon^2)\xi_{\text{re}} + \mathcal{O}(\epsilon^3)$. Define $g_{0,1/2} := g_0 \exp\big((2\pi\sigma + \epsilon^2\tilde{\nu})\xi_{\text{re}}\big)$, $\tilde{\nu}\xi_{\text{re}} := \mathcal{P}_{\xi_{\text{re}}}(\nu_{\text{error}})$, and $\nu_B := \mathcal{P}_B(\nu_{\text{error}})$, then we achieve using Proposition 3 and the Campbell-Baker-Hausdorff formula

$$\begin{aligned} g_{0,1/2}^{-1}g(2\pi) &= \exp\big(-(2\pi\sigma + \epsilon^2\tilde{\nu})\xi_{\text{re}}\big)g_0^{-1}g(0)\exp(\epsilon^2x^2(2\pi) + \mathcal{O}(\epsilon^3))\exp(2\pi\sigma\xi_{\text{re}}) \\ &= \exp\big(-(2\pi\sigma + \epsilon^2\tilde{\nu})\xi_{\text{re}}\big)\exp(\epsilon^2(\tilde{\nu}\xi_{\text{re}} + \nu_B))\exp(\epsilon^2x^2(2\pi) + \mathcal{O}(\epsilon^3))\exp(2\pi\sigma\xi_{\text{re}}) \\ &= \exp(-2\pi\sigma\xi_{\text{re}})\exp(\epsilon^2\nu_B + \mathcal{O}(\epsilon^4))\exp(\epsilon^2x^2(2\pi) + \mathcal{O}(\epsilon^3))\exp(2\pi\sigma\xi_{\text{re}}) \\ &= \exp\Big(\epsilon^2\text{Ad}_{\exp(-2\pi\sigma\xi_{\text{re}})}(\nu_B + x^2(2\pi)) + \mathcal{O}(\epsilon^3)\Big). \end{aligned}$$

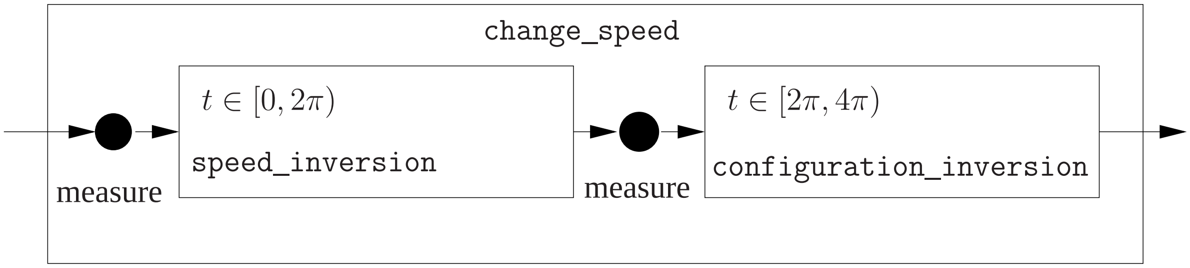

Fig. 1. Diagram of the `change_speed` motion primitive. The first measurement, of configuration and velocity, and the second measurement, of configuration alone, are represented by the black circles.

Using this, Propositions 3, 8, 10, and the Campbell-Baker-Hausdorff formula we obtain

$$
\begin{aligned}
g(4\pi) &= g(2\pi)\exp(\epsilon^2 B\mu + \mathcal{O}(\epsilon^3))\exp\big(2\pi(\sigma+\epsilon^2\rho)\xi_{\text{re}}\big) \\
&= g_0^* \exp\big(-2\pi(\sigma+\epsilon^2\rho)\xi_{\text{re}}\big) g_{0,1/2}^{-1} g(2\pi)\exp(\epsilon^2 B\mu + \mathcal{O}(\epsilon^3))\exp\big(2\pi(\sigma+\epsilon^2\rho)\xi_{\text{re}}\big) \\
&= g_0^* \exp\Big(\epsilon^2 \mathrm{Ad}_{\exp(-2\pi(\sigma+\epsilon^2\rho)\xi_{\text{re}})}\big(\mathrm{Ad}_{\exp(-2\pi\sigma\xi_{\text{re}})}(\nu_B + x^2(2\pi)) + B\mu\big) + \mathcal{O}(\epsilon^3)\Big) \\
&= g_0^* \exp\Big(\epsilon^2 \mathrm{Ad}_{\exp(-2\pi(\sigma+\epsilon^2\rho)\xi_{\text{re}})}\big(\mathrm{Ad}_{\exp(-2\pi\sigma\xi_{\text{re}})}\mathcal{P}_{\xi_{\text{re}}}(x^2(2\pi))\big) + \mathcal{O}(\epsilon^3)\Big) \\
&= g_0^* \exp\Big(\epsilon^2 \mathcal{P}_{\xi_{\text{re}}}(x^2(2\pi)) + \mathcal{O}(\epsilon^3)\Big).
\end{aligned}
$$

