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Maximizing visibility in nonconvex polygons: nonsmooth analysis and gradient algorithm design

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***Abstract*— This paper presents a motion control algorithm for a planar mobile observer such as, e.g., a mobile robot equipped with an omni-directional camera. We propose a nonsmooth gradient algorithm for the problem of maximizing the area of the region visible to the observer in a simple nonconvex polygon. First, we show that the visible area is almost everywhere a locally Lipschitz function of the observer location. Second, we provide a novel version of LaSalle Invariance Principle for discontinuous vector fields and Lyapunov functions with a finite number of discontinuities. Finally, we establish the asymptotic convergence properties of the nonsmooth gradient algorithm and we illustrate numerically its performance.**

I. INTRODUCTION

Consider a single-point mobile robot in a planar nonconvex environment modeled as a simple polygon: how should the robot move in order to monotonically increase the area of its visible region (i.e., the region within its line of sight)? This problem is the subject of this paper, together with the following modeling assumptions. The dynamical model for the robot's motion is a first order system of the form $\dot{p} = u$, where p refers to the position of the robot in the environment and u is the driving input. The robot is equipped with an omni-directional camera and range sensor; the range of the sensor is larger than the diameter of the environment. The robot does not know the entire environment and its position in it, and its instantaneous motion depends only on what is within line of sight (this assumption restricts our attention to memoryless feedback laws).

In broad terms, this problem is related to numerous optimal sensor location and motion planning problems in the computational geometry, geometric optimization, and robotics literature. In computational geometry [1], the classical Art Gallery Problem amounts to finding the optimum number of guards in a nonconvex environment so that each point of the environment is visible by at least one guard. A heuristic for this problem is to use a greedy approach wherein the first robot (guard) is placed at the point where it sees the maximum area. The next robot is placed where it sees the maximum area not visible to the first and so on. In robotics, this approach is useful for 2D map building wherein a robot moves in such a way so that its next position is the best in terms of what it can see additionally. In this robotic context, these problems are referred to as Next Best View problems. The specific problem of interest in this paper is that of optimally locating a guard in a simple polygon. To the best of our knowledge, this problem is still open and is the subject of ongoing research; see [2], [3], [4], and the surveys on geometric optimization and art gallery problems [5], [6]. However, randomized algorithms for finding the optimal location up to a constant factor approximation exist; see [4]. These algorithms can be regarded as open-loop algorithms that require knowledge of the environment. Closed-loop heuristic algorithms for the Next Best View problem are proposed and simulated in [7] and in the early work [8].

A second set of relevant references are those on nonsmooth stability analysis. Indeed, our approach to maximizing visible area is to design a nonsmooth gradient flow. To define our proposed algorithm we rely on the notions of generalized gradient [9] and of Filippov solutions for differential inclusions [10]. To study our proposed algorithm we extend recent results on the stability and convergence properties of nonsmooth dynamical systems, as presented in [11], [12].

The contributions of this paper are threefold. First, we prove some basic properties of the area visible from a point observer in a nonconvex polygon Q, see Figure 1. Namely, we show that the area of the visibility polygon, as

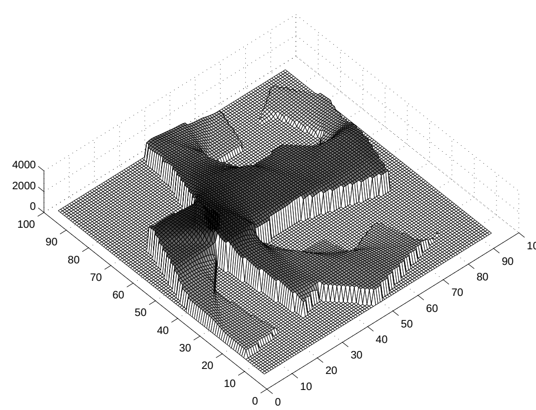

Fig. 1. The visible area function over a nonconvex polygon.

a function of the observer position, is a locally Lipschitz function almost everywhere, and that the finite point set of discontinuities are the reflex vertices of the polygon Q. Additionally, we compute the generalized gradient of the function and show that it is, in general not regular. Second, we provide a generalized version of the certain stability theorems for discontinuous vector fields available in the literature [11], [12]. Specifically, we provide a generalized nonsmooth LaSalle Invariance Principle for discontinuous vector fields, Filippov solutions, and Lyapunov functions that are locally Lipschitz almost everywhere (except for a finite set of discontinuities). Third and last, we use these novel results to design a nonsmooth gradient algorithm that monotonically increases the area visible to a point observer. To the best of our knowledge, this is the first provably correct algorithm for this version of the Next Best View problem. We illustrate the performance of our algorithm via simulations for some interesting polygons.

