Sensing limitations in the Lion and Man problem

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***Abstract*— We address the discrete-time Lion and Man problem in a bounded, convex, planar environment in which both players have identical sensing ranges, restricted to closed discs about their current locations. The evader is randomly located inside the environment and moves only when detected. The players can step inside identical closed discs, centered at their respective positions. We propose a *sweep-pursuit-capture* strategy for the pursuer to capture the evader. The *sweep* phase is a search operation by the pursuer to detect an evader within its sensing radius. In the *pursuit* phase, the pursuer employs a *greedy strategy* of moving to the *last-sensed* evader position. We show that in finite time, the problem reduces to a previously-studied problem with unlimited sensing, which allows us to use the established *Lion strategy* in the *capture* phase. We give a novel upper bound on the time required for the pursuit phase to terminate using the greedy strategy and a sufficient condition for this strategy to work in terms of the value of the ratio of sensing to stepping radius of the players.**

I. Introduction

The classical Lion-Man problem is a game posed as to determine a strategy for a pursuer(lion) to capture an evader(man) in a given environment. By *capture*, we mean that the evader and the pursuer at the same position after a finite time. The aim of the pursuer is to capture the evader for any evader trajectory. The evader wins the game if it can avoid capture indefinitely. Both the players have identical motion capabilities. An important application of this problem is in surveillance of robotic networks. It is also an interesting case-study, instructive on its own right.

A. Contributions

We address the case of limited sensing capability: the pursuer and the evader can sense each other's position only if the distance between them is less than or equal to a given sensing radius. The motion of both players is restricted to closed discs of given stepping radius, centered at their respective current positions. The game is played in a bounded, convex, planar environment which is assumed to be known to both players. The evader is at an arbitrary location inside the environment, at the start of the game. It follows a *reactive rabbit* model, i.e., does not move until it senses a pursuer [1]. The pursuer *sweeps* the environment in a definite path until the evader is sensed, which must necessarily happen in finite time. We then establish how a natural *greedy strategy* of moving to the *last-sensed* location of the evader, reduces the problem to the previously-studied problem with unlimited sensing. The analysis allows us to give a novel upper bound on the time required for the pursuit phase to terminate. Further, we give a sufficient condition on the value of the ratio of sensing to stepping radius κ of the players, so that capture takes place in a given finite, convex environment. Finally, we demonstrate using an example that for sufficiently small κ, there exists a condition on the size of the environment that guarantees escape for the evader against the greedy strategy of the pursuer.

B. Related Work

The continuous time version of this problem has been studied by Ho *et al.* [2], Lim *et al.* [3] and Pachter [4] to cite a few. Recently, significant attention has been received by the discrete-time version of the game. The paper by Sgall [5] gives sufficient conditions for a single pursuer to capture an evader in a semi-open environment. This strategy has been extended by Kopparty and Ravishankar [6] to the case of multiple pursuers,in an unbounded environment, to capture a single evader initially located inside their convex hull. Recently, Alexander *et al.* [7] proposed a simple *greedy* strategy in which the pursuer moves towards the last position of the evader and characterize environments in which the strategy is guaranteed to work. Our analysis gives upper bounds on the time in all three phases of our strategy. We also provide an improved range of values for κ that ensures capture, compared to the earlier known results on the strategy. The game has also been studied in different types of bounded environments, e.g., circular environment by Alonso *et al.* [8], curved environments by LaValle *et al.* [9]. Visibility-based pursuit evasion has been studied by Guibas *et al.* [10], Sachs *et al.* [11] and in polygonal environments by Isler *et al.* [12].

Each of above mentioned works proposes strategies which require that the pursuers have unlimited sensing capacity. In this context, Gerkey *et al.* [13] have studied a version of visibility limited to an angle, instead of the entire region. This is termed as *searching using a flashlight*. Suzuki and Yamashita have studied visibility limited to k-searchers, where the pursuer has k such finite angle search flashlights [14]. Isler *et al.* [1] have considered the problem on a graph, with the visibility of the pursuer limited to nodes adjacent to the current node of the pursuer. A framework which uses probabilistic models for sensing devices for the agents can be found in the works of Hespanha *et al.* [15] and Vidal *et al.* [16]. Our model for limited range sensing is novel and is the first of its kind in pursuit-evasion literature, to the best of our knowledge.