■

We illustrate the motion primitive `change_speed` in Fig. 1. With this motion primitive we are now able to construct the following control algorithm that speeds up, slows down, or stabilizes, a system along a relative equilibrium.

Proposition 13 (`speed_control` *algorithm):* Let Σ be a mechanical control system on a Lie group with a relative equilibrium ξ_{re} and corresponding matrix A_{re}. Assume Σ satisfies Assumptions 1, 2, 4, and 5 and take $0 < \epsilon \ll 1$. Let $g(0), g_0, \nu_{\text{error}}, \sigma, \xi_{\text{error}}, \rho$ be as in Proposition 12 and let $N \in \mathbb{N}$.

Define the algorithm $(g_0^*, \nu_{\text{error}}^*, \sigma+\epsilon^2 N\rho, \xi_{\text{error}}^*) = \texttt{speed\_control}(g_0, \nu_{\text{error}}, \sigma, \xi_{\text{error}}, \rho, N)$ by

$g_{0,1} := g_0$; $\nu_{\text{error},1} := \nu_{\text{error}}$; $\sigma_1 := \sigma$; $\xi_{\text{error},1} := \xi_{\text{error}}$;
for $k \in \{1, \ldots, N\}$ **do**
    $(g_{0,k+1}, \nu_{\text{error},k+1}, \sigma_{k+1}, \xi_{\text{error},k+1}) := \texttt{change\_speed}(g_{0,k}, \nu_{\text{error},k}, \sigma_k, \xi_{\text{error},k}, \rho)$
end for
$g_0^* = g_{0,N+1}$; $\nu_{\text{error}}^* := \nu_{\text{error},N+1}$; $\xi_{\text{error}}^* := \xi_{\text{error},N+1}$;

The final configuration and velocity after the execution of this algorithm are

$$g(N4\pi) = g_0^* \exp(\epsilon^2 \nu_{\text{error}}^*),$$
$$\xi(N4\pi) = (\sigma + \epsilon^2 N\rho)\xi_{\text{re}} + \epsilon^2 \xi_{\text{error}}^*,$$

where $\nu_{\text{error}}^*, \xi_{\text{error}}^* \in \mathbb{R}^n$, $\mathcal{P}_{\xi_{\text{re}}}(\nu_{\text{error}}^*) = \mathcal{O}(1)$, $\mathcal{P}_B(\nu_{\text{error}}^*) = \mathcal{O}(\epsilon)$, $\xi_{\text{error}}^* = \mathcal{O}(\epsilon)$, and

$$g_0^* = g_0 \exp\left(\left(\sigma T_{\text{final}} + \tfrac{1}{2}\rho\epsilon^2 N T_{\text{final}}\right)\xi_{\text{re}} + \epsilon^2 \sum_{k=1}^{N} \mathcal{P}_{\xi_{\text{re}}}(\nu_{\text{error},k})\right).$$

Proof: From Proposition 12 we have $\sigma_k = \sigma + (k-1)\rho\epsilon^2$ so we immediately obtain $\xi(N4\pi) = \sigma_{N+1}\xi_{\text{re}} + \mathcal{O}(\epsilon^3) = (\sigma + \epsilon^2 N\rho)\xi_{\text{re}} + \mathcal{O}(\epsilon^3)$. From Proposition 12 we have $g(N4\pi) = g_0^* \exp(\epsilon^2 \nu_{\text{error}}^*)$ where