The paper is organized as follows. Section II contains the analysis of the smoothness and of the generalized gradient of the function of interest. Section III contains the novel results on nonsmooth stability analysis. Section IV presents the nonsmooth gradient algorithm and the properties of the resulting closed-loop system. Finally, the simulations in Section V illustrate the convergence properties of the algorithm. In the interest of space, the proofs for the results in the paper have not been included can be found in [13].

II. The area visible from an observer

In this section we study the area of the region visible to a point observer equipped with an omnidirectional camera. We show that the visible area, as a function of the location of the observer, is locally Lipschitz, except at a finite point set. We prove that, for general nonconvex polygons, the function is not regular. We also provide expressions for the generalized gradient of the visible area function wherever it is locally Lipschitz. We refer the reader to [9] for the notion of locally Lipschitz functions and related concepts.

Let us start by introducing the set of lines on the plane $\mathbb{R}^2$. For $(a,b,c) \in \mathbb{R}^3 \setminus \{(0,0,c) \in \mathbb{R}^3 \mid c \in \mathbb{R}\}$, we define the equivalence class $[(a,b,c)] = \{(a',b',c') \in \mathbb{R}^3 \mid (a,b,c) = \lambda(a',b',c'),\ \lambda \in \mathbb{R}\}$. The *set of lines on* $\mathbb{R}^2$ is defined as

$$\mathbb{L} = \{[(a,b,c)] \subset \mathbb{R}^3 \mid (a,b,c) \in \mathbb{R}^3,\ a^2+b^2 \neq 0\}.$$

Next, two simple and useful functions are introduced. Let $f_{\rm pl} : \mathbb{R}^2 \times \mathbb{R}^2 \setminus \{(p,p) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid p \in \mathbb{R}^2\} \to \mathbb{L}$ map two distinct points in $\mathbb{R}^2$ to the line passing through them. For $(x_1,y_1),(x_2,y_2) \in \mathbb{R}^2$, the function $f_{\rm pl}$ admits the expression

$$f_{\rm pl}((x_1,y_1),(x_2,y_2)) = [(y_2-y_1, x_1-x_2, y_1x_2-x_1y_2)].$$

If $l_1 \parallel l_2$ denotes that the two lines $l_1, l_2 \in \mathbb{L}$ are parallel, let $f_{\rm lp} : \mathbb{L}^2 \setminus \{(l_1,l_2) \in \mathbb{L}^2 \mid l_1 \parallel l_2\} \to \mathbb{R}^2$ map two lines that are not parallel to their unique intersection point. Given two lines $l_1 = [(a_1,b_1,c_1)]$ and $l_2 = [(a_2,b_2,c_2)]$ that are not parallel, the function $f_{\rm lp}$ admits the expression

$$f_{\rm lp}(l_1,l_2) = \left(\frac{b_2c_1-b_1c_2}{a_2b_1-a_1b_2}, \frac{a_1c_2-a_2c_1}{a_2b_1-a_1b_2}\right).$$

Note that the functions $f_{\rm pl}$ and $f_{\rm lp}$ are class C^ω, i.e., they are analytic over their domains.

Now, let us turn our attention to the polygonal environment. Let Q be a simple polygon, possibly nonconvex. A polygon is said to be simple if the only points in the plane belonging to two polygon edges are the polygon vertices. Such a polygon has a well defined interior and exterior. *Note that a simple polygon can contain holes*. Let $\mathring{Q}$ and ∂Q denote the interior and the boundary of Q, respectively. Let $\mathrm{Ve}(Q) = (v_1,\dots,v_n)$ be the list of vertices of Q ordered counterclockwise. The *interior angle of a vertex* v of Q is the angle formed inside Q by the two edges of the boundary of Q incident at v. The point $v \in \mathrm{Ve}(Q)$ is a *reflex vertex* if its interior angle is strictly greater than π. Let $\mathrm{Ve}_r(Q)$ be the list of reflex vertices of Q. If S is a finite set, then let $|S|$ denote its cardinality.

