Synchronous robotic networks and complexity of control and communication laws

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Abstract

This paper proposes a formal model for a network of robotic agents that move and communicate. Building on concepts from distributed computation, robotics and control theory, we define notions of robotic network, control and communication law, coordination task, and time and communication complexity. We then analyze a number of basic motion coordination algorithms such as pursuit, rendezvous and deployment.

1 Introduction

Problem motivation The study of networked mobile systems presents new challenges that lie at the confluence of communication, computing, and control. In this paper we consider the problem of designing joint communication protocols and control algorithms for groups of agents with controlled mobility. For such groups of agents we define the notion of communication and control law by extending the classic notion of distributed algorithm in synchronous networks. Decentralized control strategies are appealing for networks of robots because they can be scalable and they provide robustness to vehicle and communication failures.

One of our key objectives is to develop a computable theory of time and communication complexity for motion coordination algorithms. Hopefully, our formal model will be suitable to analyze objectively the performance of various coordination algorithms. It is our contention that such a theory is required to assess the complex trade-offs between computation, communication and motion control or, in other words, to establish what algorithms are *scalable* and practically implementable in large networks of mobile autonomous agents. The need for modern models of computation in wireless and sensor network applications is discussed in the well-known report [36].

Literature review To study complexity of motion coordination, our starting points are the standard notions of *synchronous and asynchronous networks* in distributed and parallel computation, e.g., see Lynch [27] and, with an emphasis on numerical methods, Bertsekas and Tsitsiklis [3]. This established body of knowledge, however, is not applicable to the robotic network setting because of the agents' mobility and the ensuing dynamic communication topology.

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An important contribution towards a network model of mobile interacting robots is introduced by Suzuki and Yamashita [43], see also [1, 42]. The Suzuki-Yamashita model consists of a group of "distributed anonymous mobile robots" that interact by sensing each other's relative position. A related model is presented in [13, 14]. A brief survey of models, algorithms, and the need for appropriate complexity notions is presented in [40]. This model is discussed in Section 2.4.

Recently, a notion of communication complexity for control and communication algorithms in multi-robot systems is analyzed in [19], see also [20] where a formal model of communication and control laws for multi-agent networks is proposed. A general modeling paradigm is discussed in [28]. The time complexity of a distributed algorithm for coordinated motion planning is computed in [41].

Finally, we conclude this literature review session with a general overview of the recent large research effort dedicated to multi-vehicle systems in the robotics and control domains. A survey on cooperative mobile robotics is presented in [4] and an overview of control and communication issues is contained in [21]. Specific topics of interest include formation control [2, 9, 11, 18, 38, 43], rendezvous [1, 5, 14, 25, 26], flocking [17], swarm aggregation [16], gradient climbing [33], cyclic pursuit [29], robotic exploration [45], deployment [6, 7], and foraging [15, 37]. Consensus and control theoretical problems on dynamic graphs are discussed in [32, 35, 39] and in [30], respectively.

Statement of contributions We summarize our approach as follows. A *robotic network* is a group of robotic agents moving in space and endowed with communication capabilities. The agents position obey a differential equation and the communication topology is a function of the agents' relative positions. Each agent repeatedly performs communication, computation and physical motion as follows. At predetermined time instants, the agents exchange information along the communication graph and update their internal state. Between successive communication instants, the agents move according to a motion control law, computed as a function of the agent location and of the available information gathered through communication with other agents. In short, a *control and communication law* for a robotic network consists of a message-generation function (what do the agents communicate?), a state-transition function (how do the agents update their internal state with the received information?), and a motion control law (how do the agents move between communication rounds?). We then define the notion of *time complexity* of a control and communication law (aimed at solving a given coordination task) as the minimum number of communication rounds required by the agents to achieve the task. The *time complexity of a coordination task*, as opposed to that of an algorithm, is the minimum time complexity of any algorithm achieving the task. We also provide similar definitions for notions of mean and total communication complexity. In particular, we show that our model of time and communication complexity satisfies a basic well-posedness property that we refer to as "invariance under reschedulings."