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C. Organization

The problem formulation is described in section II. The pursuer strategy is described using three phases given in sections III-A, III-B and III-C, with section III-A describing the sweep strategy, section III-B discussing the aspects of the greedy strategy and finally section III-C showing the application of the Lion strategy [5] for completing the capture.

II. Problem Set-up

We assume that initially the evader is arbitrarily located inside a bounded, convex, planar environment, $\mathcal{Q} \subset \mathbb{R}^2$. We assume a discrete-time model with alternate motion of the evader and the pursuer. Both players know the entire environment. Define $\mathcal{Q}_\text{e} = \mathcal{Q} \cup \phi$, where $\mathcal{Q}$ is the environment and ϕ is the null element. Here, the null element refers to the fact that during sensing, the measurement of the position of an evader may not be available to the pursuer. Let $e[t]$ and $p[t]$ denote the absolute positions of the evader and the pursuer respectively, at time t. The equations of motion, in discrete-time, can be written as,

$$\begin{aligned} e[t+1] &= e[t] + u^e(e[t], y^p[t]), \\ p[t+1] &= p[t] + u^p(e[t], y^e[t+1], p[t]), \end{aligned}$$

where $y^p[t] \in \mathcal{Q}_\text{e}$ is the measurement of the pursuer position taken by the evader at the t^{th} time instant and $y^e[t+1] \in \mathcal{Q}_\text{e}$ is the measurement of the evader position taken by the pursuer at the $(t+1)^{th}$ time instant. We assume that the players can sense each other only if the distance between them is less than or equal to the sensing radius r_sens. Thus,

$$y^p[t] = \begin{cases} p[t], & \text{if } \|p[t] - e[t]\| \le r_\text{sens}, \\ \phi, & \text{otherwise.} \end{cases}$$

Similarly,

$$y^e[t+1] = \begin{cases} e[t+1], & \text{if } \|p[t] - e[t+1]\| \le r_\text{sens}, \\ \phi, & \text{otherwise.} \end{cases}$$

The functions $u^e : \mathcal{Q}_\text{e} \times \mathcal{Q} \to \mathcal{Q}$ and $u^p : \mathcal{Q}_\text{e} \times \mathcal{Q}_\text{e} \times \mathcal{Q} \to \mathcal{Q}$ are termed as *strategies* for the evader and pursuer respectively. We assume that both players can move with a maximum step size of r_step. This gives,

$$\|u^e\|, \|u^p\| \le r_\text{step}.$$

The sensing radius, r_sens, is κ times the motion radius, r_step. Throughout this paper, we assume κ is greater than 1, i.e., both players can sense further than they can move. From the *reactive rabbit* model for the evader, we have $u^e = 0$ until the evader is detected. After this happens, the problem is to determine u^p that guarantees capture for any evader strategy, u^e.

III. The Sweep-Pursuit-Capture Strategy

We establish sufficient conditions on the parameter κ so that the pursuer can capture the evader. We formally define two notions of *capture* as follows.

Definition III.1 (Capture notions) *The pursuer is said to have* captured *the evader if after finite time, independent of any evader policy u^e, the pursuer is at the same position as the evader.*

Similarly, the pursuer is said to have trapped *the evader within its sensing radius if after a finite time, independent of any evader policy u^e, the motion circle of the evader is completely contained within the sensing circle of the pursuer. After sensing the evader, the* trapping time *t^* is defined as the time taken by the pursuer to trap the evader within its sensing radius.*

We now describe the *sweep-pursuit-capture* strategy.