A point $q \in Q$ is *visible from* $p \in Q$ if the segment between q and p is contained in Q. The *visibility polygon* $S(p) \subset Q$ from a point $p \in Q$ is the set of points in Q visible from p. It is convenient to think of $p \mapsto S(p)$ as a map from Q to the set of polygons contained in Q. It must be noted that the visibility polygon is not necessarily a simple polygon.

Definition 2.1: Let v be a reflex vertex of Q, and let $w \in \mathrm{Ve}(Q)$ be visible from v. The (v,w)-*generalized inflection segment* $I(v,w)$ is the set

$$I(v,w) = \{q \in S(v) \mid q = \lambda v + (1-\lambda)w, \lambda \geq 1\}.$$

Also v is an *anchor of* $p \in Q$ if it is visible from p and if $\{q \in S(v) \mid q = \lambda v + (1-\lambda)p, \lambda > 1\}$ is not empty.

In other words, a reflex vertex is an anchor of p if it occludes a portion of the environment from p. Figure 2 illustrates the various quantities defined above. Given a point q and a line l, let $\mathrm{dist}(q,l)$ denote the distance between them.

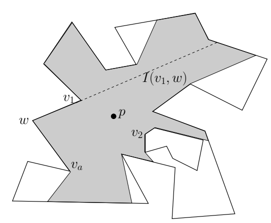

Fig. 2. Reflex vertices v_1 and v_2, a generalized inflection segment $I(v_1,w)$, an anchor v_a of p and the visibility polygon (shaded region) from p. Note that the polygonal environment has a hole.

Theorem 2.2: Let $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ be the set of generalized inflection segments of Q, and let P be a connected component of $Q \setminus \bigcup_{\alpha \in \mathcal{A}} I_\alpha$ For all $p \in P$, the visibility polygon $S(p)$ is simple and has a constant number of vertices, say $\mathrm{Ve}(S(p)) = \{u_1(p),\dots,u_k(p)\}$. For all $i \in \{1,\dots,k\}$, the map $P \ni p \mapsto u_i(p)$ is C^ω and either

$$\mathrm{d}u_i(p) = 0$$

if $u_i(p) \in \mathrm{Ve}(Q)$, or

$$\mathrm{d}u_i(p) = \frac{\mathrm{dist}(v_a,l)}{(\mathrm{dist}(p,l)-\mathrm{dist}(v_a,l))^2\sqrt{a^2+b^2}} \begin{bmatrix} -b \\ a \end{bmatrix} \begin{bmatrix} y-y_a \\ x_a-x \end{bmatrix}^T,$$

if $u_i(p) = f_{\mathrm{lp}}(f_{\mathrm{pl}}(v_a, p), l)$, where $v_a = (x_a, y_a)$ is an anchor of p and $l = [(a, b, c)]$ is a line defined by an edge of Q.

Next, the area of a visibility polygon as a function of the observer location is studied, see Figure 1. Recall that the area of a simple polygon Q with counterclockwise-ordered vertices $\mathrm{Ve}(Q) = ((x_1, y_1), \ldots, (x_n, y_n))$ is given by

$$A(Q) = \frac{1}{2} \sum_{i=1}^{n} x_i (y_{i-1} - y_{i+1}),$$

where $(x_0, y_0) = (x_n, y_n)$ and $(x_{n+1}, y_{n+1}) = (x_1, y_1)$. As in the previous theorem, let $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ be the set of generalized inflection segments of Q and let P be a connected component of $Q \setminus \bigcup_{\alpha \in \mathcal{A}} I_\alpha$. Next, if $p \in P$, the visibility polygon from p has a constant number of vertices, say $k = |\mathrm{Ve}(S(p))|$, is simple, and satisfies $A \circ S(p) = \sum_{i=1}^{k} x_i (y_{i-1} - y_{i+1})$ where $\mathrm{Ve}(S(p)) = (u_1, \ldots, u_k)$ are ordered counterclockwise, $u_i(p) = (x_i, y_i)$, $u_0 = u_k$, and $u_{k+1} = u_1$. Therefore, $P \ni p \mapsto A \circ S(p)$ is also C^ω and

$$\mathrm{d}(A \circ S)(p) = \sum_{i=1}^{k} \frac{\partial A(u_1, \ldots, u_k)}{\partial u_i} \mathrm{d}u_i(p). \qquad (1)$$