Building on these notions we establish complexity estimates for a few basic motion coordination algorithms such as pursuit, rendezvous, and deployment. First, for a network of agents moving on the circle, we introduce a dynamic law, called agree-and-pursue, that combines elements of the classic cyclic pursuit and leader election problems; see [27, 29], respectively. We show that this law achieves consensus on the agents' direction of motion and equidistance between the agents' positions. Furthermore, we show that these tasks are complete in time $O(N)$ and $O(N^2 \log N)$, respectively. Second, we analyze a simple averaging law for a network of locally-connected agents moving on a line. This law is related to the widely known Vicsek's model, see [17, 44]. We show that this law achieves rendezvous (without preserving connectivity) and that its time complexity belongs to $\Omega(N)$ and $O(N^5)$. Third, for a network of locally-connected agents moving on a line or on a segment, we show that the well-known circumcenter algorithm by [1] has time complexity of order $\Theta(N)$. (This algorithm achieves rendezvous while preserving connectivity with a communication graph with $O(N^2)$ links.) We then consider a network based on a different communication graph, called the limited Delaunay graph, that arises naturally in computational geometry and in the study of wireless communication topologies. For this less dense graph with $O(N)$ communication links, we show that the time complexity of the circumcenter algorithm grows to $\Theta(N^2 \log N)$. For a network of agents moving on $\mathbb{R}^d$ (with a certain communication graph) we introduce a novel "parallel-circumcenter algorithm" and establish its time complexity of order $\Theta(N)$. Fourth and last, for a network of agents in a one-dimensional environment, we show that the time complexity of the deployment algorithm introduced in [7] is $O(N^3 \log N)$. To obtain these complexity estimates, we develop some novel analysis methods. In particular, we develop a key set of results on tridiagonal Toeplitz and circulant matrices that characterize their convergence rates as a function of the matrices dimensions.

We refer the reader to the Conclusions (Section 5) for a discussion about the limitations of the proposed results and about avenues for future research.

Organization Section 2 presents a general approach to the modeling of robotic networks by formally introducing various notions including, for example, those of communication graph, control and communication law, and network evolution. Section 3 defines the notions of task and of time and communication complexity for a control and communication law. Section 4 presents a few motion coordination algorithms performing the basic tasks of rendezvous and deployment. For each algorithm, we characterize time and communication complexity, along with asymptotic correctness. Finally, we present our conclusions in Section 5. In the appendices, we review some basic computational geometric structures and we prove some key facts about the discrete-time dynamical systems defined by tridiagonal Toeplitz and circulant matrices.

Notation We let `BooleSet` be the set $\{\texttt{true}, \texttt{false}\}$. We let $\prod_{i\in\{1,\dots,N\}} S_i$ denote the Cartesian product of sets $S_1, \dots, S_N$. We let $\mathbb{R}_+$ and $\overline{\mathbb{R}}_+$ denote the set of strictly positive and non-negative real numbers, respectively. We let $\mathbb{N}$ denote the set of positive natural numbers and $\mathbb{N}_0$ denote the set of non-negative integers. If S is a set, then $\operatorname{diag}(S \times S) = \{(s, s) \in S \times S \mid s \in S\}$. For $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ denote the floor of x. For $x \in \mathbb{R}^d$, we let $\|x\|_2$ and $\|x\|_\infty$ denote the Euclidean and the ∞-norm of x, respectively. Recall that $\|x\|_\infty \le \|x\|_2 \le \sqrt{d}\|x\|_\infty$ for all $x \in \mathbb{R}^d$. For $x \in \mathbb{R}^d$ and $r \in \mathbb{R}_+$, we let $B(x, r)$ and $\overline{B}(x, r)$ denote the open and closed ball in $\mathbb{R}^d$ centered at x of radius r, respectively. We let $\mathbf{e}_1, \dots, \mathbf{e}_d$ be the standard orthonormal basis of $\mathbb{R}^d$. Also, we define the vectors $\mathbf{0} = (0, \dots, 0)$ and $\mathbf{1} = (1, \dots, 1)$ in $\mathbb{R}^d$. For $f, g \colon \mathbb{N} \to \mathbb{R}$, we say that $f \in O(g)$ (respectively, $f \in \Omega(g)$) if there exist $N_0 \in \mathbb{N}$ and $k \in \mathbb{R}_+$ such that $|f(N)| \le k|g(N)|$ for all $N \ge N_0$ (respectively, $|f(N)| \ge k|g(N)|$ for all $N \ge N_0$). If $f \in O(g)$ and $f \in \Omega(g)$, then we use the notation $f \in \Theta(g)$. Finally, we refer the reader to Appendix A for some useful geometric concepts.

2 A formal model for synchronous robotic networks

In this section we introduce a notion of robotic network, as a group of robotic agents with the ability to move and to communicate according to a specified communication topology. Our notion of control and communication law for a robotic network parallels the definition in [27] of synchronous distributed algorithm.

2.1 The physical components of a robotic network

Here we introduce our basic definition of physical quantities such as the agents and such as the ability of agents to communicate. We begin by defining a control system as our basic model for how each robotic agent moves in space.