A. The Sweep phase

Let $\operatorname{diam}(\mathcal{Q})$ denote the diameter of $\mathcal{Q}$. The pursuer moves along a path with maximum step size such that the union of the sensing discs of the pursuer at the end of each step contains $\mathcal{Q}$. We term such a path as a *sweeping path* for $\mathcal{Q}$. Consider placing $\mathcal{Q}$ inside a square region of length $\operatorname{diam}(\mathcal{Q})$ and the pursuer moving along a sweeping path for the square region, as shown in Figure 1. The sweeping path is between strips of width $2r_\text{step}\sqrt{\kappa^2 - \frac{1}{4}}$, parallel to the side. There would be $\left\lceil \frac{\operatorname{diam}(\mathcal{Q})}{2r_\text{step}\sqrt{\kappa^2 - \frac{1}{4}}} \right\rceil$ such strips and it takes at most $\left\lceil \frac{\operatorname{diam}(\mathcal{Q})}{r_\text{step}} \right\rceil + \left\lceil \sqrt{\kappa^2 - \frac{1}{4}} \right\rceil$ to sweep one strip completely and position the pursuer at the start of a new strip. We obtain the following result.

Lemma III.2 *Along a sweeping path, the pursuer senses the evader in at most $\left\lceil \frac{\operatorname{diam}(\mathcal{Q})}{2r_\text{step}\sqrt{\kappa^2 - \frac{1}{4}}} \right\rceil \left(\left\lceil \frac{\operatorname{diam}(\mathcal{Q})}{r_\text{step}} \right\rceil + \left\lceil \sqrt{\kappa^2 - \frac{1}{4}} \right\rceil \right)$ time steps.*

Fig. 1. A sweeping path

This phase ends when the pursuer senses the evader.

B. The Pursuit phase

Once the pursuer has sensed the evader (and vice-versa), the evader needs to move in such a way that its new position is not within the current sensing radius of the pursuer. Otherwise the pursuer can move towards the new position of the evader, with maximum step and thus trap it within its sensing disc. We now propose a greedy policy for the pursuer according to which the pursuer must move towards the last sensed position of the evader so as to ensure that it would sense the evader again. This *greedy* strategy applies very naturally in our problem set-up. We define it formally using the following control input for the pursuer,

$$u^p_{\text{greedy}} = \begin{cases} r_{\text{step}} \operatorname{vers}(y^e[t+1] - p[t]), & \text{if } y^e[t+1] \neq \phi, \\ r_{\text{step}} \operatorname{vers}(e[t] - p[t]), & \text{otherwise.} \end{cases}$$

where,

$$\operatorname{vers}(v) = \begin{cases} \frac{v}{\|v\|}, & \text{if } v \neq 0, \\ 0, & \text{if } v = 0. \end{cases}$$

We now present our main result.

Theorem III.3 (Greedy Pursuit) *After sensing the evader, the pursuer will trap it within its sensing radius using the* greedy strategy *if*

$$\kappa \in (\sqrt{2 + 2\cos\beta_{\text{c}}}, \infty),$$

and the trapping time t^* satisfies,

$$t^* \leq \left(\left\lceil \frac{\log\left(\frac{\kappa - 1}{\sqrt{\kappa^2 - \sin^2\beta_{\text{c}}} - \cos\beta_{\text{c}} - 1} \right)}{\log \frac{1}{\lambda}} \right\rceil + 1 \right) N^*, \qquad (1)$$

where

$$\beta_{\text{c}} = \frac{1}{\left\lceil \frac{\operatorname{diam}(\mathcal{Q})}{\frac{\sqrt{3}}{2} r_{\text{step}}} \right\rceil} \tan^{-1}\left(\frac{1}{4} \frac{r_{\text{step}}}{\operatorname{diam}(\mathcal{Q})} \right),$$

$$\lambda = 1 - \frac{1 - \cos\beta_{\text{c}}}{\kappa}, \text{ and}$$

$$N^* = \left\lceil \frac{\operatorname{diam}(\mathcal{Q})}{\frac{\sqrt{3}}{2} r_{\text{step}}} \right\rceil.$$

To prove Theorem III.3, we need some preliminary definitions and results which we present now.

Definition III.4 (Deviation and Evasion angles)

Angle of deviation $\alpha[t] \triangleq \angle e[t+1]p[t+1]e[t]$,

Angle of evasion $\beta[t] \triangleq \alpha[t] + \angle p[t+1]e[t+1]e[t]$,

where the notation $\angle ABC$ refers to the angle between segments AB and BC. These angles are illustrated in Figure 2. We have the following result.