To illustrate this equality, it is convenient to introduce the *versor* operator defined by $\mathrm{vers}(X) = X/\|X\|$ if $X \in \mathbb{R}^2 \setminus \{0\}$ and by $\mathrm{vers}(0) = 0$. We depict the normalized gradient $\mathrm{vers}(\mathrm{d}(A \circ S))$ of the visible area function in Figure 3.

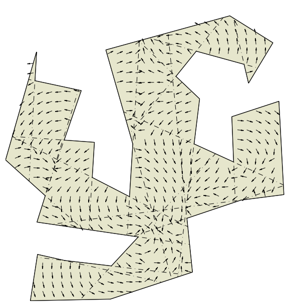

Fig. 3. Normalized gradient of the visible area function over the nonconvex polygon depicted in Figure 1. The dashed lines represent some of the generalized inflection segments.

Theorem 2.3: The map $A \circ S$ restricted to $Q \setminus \mathrm{Ve}_r(Q)$ is locally Lipschitz.

To obtain the expression for the generalized gradient of $A \circ S$, the polygon Q is partitioned as follows.

Lemma 2.4: Let $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ be the set of generalized inflection segments of Q. There exists a unique partition $\{\overline{P}_\beta\}_{\beta \in \mathcal{B}}$ of Q where P_β is a connected component of $Q \setminus \bigcup_{\alpha \in A} I_\alpha$ and $\overline{P}_\beta$ denotes its closure.

Figure 4 illustrates this partition for the given nonconvex polygon. For $\beta \in \mathcal{B}$, define $A_\beta : \overline{P}_\beta \to \mathbb{R}_+$ by

$$A_\beta(p) = A \circ S(p), \quad \text{for } p \in P_\beta,$$

and by continuity on the boundary of P_β. It turns out that the maps A_β, $\beta \in \mathcal{B}$, are continuously differentiable[1] on $\overline{P}_\beta$. Equation (1) gives the value of the gradient for $p \in P_\beta$. However, in general, for $p \in \overline{P}_{\beta_1} \bigcap \ldots \bigcap \overline{P}_{\beta_m} \setminus \mathrm{Ve}_r(Q)$, based on Theorem 2.3 and Lemma 2.4, we can write that

$$\partial(A \circ S)(p) = \mathrm{co}\left\{\mathrm{d}A_{\beta_1}(p), \ldots, \mathrm{d}A_{\beta_m}(p)\right\}. \qquad (2)$$

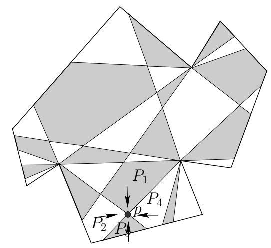

Fig. 4. Partition of Q. The generalized gradient of the area function at p is the convex hull of the gradient of four functions $A_1, \ldots, A_4$ at p.

This completes our study of the generalized gradient of the locally Lipschitz function $A \circ S$. The following lemma concerns the regularity of this function.

Lemma 2.5: There exists a nonconvex polygon Q such that the maps $A \circ S$ and $-A \circ S$ restricted to $Q \setminus \mathrm{Ve}_r(Q)$ are not regular.

III. An Invariance Principle in Nonsmooth Stability Analysis

This section presents results on stability analysis for discontinuous vector fields via nonsmooth Lyapunov functions. The results extend the work in [12] and will be useful in the next control design section. We refer the reader to [10] for some useful nonsmooth analysis concepts.

In what follows we shall study differential equations of the form

$$\dot{x}(t) = X(x(t)),$$

where X is a discontinuous vector field on $\mathbb{R}^N$.