$$\begin{aligned}
g_0^* &= g_0 \left(\prod_{k=1}^{N} \exp\left(2\pi(2\sigma_k + \rho\epsilon^2)\xi_{\text{re}} + \epsilon^2 \mathcal{P}_{\xi_{\text{re}}}(\nu_{\text{error},k})\right)\right) \\
&= g_0 \exp\left(\sum_{k=1}^{N} \left(2\pi(2\sigma_k + \rho\epsilon^2)\xi_{\text{re}} + \epsilon^2 \mathcal{P}_{\xi_{\text{re}}}(\nu_{\text{error},k})\right)\right) \\
&= g_0 \exp\left(2\pi N \left(2\sigma + N\rho\epsilon^2\right)\xi_{\text{re}} + \epsilon^2 \sum_{k=1}^{N} \mathcal{P}_{\xi_{\text{re}}}(\nu_{\text{error},k})\right) \\
&= g_0 \exp\left(\left(\sigma T_{\text{final}} + \tfrac{1}{2}\rho\epsilon^2 N T_{\text{final}}\right)\xi_{\text{re}} + \epsilon^2 \sum_{k=1}^{N} \mathcal{P}_{\xi_{\text{re}}}(\nu_{\text{error},k})\right).
\end{aligned}$$

From Proposition 12, its proof, and Proposition 3, we have that `change_speed` gives the map $(\xi_{\text{error},k}, \mathcal{P}_B(\nu_{\text{error},k}), \sigma) \mapsto (\xi_{\text{error},k+1}, \mathcal{P}_B(\nu_{\text{error},k+1}), \sigma + \epsilon^2\rho)$ independent of g_0 and $\mathcal{P}_{\xi_{\text{re}}}(\nu_{\text{error},k})$. Because $(\xi_{\text{error},k}, \mathcal{P}_B(\nu_{\text{error},k})) = \mathcal{O}(1)$ gives $(\xi_{\text{error},k+1}, \mathcal{P}_B(\nu_{\text{error},k+1})) = \mathcal{O}(\epsilon)$ we obtain that $\mathcal{P}_B(\nu_{\text{error},k}) = \mathcal{O}(\epsilon, k) = \mathcal{O}(\epsilon)$, $\mathcal{P}_{\xi_{\text{re}}}(\nu_{\text{error},k}) = \mathcal{O}(1, k) = \mathcal{O}(1)$, and $\xi_{\text{error},k} = \mathcal{O}(\epsilon, k) = \mathcal{O}(\epsilon)$. ■

Note that $\rho > 0$ speeds up the system along the relative equilibrium, $\rho < 0$ slows down the system, and $\rho = 0$ stabilizes the system's motion along the relative equilibrium. We may select $N = \mathcal{O}(\frac{1}{\epsilon^2})$ in Proposition 13 so that the absolute change in velocity along the relative equilibrium is of order $\mathcal{O}(1)$. Thus, it is possible to use the algorithm `speed_control` to change the velocity along the relative equilibrium from a given value to another independent of ϵ.

In summary, the algorithm `speed_control` consists of the repeated use of the `change_speed` motion primitive which, in turn, invokes the two inversion primitives `speed_inversion` and `configuration_inversion` in succession.

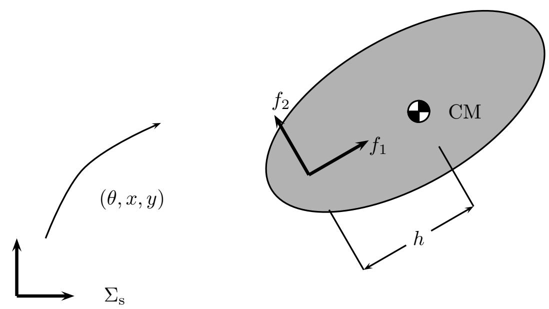

Fig. 2. The planar rigid body with two forces applied at a point a distance h from the center of mass CM. Σ_s denotes an inertial reference frame. (θ, x, y) are coordinates for the configuration of the body. The body reference frame (not depicted) is aligned with the direction of application of f_1 and f_2.

V. Examples

The usefulness of the theory is illustrated in the following examples.