Proposition III.5 *When the pursuer uses the greedy strategy, for every instant of time t,*

$$|\beta[t]| \geq |\alpha[t]|. \qquad (2)$$

Note that equality in Equation (2) only holds when the evader moves along the line $\overline{p[t]e[t]}$.

It can be deduced that when the pursuer employs the *greedy strategy*, the distance between the pursuer and evader is non-increasing.

Definition III.6 (Cone) *A sequence of cones $\mathcal{C}_k$ for $k \in \mathbb{Z}_{\geq 0}$ are defined as follows:*

(i) *Let t_0 denote the time at the end of the sweep phase. Define cone $\mathcal{C}_0$ with $p[t_0]$ as its vertex and the line $\overline{p[t_0]e[t_0]}$ as the angle bisector, extended to meet the $\mathcal{Q}$ at point X as shown in Figure 3. Let $\overline{YZ}$ be of length $\frac{r_{\text{step}}}{2}$ and perpendicular to $\overline{p[t_0]e[t_0]}$ at X and with X as its midpoint. $\angle Y p[t_0] Z$ is called the* cone angle. *The cone is fixed as long as the evader is in the interior of the cone.*

(ii) *For $k > 0$, let t_k denote the time at which the evader steps out of the $(k-1)^{th}$ cone. Construct $\mathcal{C}_k$ analogous to part (i) of this definition by replacing $p[t_0]$ by $p[t_k]$ and $e[t_0]$ by $e[t_k]$. Once the evader leaves the cone, a new cone is constructed which has the same properties as described in part (i).*

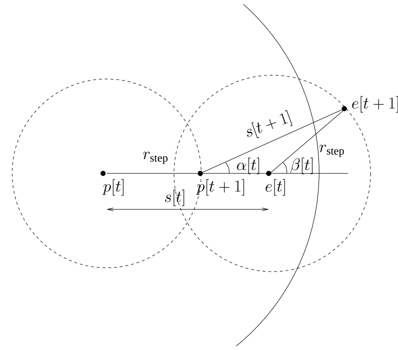

Fig. 2. Relation between angle of deviation and evasion angle

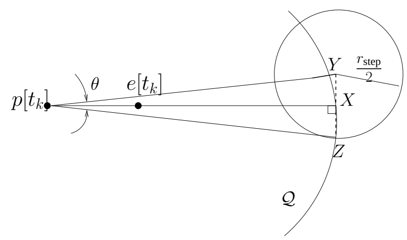

Fig. 3. Construction of cone $\mathcal{C}_k$.

The cone described above can be shown to have some useful properties such as,

Proposition III.7 (Cone properties) (i) *There exists a positive angle θ less than or equal to any cone angle.*

(ii) *The number of steps, N^*, for which the evader can remain inside the cone without being captured, is upper-bounded by,*

$$N^* \leq \left\lceil \frac{\operatorname{diam}(\mathcal{Q})}{\frac{\sqrt{3}}{2} r_{\text{step}}} \right\rceil. \qquad (3)$$

Proof:

(i) The first claim follows from the fact that the region is bounded has a finite diameter, $\operatorname{diam}(\mathcal{Q})$. Hence,

$$\min_{p[0], e[0] \in \mathcal{Q}} \theta = 2\tan^{-1}\left(\frac{r_{\text{step}}}{4\operatorname{diam}(\mathcal{Q})} \right).$$

(ii) As the pursuer moves in the cone, its step radius always divides the cone into 2 disjoint regions. So, the evader cannot go from one disjoint region into the other as it will have to move into the step radius of the pursuer, in which case the problem is over. We claim that the worst time path for the pursuer inside the cone, with maximum step size at each time instant, is as shown in Figure 4, where the length of each dotted segment is r_{step}. The motion disc is never tangent to any boundary of the cone and hence after a finite number of steps, the pursuer will sweep the entire cone. This can be seen by comparing the path in Figure 4 to the path in Figure 5, where we consider a path inside a rectangle, which is clearly greater than the path in the cone. The length L of the rectangle is at most equal to $\operatorname{diam}(\mathcal{Q})$. Thus, equation (3) follows. ■