Lemma 3.1: Let $X : \mathbb{R}^N \to \mathbb{R}^N$ be measurable and essentially locally bounded and let $f : \mathbb{R}^N \to \mathbb{R}$ be locally Lipschitz. Let $\gamma : [t_0, t_1] \to \mathbb{R}^N$ be a Filippov solution of X such that $f(\gamma(t))$ is regular for almost all $t \in [t_0, t_1]$. Then

(i) $\frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t)))$ exists for almost all $t \in [t_0, t_1]$, and
(ii) $\frac{\mathrm{d}}{\mathrm{d}t}(f(\gamma(t))) \in \widetilde{\mathcal{L}}_X f(\gamma(t))$ for almost all $t \in [t_0, t_1]$.

The following result is a generalization of the classic LaSalle Invariance Principle for smooth vector fields and smooth Lyapunov functions to the setting of discontinuous vector fields and nonsmooth Lyapunov functions.

Theorem 3.2 (LaSalle Invariance Principle): Let $X : \mathbb{R}^N \to \mathbb{R}^N$ be measurable and essentially locally bounded and let $S \subset \mathbb{R}^N$ be compact and strongly invariant for X. Let $C \subset S$ consist of a finite number of points and let $f : S \to \mathbb{R}$ be locally Lipschitz on $S \setminus C$ and bounded from below on S. Assume the following properties hold:

[1]A function is continuously differentiable on a closed set if (1) it is continuously differentiable on the interior, and (2) the limit of the derivative at a point in the boundary does not depend on the direction from which the point is approached.

(A1) if $x \in S \setminus C$, then either $\max \widetilde{\mathcal{L}}_X f(x) \leq 0$ or $\widetilde{\mathcal{L}}_X f(x) = \emptyset$,

(A2) if $x \in C$ and if γ is a Filippov solution of X with $\gamma(0) = x$, then $\lim_{t \to 0^-} f(\gamma(t)) \geq \lim_{t \to 0^+} f(\gamma(t))$, and

(A3) if $\gamma : \overline{\mathbb{R}}_+ \to S$ is a Filippov solution of X, then $f \circ \gamma$ is regular almost everywhere.

Define $Z_{X,f} = \left\{x \in S \setminus C \mid 0 \in \widetilde{\mathcal{L}}_X f(x)\right\}$ and let M be the largest weakly invariant set contained in $(\overline{Z}_{X,f} \cup C)$. Then the following statements hold:

(i) if $\gamma : \overline{\mathbb{R}}_+ \to S$ is a Filippov solution of X, then $f \circ \gamma$ is monotonically nonincreasing;

(ii) each Filippov solution of X with initial condition in S approaches M as $t \to +\infty$;

(iii) if M consists of a finite number of points, then each Filippov solution of X with initial condition in S converges to a point of M as $t \to +\infty$.

IV. Maximizing the area visible from a mobile observer

In this section we build on the analysis results obtained thus far to design an algorithm that maximizes the area visible to a mobile observer. We aim to reach local maxima of the discontinuous visible area $A \circ S$ by designing some appropriate form of a gradient flow for it. We now present an *introductory and incomplete* version of the algorithm: the objective is to steer the mobile observer along a path for which the visible area is guaranteed to be nondecreasing.

Name:	Increase visible area for Q
Goal:	Maximize the area visible to a mobile observer
Assumption:	Generalized inflection segments of Q do not intersect. Initial position does not belong to a generalized inflection segment.

Let $p(t)$ denote the observer position at time t inside the nonconvex polygon Q. The observer performs the following tasks at each time instant:

compute visibility polygon $S(p(t)) \subset Q$,

if $p(t)$ does not belong to any generalized inflection segment or to the boundary of Q **then**

move along the versor of the gradient d$(A \circ S)$

else if $p(t)$ belongs to a generalized inflection segment but not to the boundary of Q **then**

depending on the generalized gradient $\partial(A \circ S)$, either slide along the segment or leave the segment in an appropriate direction

else if $p(t)$ belongs to the boundary of Q but not to a reflex vertex, **then**

depending on the projection of $\partial(A \circ S)$ along the boundary, either slide along the boundary or move in an appropriate direction toward the interior of Q

else

either follow a direction of ascent of $A \circ S$ or stop

end if

The remainder of this section is dedicated to formalizing this loose description.