Example 14 (Planar rigid body): Consider a rigid body moving in the plane as described in [5]. The configuration manifold is $G = SE(2)$ with local coordinates (θ, x, y). Let m denote the mass of the body, J its moment of inertia and h the distance from the center of mass to the control forces. For $(\omega, v_1, v_2)^T \in \mathbb{R}^3$ we have that the adjoint operator is given by $\mathrm{ad}_{(\omega,v_1,v_2)^T} = \begin{bmatrix} 0 & 0 & 0 \\ v_2 & 0 & -\omega \\ -v_1 & \omega & 0 \end{bmatrix}$. The inertia tensor has the representation $\mathbb{I} = \mathrm{diag}(J, m, m)$. With controls as in Fig. 2 we have $b_1 = \frac{1}{m} e_2$ and $b_2 = -\frac{h}{J} e_1 + \frac{1}{m} e_3$, which gives $\langle b_1 : b_1 \rangle = 0$, $\langle b_2 : b_2 \rangle = \frac{2h}{Jm} e_2$, and $\langle b_1 : b_2 \rangle = -\frac{h}{Jm} e_3$. Assumption 2 is immediately seen to be satisfied. Choosing the relative equilibrium $\xi_{\mathrm{re}} = e_3$ we have $A_{\mathrm{re}} = \mathrm{ad}_{\xi_{\mathrm{re}}} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ so Assumptions 3 and 4 are met. It is straightforward to calculate that $A_{\mathrm{re}} B = BQ$, with $Q = -\frac{hm}{J} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, so Assumption 1 is satisfied.

The γ-values can be calculated using Definition 5 to be $\alpha_{11} = \alpha_{12} = \alpha_{22} = 0$, $\alpha_{21} = 1$, $\gamma_{11} = \gamma_{12} = \gamma_{22} = 0$, and $\gamma_{21} = -\alpha_{21} \lambda_{21}(Z)/\beta$, where $\beta \in \{2, 3, 4, 5\}$. Finally, the components of χ are found to be $\chi_1 = \pi h (\lambda_{21}(Z)^2 + \gamma_{21}^2)/J$ and $\chi_2 = 0$.

Assumption 5 is immediately seen to be satisfied, so all the assumptions are met, and therefore we can apply the `speed_control` algorithm to speed up the system along e_3. The result of the `speed_control` algorithm applied to the planar rigid body can be seen in Fig. 3.

Example 15 (Satellite with two thrusters): Consider a satellite with two thrusters aligned with the first and second principal axes. The configuration manifold is $G = SO(3)$ and the equations of motion are of the form (1) and (3) where the symmetric product is given by $\langle \xi : \eta \rangle =$

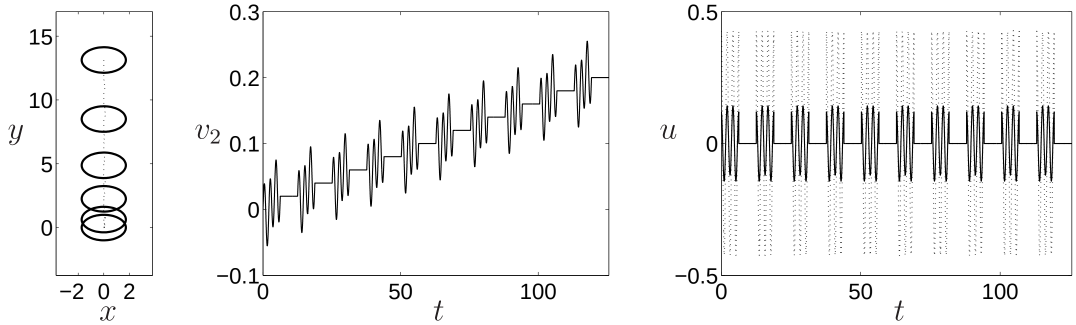

Fig. 3. `speed_control` applied to the planar rigid body with $\xi_{\text{re}} = e_3$, $\epsilon = 0.1$, and $\rho = 2$ and with initial conditions $(\theta, x, y)(0) = 0$, $g_0 = g(0)$, and $(\omega, v_1, v_2)(0) = 0$. The dotted curve in the left figure corresponds to the motion of the center of mass and the ellipses corresponds to the planar body at time equidistant instances. In the right figure the dashed curve corresponds to $u_1(t)$ and the solid curve corresponds to $u_2(t)$.