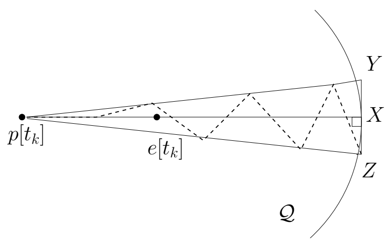

Fig. 4. A maximum length path inside a cone

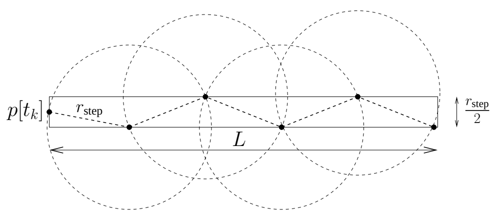

Fig. 5. Upper bound on the number of steps inside a cone for the pursuer

We now state two key results which would be used shortly.

Lemma III.8 (Maximum evasion angle) *While the pursuer employs the greedy policy, the maximum value of the evasion angle, β_{max}, for the evader without stepping inside the pursuer's sensing disc is given by,*

$$\beta_{\text{max}} = \cos^{-1}\left(\frac{(\kappa^2 - 1)r_{\text{step}}^2 - s^2[t]}{2s[t]r_{\text{step}}}\right), \tag{4}$$

where $s[t] = \|p[t] - e[t]\|$.

This can be seen by applying cosine rule to $\triangle p[t]e[t]e[t+1]$, shown in Figure 6. The notation $\triangle ABC$ stands for triangle formed by points A, B and C.

Lemma III.9 (Constraint on maximum evasion angle) *For the evader to move out of any cone, described in Definition III.6, the maximum evasion angle, β_{max}, must satisfy,*

$$|\beta_{\text{max}}| > \frac{\theta}{2N^*} \triangleq \beta_{\text{c}}, \tag{5}$$

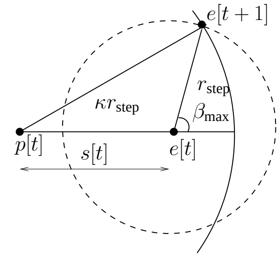

Fig. 6. Constraint on maximum evasion angle

where θ and N^* are given in Proposition III.7

Proof: For the evader to step out of cone $\mathcal{C}_k$, the sum of the angles of deviation for the pursuer must satisfy,

$$\sum_{t=t_k}^{t_{k+1}} |\alpha[t]| > \frac{\theta}{2}.$$

This is illustrated in Figure 7. From Proposition III.5, we have,

$$\sum_{t=t_k}^{t_{k+1}} |\beta[t]| > \frac{\theta}{2}.$$

Equation (5) now follows from using the lower bound on θ, derived in part 1 of Proposition III.7 and with N^* as the upper bound on the number of steps for the evader to remain in the cone without being captured, as derived in part 2 of Proposition III.7. ■

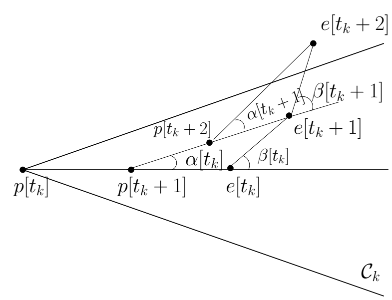

Fig. 7. Illustrating Lemma III.9

We are now ready to prove Theorem III.3.

Proof of Theorem III.3: Two cases need to be considered:

(i) *Evader does not move out of a cone*: The cone has been so defined that the stepping disc of the pursuer sweeps through the cone and the stepping disc of the evader falls completely in the sensing disc of the pursuer, in finite time, as a result of part 2 of Proposition III.7. Thus, the result follows.