A. A modified gradient vector field

Before describing the algorithm to maximize the area visible to the mobile observer, we introduce the following useful notions. Given a simple polygon Q with $\mathrm{Ve}(Q) = (v_1, \dots, v_n)$ and $\epsilon > 0$, define the following quantities:

(i) let the *ϵ-expansion of* Q be $Q^\epsilon = \{p \mid ||p - q|| \leq \epsilon \text{ for some } q \in Q\}$,

(ii) for $i \in \{1, \dots, n\}$, let P_i^ϵ be the open set delimited by the edge $\overline{v_i v_{i+1}}$, the bisectors of the external angles at v_i and v_{i+1} and the boundary of Q^ϵ,

(iii) for ϵ small enough and for any point p in Q^ϵ, let $\mathrm{prj}_Q(p)$ be uniquely equal to $\arg\min\{||p' - p|| \mid p' \in \partial Q\}$, and

(iv) let the *outward normal* $n(\mathrm{prj}_Q(p))$ be the unit vector directed from $\mathrm{prj}_Q(p)$ to p.

We illustrate these notions in Figure 5. Note that $\mathrm{prj}_Q(p)$ can never be a reflex vertex. We can now define a vector field on Q^ϵ as follows:

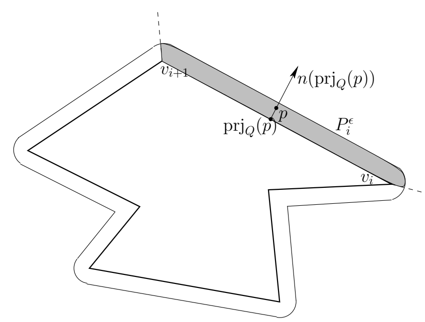

Fig. 5. The ϵ-expansion Q^ϵ of the simple polygon Q, an open set P_i^ϵ and the corresponding outward normal $n(\mathrm{prj}_Q(p))$.

$$X_Q(p) = \begin{cases} \mathrm{vers}(\mathrm{d}(A \circ S)(p)), & \text{if } p \in \mathring{Q} \setminus \{I_\alpha\}_{\alpha \in \mathcal{A}}, \\ -n(\mathrm{prj}_Q(p)), & \text{if } p \in P_i^\epsilon, \\ 0, & \text{otherwise.} \end{cases}$$

(Recall that the versor operator is defined by $\mathrm{vers}(Y) = Y/\|Y\|$ if $Y \in \mathbb{R}^2 \setminus \{0\}$ and by $\mathrm{vers}(0) = 0$.) Note that X_Q is well-defined because at $p \in \mathring{Q} \setminus \{I_\alpha\}_{\alpha \in \mathcal{A}}$ the function $A \circ S$ is analytic. Clearly, X_Q is not continuous on Q^ϵ. However, the set of points where it is discontinuous is of measure zero. Almost everywhere in the interior of Q, the vector field X_Q is equal to the normalized gradient of $A \circ S$ as depicted in Figure 3. We now present the differential equation describing the motion of the observer:

$$\dot{p}(t) = X_Q(p(t)). \tag{3}$$

A Filippov solution of (3) on an interval $[t_0, t_1] \subset \mathbb{R}$ is defined as a solution of the differential inclusion

$$\dot{p}(t) \in K[X_Q](p(t)), \tag{4}$$

where $K[X_Q]$ is the usual Filippov differential inclusion associated with X_Q, see [10]. Since X_Q is measurable and bounded, the existence of a Filippov solution is guaranteed. We study uniqueness and completeness of Filippov solutions in the following lemma.

Lemma 4.1: The following statements hold true:

(i) there exists a simple polygon Q for which the corresponding vector field X_Q admits multiple Filippov solutions;

(ii) any simple polygon Q is a strongly invariant set for the corresponding vector field X_Q and, therefore, any Filippov solution is defined over $\overline{\mathbb{R}}_+$.

We now claim that any solution of the differential inclusion (4) has the property that the visible area increases monotonically. To prove these desirable properties, we first present the following results in nonsmooth analysis.