Definition 2.1 *A* control system *is a tuple* (X, U, X_0, f) *consisting of*

(i) X *is a differentiable manifold, called the* state space*;*

(ii) U *is a compact subset of* $\mathbb{R}^m$ *containing* 0*, called the* input space*;*

(iii) X_0 *is a subset of* X*, called the* set of allowable initial states*;*

(iv) $f\colon X \times U \to TX$ *is a* C^∞*-map with* $f(x,u) \in T_xX$ *for all* $(x,u) \in X \times U$*.* •

We refer to $x \in X$ and $u \in U$ as a *state* and an *input* of the control system, respectively. We will often consider control-affine systems, i.e., control systems for which $f(x,u) = f_0(x) + \sum_{a=1}^{m} f_a(x)\, u_a$. In such a case, with a slight abuse of notation, we will represent the map f as the ordered family of C^∞-vector fields $(f_0, f_1, \ldots, f_m)$ on X. We will also sometime consider *driftless* systems, i.e., control systems for which $f(x, 0) = 0$.

Definition 2.2 *A* network of robotic agents *(or* robotic network*)* $\mathcal{S}$ *is a tuple* $(I, \mathcal{A}, E_{\text{cmm}})$ *consisting of*

(i) $I = \{1, \ldots, N\}$*;* I *is called the* set of unique identifiers (UIDs)*;*

(ii) $\mathcal{A} = \{A^{[i]}\}_{i\in I} = \{(X^{[i]}, U^{[i]}, X_0{}^{[i]}, f^{[i]})\}_{i\in I}$ *is a set of control systems; this set is called the* set of physical agents*;*

(iii) E_{cmm} *is a map from* $\prod_{i\in I} X^{[i]}$ *to the subsets of* $I \times I \setminus \operatorname{diag}(I \times I)$*; this map is called the* communication edge map*.*

If $A^{[i]} = (X, U, X_0, f)$ *for all* $i \in I$*, then the robotic network is called* uniform*.* •

Let us comment on this definition and on how robotic agents communicate in a robotic network $(I, \mathcal{A}, E_{\text{cmm}})$.

Remarks 2.3 (i) By convention, we let the superscript $[i]$ denote the variables and spaces which correspond to the agent with unique identifier i; for instance, $x^{[i]} \in X^{[i]}$ and $x_0^{[i]} \in X_0^{[i]}$ denote the state and the initial state of agent $A^{[i]}$, respectively. We refer to $(x^{[1]}, \ldots, x^{[N]}) \in \prod_{i\in I} X^{[i]}$ as a *state* of the network.

(ii) The map E_{cmm} models the topology of the communication service between the agents. In other words, at a network state $x = (x^{[1]}, \ldots, x^{[N]})$, two agents at locations $x^{[i]}$ and $x^{[j]}$ can communicate if the pair (i, j) is an edge in $E_{\text{cmm}}(x^{[1]}, \ldots, x^{[N]})$. Accordingly, we refer to the pair $(I, E_{\text{cmm}}(x^{[1]}, \ldots, x^{[N]}))$ as the *communication graph* at x. When and what agents communicate is discussed below.

Maps of the form $E\colon \prod_{i\in I} X^{[i]} \to 2^{I\times I\setminus\operatorname{diag}(I\times I)}$ are called *proximity edge maps*, they arise in wireless communication and in computational geometry, we refer the reader to Appendix A for more details. Excluding edges of the form (i,i), for $i \in I$, means that any individual agent does not communicate with itself. •

To make things concrete, let us now present some useful examples of robotic networks; we will use these examples throughout the paper. We start with a fairly common example and define some interesting variations.

Example 2.4 (Locally-connected first-order agents in $\mathbb{R}^d$) Consider N points $x^{[1]}, \dots, x^{[N]}$ in the Euclidean space $\mathbb{R}^d$, $d \geq 1$, obeying a first-order dynamics $\dot{x}^{[i]}(t) = u^{[i]}(t)$. According to Definition 2.1, these are identical agents of the form $A = (\mathbb{R}^d, \mathbb{R}^d, \mathbb{R}^d, (\mathbf{0}, \mathbf{e}_1, \dots, \mathbf{e}_d))$. Assume that each agent can communicate to any other agent within Euclidean distance r, that is, adopt as communication edge map the r-disk proximity edge map $E_{r\text{-disk}}$ defined in Appendix A. These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}} = (I, \mathcal{A}, E_{r\text{-disk}})$. •