(ii) *Evader moves out of a cone*: In this case, we seek to show that the evader cannot leave an arbitrarily large number of cones. If the evader steps outside the cone $\mathcal{C}_k$, then for some $\tau \in \{t_k, \ldots, t_{k+1} - 1\}, \beta[\tau] > \beta_{\text{c}}$. Applying cosine rule to $\triangle p[\tau]e[\tau]e[\tau+1]$, we obtain,

$$\begin{aligned} s^2[\tau+1] &= r_{\text{step}}^2 + (s[\tau] - r_{\text{step}})^2 \\ &\quad + 2r_{\text{step}}(s[\tau] - r_{\text{step}})\cos\beta[\tau], \\ \Rightarrow s^2[\tau] - s^2[\tau+1] &= 2r_{\text{step}}(s[\tau] - r_{\text{step}})(1 - \cos\beta[\tau]). \end{aligned}$$

Using Equation (5) and the fact that,

$$s[\tau] + s[\tau+1] \leq 2\kappa r_{\text{step}},$$

we obtain,

$$s[\tau+1]-r_{\text{step}} \leq \left(1 - \frac{(1-\cos(\frac{\theta}{2N^*}))}{\kappa}\right)(s[\tau]-r_{\text{step}}). \tag{6}$$

Define $\chi_k = s[t_k] - r_{\text{step}}$. Thus,

$$\begin{aligned} \chi_{k+1} &\leq s[\tau+1] - r_{\text{step}}, \\ &\leq \left(1 - \frac{(1-\cos(\frac{\theta}{2N^*}))}{\kappa}\right)(s[\tau] - r_{\text{step}}), \\ &\leq \left(1 - \frac{(1-\cos(\frac{\theta}{2N^*}))}{\kappa}\right)\chi_k, \end{aligned} \tag{7}$$

where the first and third inequalities follow from the fact that distance $s[t]$ is non-increasing in the greedy policy and the second inequality follows from Equation (6). Since the term in the parenthesis is strictly less than 1, the $\chi_k \to 0$ asymptotically, which means that the distance between the pursuer and evader tends to r_{step} asymptotically. For $\kappa > 2$, the distance will reduce to $(\kappa-1)r_{\text{step}}$ after finite time and thus, the motion circle of the evader is completely contained within the sensing circle of the pursuer. Hence, the result follows.

The case of $\kappa = 2$: We have seen that the distance $s[t]$ between the pursuer and evader tends asymptotically to r_{step}. From Lemma III.9, we obtain that as $s[t] \to r_{\text{step}}$, the angle $\beta_{\text{max}} \to 0$. So, after some finite time,

$$\beta_{\text{max}} < \frac{\theta}{2N^*} \triangleq \beta_{\text{c}}.$$

Thus, evader is confined to the current cone according to Lemma III.9 and from Proposition III.7 and we can see from part (i) of this proof, that the pursuer will trap the evader within its sensing radius.

If $\kappa < 2$: There exists a maximum value of the evasion angle at each step, so that the evader's next step is not in the pursuer's present sensing disc. This is shown in Figure 8. The key idea of this part of the proof is that if we ensure that this maximum angle is less than the minimum value needed for the evader to escape a cone, then the evader is forced to remain inside the cone and trapping follows from part (i). The pursuer employs the *greedy strategy* until the distance is reduced to such a value that the maximum evasion angle is less than or equal to $\gamma(1+\delta)$, where γ is the maximum evasion angle if the evader is at $e'[t]$, on the stepping radius of the pursuer and δ is some positive number. At this time instant t_{final}, let the pursuer construct a new cone, $\mathcal{C}_{\text{final}}$. Now if,

$$\gamma(1+\delta)N^* = \min_{p,e\in\mathcal{Q}} \frac{\theta}{2}, \tag{8}$$

where N^* and θ are defined in Proposition III.7, then for some $\tau \in \{t_{\text{final}}, \ldots, t_{\text{final}} + N^*\}$, $\beta[\tau] \geq \gamma(1+\delta) = \beta_{\text{c}}$ for the evader to leave $\mathcal{C}_{\text{final}}$. This means that the evader is forced to step inside the current sensing radius of the pursuer or remain inside the final cone $\mathcal{C}_{\text{final}}$. In both cases, the pursuer traps the evader within its sensing radius. From Equation (8),