B. Properties of solutions and convergence analysis

To prove the convergence properties of the solution of (4) using the results presented in Section III, we must first define a suitable Lyapunov function. Intuitively since our objective is to maximize the visible area, our Lyapunov function should be closely related to it. For $\epsilon > 0$, we now define the *extended area function* A_Q^ϵ at all points $p \in Q \bigcup \{\cup_i P_i^\epsilon\}$. The extended function coincides with the original function on the interior and on the boundary of Q and is defined appropriately outside:

$$A_Q^\epsilon(p) = \begin{cases} A \circ S(p), & p \in Q, \\ A \circ S(\mathrm{prj}_Q(p)) - ||p - \mathrm{prj}_Q(p)||, & p \in \cup_i P_i^\epsilon. \end{cases}$$

For all $p \in \partial Q \setminus \mathrm{Ve}\, Q$, A_Q^ϵ satisfies (see Figure 6):

$$A_Q^{\epsilon\,\prime}(p; n(\mathrm{prj}_Q(p))) = -1.$$

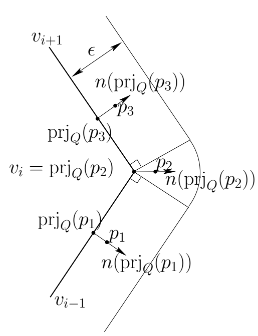

Fig. 6. Extending the function $A \circ S$ to A_Q^ϵ. Note the direction of $n(\mathrm{prj}_Q(p_i))$ at all points p_i.

Remark 4.2: The extended area function A_Q^ϵ is locally Lipschitz on $(Q \setminus \mathrm{Ve}_r(Q)) \bigcup \{\cup_i P_i^\epsilon\}$ and analytic almost everywhere on $Q \bigcup \{\cup_i P_i^\epsilon\}$.

The following theorem is important to prove that such a function leads to a monotonically nondecreasing value of the area of the visibility polygon.

Theorem 4.3: Let $G(Q)$ be the subset of Q where both maps $p \mapsto -A_Q^\epsilon(p)$ and $p \mapsto A_Q^\epsilon(p)$ are not regular. Then any Filippov solution $\gamma : \overline{\mathbb{R}}_+ \to Q$ of X_Q has the property that $\gamma(t) \notin G(Q)$ for almost all $t \in \overline{\mathbb{R}}_+$ unless γ reaches a critical point of $K[X_Q]$.

In the following theorem, the functions A_Q^ϵ and $-A_Q^\epsilon$ are used as candidate Lyapunov functions to show the convergence properties of Filippov solutions of X_Q.

Theorem 4.4: Any Filippov solution $\gamma : \overline{\mathbb{R}}_+ \to Q$ of X_Q has the following properties:

(i) $t \mapsto A \circ S(\gamma(t))$ is continuous and monotonically nondecreasing,

(ii) γ approaches the set of critical points of $K[X_Q]$.

Theorem 4.4 implies that the single observer converges to a critical point of $A \circ S$ or to a reflex vertex of Q. However, as shown in Figure 7, the presence of noise or computational inaccuracies actually works to drive the observer away from a reflex vertex that is not a local maximum. This will also be true for other critical points that are not local maxima.

V. Simulation Results

Figures 7 and 9 illustrate the performance of the gradient algorithm in equation (4). The algorithm is implemented in `Matlab`®. The vertices of the visibility polygon are obtained by means of an $O(n^2)$ algorithm, where n is the number of vertices of the polygonal environment. These are then sorted in counterclockwise order to compute the visibility polygon. The calculation of the generalized gradient of the visible area function is then a natural outcome of (1) and (2). Computational inaccuracies in the implementation of the algorithm to calculate the visibility polygon have been noticed in some configurations; see the plot of the variation of visible area with time in Figure 7. See Figure 8(b) for the phase portrait of the vector field X_Q for the polygon in Figure 8(a). Our experiments suggest that the observer reaches a local maximum of the visible area in finite time, however this can be shown not to be true in general.

VI. Conclusions

This paper introduces a gradient-based algorithm to optimally locate a mobile observer in a nonconvex environment. We presented nonsmooth analysis and control design results. The simulation results illustrate that, in the presence of noise, the observer reaches a local maximum of the visible area. In an “highly nonconvex” environment, a single observer may not be able to see a large fraction of the environment. In such a case, a team of observers can be deployed to achieve the same task. We therefore plan to investigate this same visibility objective for teams of observers. Other directions of future research include practical robotic implementation issues as well as other combined mobility and visibility problems.