$\mathbb{I}^{-1}\big(\xi \times (\mathbb{I}\eta) + \eta \times (\mathbb{I}\xi)\big)$, where $\mathbb{I} = \operatorname{diag}(J_1, J_2, J_3)$, J_i being the moment of inertia along the ith principal axis, and $\times$ is the cross product. We have that $\langle e_3 : e_3 \rangle = 0$, so e_3 is a relative equilibrium, and since $b_1 = \frac{1}{J_1} e_1$ and $b_2 = \frac{1}{J_2} e_2$ it is not possible to directly control the motion in the e_3 direction. With $\xi_{\text{re}} = e_3$ we compute $A_{\text{re}} = \left[\begin{smallmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right]$ where $a_{12} = \frac{J_2 - J_3}{J_1}$ and $a_{21} = \frac{J_3 - J_1}{J_2}$. It is straightforward to calculate that $A_{\text{re}} B = BQ$, with $Q = \left[\begin{smallmatrix} 0 & \frac{J_2 - J_3}{J_2} \\ \frac{J_3 - J_1}{J_1} & 0 \end{smallmatrix}\right]$, so Assumption 1 is satisfied. From $\langle b_1 : b_1 \rangle = \langle b_2 : b_2 \rangle = 0$ and $\langle b_1 : b_2 \rangle = \frac{J_2 - J_1}{J_1 J_2 J_3} e_3$ we see that Assumption 2 is fulfilled if $J_1 \neq J_2$. Assumption 3 is satisfied because $\langle e_3 : \langle b_1 : b_2 \rangle \rangle = \frac{J_2 - J_1}{J_1 J_2 J_3} \langle e_3 : e_3 \rangle = 0$. Since $\operatorname{ad}_\xi \eta = \xi \times \eta$ we see that also Assumption 4 is satisfied. Assumption 5 is immediately seen to be met. Thus, if $J_1 \neq J_2$ in a satellite with thrusters along the first and second principal axis, then the theory presented in this paper can be used to speed up the satellite along the third (un-actuated) principal axis. The result of the `speed_control` algorithm applied to this example can be seen in Fig. 4.

VI. Conclusion

In this note we have designed a motion control algorithm suitable for a class of invariant mechanical systems on Lie groups. Using small-amplitude control forces the algorithm solves the tasks of accelerating along, decelerating along, and stabilizing relative equilibria. The algorithm has been applied numerically to two example systems to illustrate the theory.

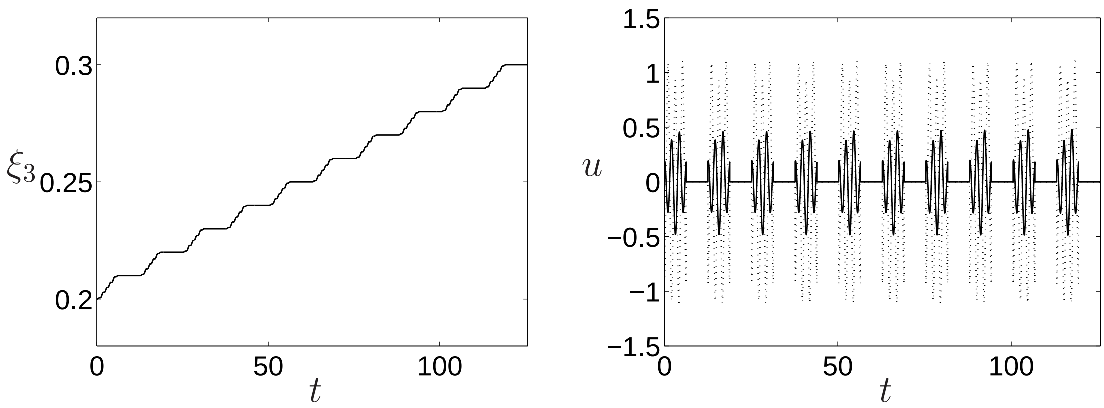

Fig. 4. `speed_control` applied to the satellite with two thrusters with $\xi_{\rm re} = e_3$, $\epsilon = 0.1$, and $\rho = 1$ and with initial conditions $\xi(0) = (0, 0, 0.2)$ and $g_0 = g(0)$. In the right figure the dashed curve corresponds to $u_1(t)$ and the solid curve corresponds to $u_2(t)$.

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