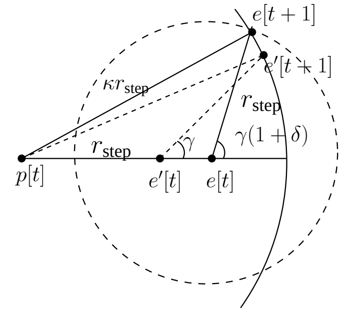

Fig. 8. Illustrating parameters in Equation (8)

$$\gamma < \min_{p,e\in\mathcal{Q}} \frac{\theta}{2N^*} = \beta_{\text{c}}.$$

Applying cosine rule to $\triangle p[t]e'[t]e'[t+1]$,

$$\begin{aligned} \kappa &= \sqrt{2 + 2\cos\gamma}, \\ &> \sqrt{2 + 2\cos\beta_{\text{c}}}. \end{aligned}$$

Computing upper bound on time: We have seen that when the pursuer uses the greedy policy, the evader cannot leave arbitrarily large number of cones. Thus, to compute an upper bound on the trapping time, we compute an upper bound on the number of cones the evader can leave. We have seen that using *greedy strategy*, $\beta_{\text{max}} \leq \beta_{\text{c}}$, after finite time. From Equation (4), we want to determine that distance s_{c} for which $\beta_{\text{max}} = \beta_{\text{c}}$, so that subsequently, the evader is confined to the same cone. Thus,

$$s_{\text{c}} = (\sqrt{\kappa^2 - \sin^2\beta_{\text{c}}} - \cos\beta_{\text{c}})r_{\text{step}}.$$

If k is the final cone index, then using Equation (7),

$$s_{\text{c}} - r_{\text{step}} \leq \chi_k \leq \lambda\chi_{k\text{-}1} \leq \cdots \leq \lambda^k(\kappa-1)r_{\text{step}},$$

where $\lambda = 1 - \frac{1-\cos(\frac{\theta}{2N^*})}{\kappa}$ and the worst-case $\chi_0 = (\kappa-1)r_{\text{step}}$. Upon simplifying,

$$k \leq \left\lceil \frac{\log(\frac{\kappa-1}{\sqrt{\kappa^2-\sin^2\beta_{\text{c}}}-\cos\beta_{\text{c}}-1})}{\log\frac{1}{\lambda}} \right\rceil.$$

The result now follows from the fact that for the case of $\kappa < 2$, we construct an extra final cone and the maximum number of steps in each cone can be N^*. ■

We now provide a sufficient condition to ensure evasion, if the pursuer uses the *greedy strategy*.

Remark III.10 (Example of evasion) For $\kappa < \sqrt{2}$, if the pursuer is following the *greedy strategy* and if there exists a closed curve $\Omega \in \mathcal{Q}$ satisfying,

$$\rho \geq \frac{r_{\text{step}}}{\sqrt{4-\kappa^2}}, \tag{9}$$

where ρ is the radius of curvature at any point on Ω, then the evader can avoid capture.

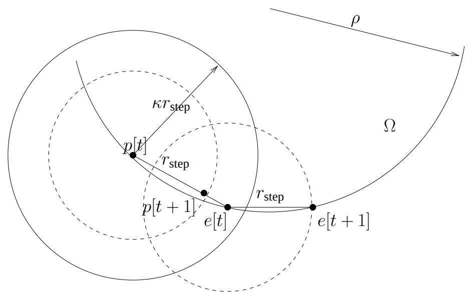

Fig. 9. Evasion using the closed curve Ω

This can be deduced from the following evader strategy: Consider a closed curve Ω to be a circle of radius equal to ρ, which satisfies equation (9). Suppose the pursuer and the evader are on Ω as shown in Figure 9. The evader strategy would be to choose a point $e[t+1]$ on Ω such that $\|e[t]-e[t+1]\| = r_{\text{step}}$. Since ρ satisfies equation (9), $e[t+1]$ will lie outside the pursuer's sensing disc at time t. In Theorem III.3, we have shown that using the *greedy strategy*, $s[t] \to r_{\text{step}}$ asymptotically. Thus, using this strategy, the evader can avoid stepping inside the pursuer's current sensing disc indefinitely.

The pursuit phase ends once the pursuer traps the evader within its sensing radius.