Acknowledgments

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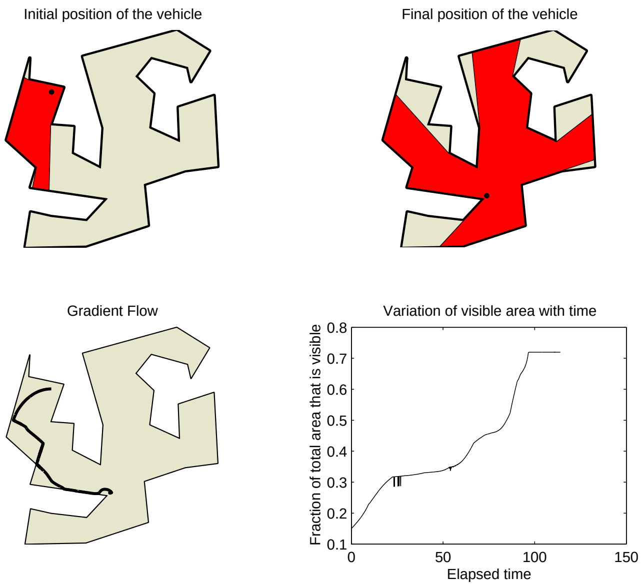
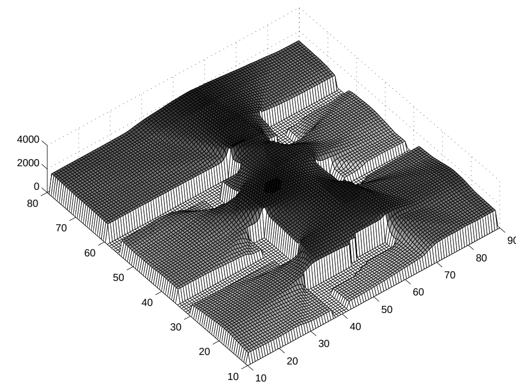
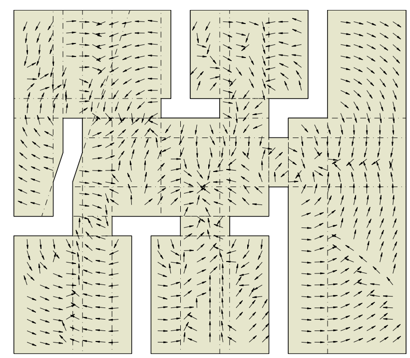


Fig. 7. Simulation results of the gradient algorithm for the nonconvex polygon depicted in Figure 1. The observer arrives, in finite time, at a local maximum. Note here that the observer visits a reflex vertex at some point in its trajectory but comes out of it due to computational inaccuracies because it is not a local maximum.



(a) The visible area function.



(b) The vector field.

Fig. 8. Illustration of the visible area function and the vector field over a polygon in the shape of a floor plan of a building.

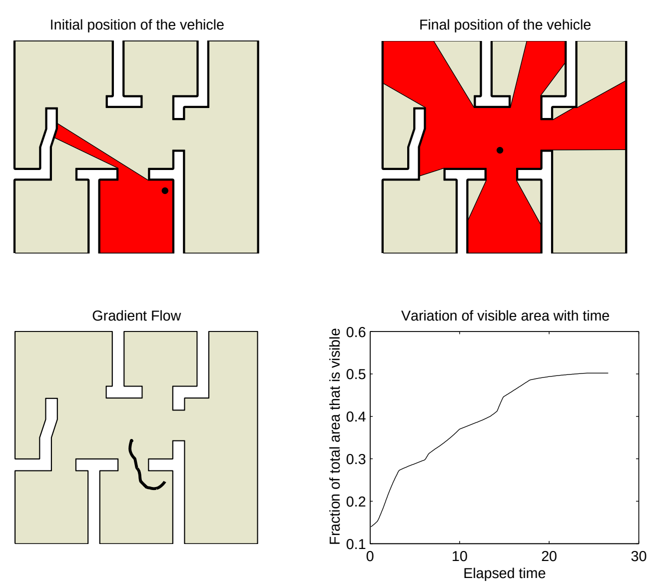


Fig. 9. Simulation results of the gradient algorithm. The observer arrives, in finite time, at a local maximum.

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