C. The Capture phase

Once the evader is trapped within the sensing range of the pursuer, the pursuer employs the *Lion strategy* [5] to complete the capture. For the sake of completion, we now give a brief description of the Lion strategy, adapted to the present problem setting and an upper bound on the time to capture. An upper bound for capture time has been obtained by Isler *et al.* for polygonal environments [12]. Consider a single pursuer and a single evader inside a bounded, convex environment. In this phase, the next position of the evader is within the current sensing range of the pursuer and hence,

$$y^e[t+1] = e[t+1].$$

The Lion strategy can be applied to this phase as follows,

(i) At the beginning of the $(t+1)^{th}$ move of the pursuer, the pursuer constructs the line $e[t]p[t]$, as shown in Figure 10. Let this line intersect the environment at $X[t]$ such that $p[t]$ lies between $e[t]$ and $X[t]$.

(ii) The pursuer constructs the line $e[t+1]X[t]$. It moves to the intersection of this line and the circle centered at $p[t]$ and of radius r_{step}. Of the two possible points, the pursuer moves to the point closer to $e[t+1]$.

A simple analysis reveals that $X[t]$ is the same as $X[t_0+t^*]$, for $t \geq (t_0+t^*)$, where (t_0+t^*) is the time at the end of the pursuit phase. The Lion strategy gives us the following result.

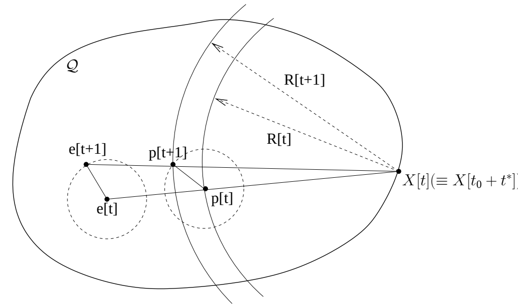

Fig. 10. Using the Lion strategy

Theorem III.11 (Lion strategy [5]) *After trapping the evader within the sensing radius, the pursuer captures the evader using the* Lion strategy *in at most* $\left\lceil \frac{\operatorname{diam}^2(\mathcal{Q})}{r^2_{\text{step}}} \right\rceil$ *time steps.*

The following result follows from a simple analysis of the Lion strategy.

Lemma III.12 *The distance,* $s[t] = \|p[t]-e[t]\|$, *is non-increasing after every move using the Lion strategy by the pursuer.*

The pursuit-evasion problem with limited sensing is now solved when we state the final result.

Theorem III.13 *When the pursuer employs the Lion strategy, the motion disc of the evader is always contained inside the sensing disc of the pursuer.*

$$\mathcal{B}_{e[t]}(r_{\text{step}}) \subset \mathcal{B}_{p[t]}(r_{\text{sens}}), \tag{10}$$

for every time instant t in the capture phase.

Proof: Equation (10) is satisfied at the end of pursuit phase from the definition of trapping within the sensing radius. The distance between pursuer and evader never increases during the greedy pursuit and the Lion strategy. So, equation (10) will continue to hold at each time instant in the capture phase. ■

IV. Conclusion and Future Directions

We have shown that even with sensing constrained to a closed disc, it is possible for a pursuer to capture an evader in a bounded, convex environment. An interesting direction for future research is to determine the expected time the pursuer would take to capture an evader, when the evader is allowed to move randomly until it first senses the pursuer, instead of the reactive rabbit model. This problem has been solved on a finite graph by Isler *et al.* [1].

We have shown that given any bounded, convex environment, there exists a range of values for the ratio of sensing to stepping radius of the players, κ, for which the *greedy strategy* is guaranteed to work. We further noted that if

$\kappa < \sqrt{2}$, then there exists environments large enough for which the evader has an escape policy if the pursuer uses the greedy strategy proposed in this paper. A natural direction for future research consists of searching for alternative pursuit strategies that guarantee capture for every bounded convex environment when $\kappa \leq \sqrt{2}$. Also, in the Lion strategy, the pursuer needs to memorize either the environment or the location of the centers, $X[t]$. It is not clear whether there exists any pursuer strategy which relies solely on the current and possibly next positions of the evader.

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