Example 2.5 (LD-connected first-order agents in $\mathbb{R}^d$) Consider the set of physical agents defined in the previous example. For $r \in \mathbb{R}_+$, recall from Appendix A the r-limited Delaunay map $E_{r\text{-LD}}$ defined by

$$(i,j) \in E_{r\text{-LD}}(x^{[1]}, \dots, x^{[N]}) \quad \text{if and only if} \quad \big(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})\big) \cap \big(V^{[j]} \cap \overline{B}(x^{[j]}, \tfrac{r}{2})\big) \neq \emptyset, \ i \neq j,$$

where $\{V^{[1]}, \dots, V^{[N]}\}$ is the Voronoi partition of $\mathbb{R}^d$ generated by $\{x^{[1]}, \dots, x^{[N]}\}$. These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}} = (I, \mathcal{A}, E_{r\text{-LD}})$. •

Example 2.6 (Locally-∞-connected first-order agents in $\mathbb{R}^d$) Consider the set of physical agents defined in the previous two examples. For $r \in \mathbb{R}_+$, define the proximity edge map $E_{r\text{-}\infty\text{-disk}}$ by

$$(i,j) \in E_{r\text{-}\infty\text{-disk}}(x^{[1]}, \dots, x^{[N]}) \quad \text{if and only if} \quad \|x^{[i]} - x^{[j]}\|_\infty \leq r, \ i \neq j.$$

These data define the uniform robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-}\infty\text{-disk}} = (I, \mathcal{A}, E_{r\text{-}\infty\text{-disk}})$. •

In what follows, let $\mathbb{S}^1$ be the unit circle, and measure positions on the circle counterclockwise from the positive horizontal axis. Let us be specific about distances on the circle and related concepts. For $x, y \in \mathbb{S}^1$, we let $\operatorname{dist}(x,y) = \min\{\operatorname{dist}_{\mathsf{c}}(x,y), \operatorname{dist}_{\mathsf{cc}}(x,y)\}$. Here, $\operatorname{dist}_{\mathsf{c}}(x,y) = (x-y) \pmod{2\pi}$ is the clockwise distance, that is, the path length from x to y traveling clockwise. Similarly, $\operatorname{dist}_{\mathsf{cc}}(x,y) = (y-x) \pmod{2\pi}$ is the counterclockwise distance, i.e., the path length from x to y traveling counterclockwise. Here $x \pmod{2\pi}$ is the remainder of the division of x by 2π.

Example 2.7 (Locally-connected first-order agents on the circle) For $r \in \mathbb{R}_+$, consider the uniform robotic network $\mathcal{S}_{\mathbb{S}^1, r\text{-disk}} = (I, \mathcal{A}, E_{r\text{-disk}})$ composed of identical agents of the form $(\mathbb{S}^1, (0, \mathbf{e}))$. Here $\mathbf{e}$ is the vector field on $\mathbb{S}^1$ describing unit-speed counterclockwise rotation. We define the r-disk proximity edge map $E_{r\text{-disk}}$ on the circle by

$$(i,j) \in E_{r\text{-disk}}(\theta^{[1]}, \dots, \theta^{[N]}) \quad \text{if and only if} \quad \operatorname{dist}(\theta^{[i]}, \theta^{[j]}) \leq r\,,$$

where $\operatorname{dist}(x,y)$ is the geodesic distance between the two points x, y on the circle. •

2.2 Control and communication laws for robotic networks

Here we present a discrete-time communication, continuous-time motion model for the evolution of a robotic network. In our model, the robotic agents evolve in the physical domain in continuous-time and have the ability to exchange information (position and/or dynamic variables) that affect their motion at discrete-time instants.

Definition 2.8 (Control and communication law) *Let $\mathcal{S}$ be a robotic network. A* (synchronous, dynamic, feedback) control and communication law *$\mathcal{CC}$ for $\mathcal{S}$ consists of the sets:*

(i) $\mathbb{T} = \{t_\ell\}_{\ell \in \mathbb{N}_0} \subset \overline{\mathbb{R}}_+$ *is an increasing sequence of time instants, called* communication schedule*;*

(ii) *L is a set containing the* `null` *element, called the* communication language*; elements of L are called* messages*;*

(iii) *$W^{[i]}$, $i \in I$, are sets of values of some* logic variables *$w^{[i]}$, $i \in I$;*

(iv) *$W_0^{[i]} \subseteq W^{[i]}$, $i \in I$, are subsets of* allowable initial values*;*

and of the maps:

(i) $\mathrm{msg}^{[i]} \colon \mathbb{T} \times X^{[i]} \times W^{[i]} \times I \to L$, *$i \in I$, are called* message-generation functions*;*

(ii) $\mathrm{stf}^{[i]} \colon \mathbb{T} \times W^{[i]} \times L^N \to W^{[i]}$, *$i \in I$, are called* state-transition functions*;*

(iii) $\mathrm{ctl}^{[i]} \colon \overline{\mathbb{R}}_+ \times X^{[i]} \times X^{[i]} \times W^{[i]} \times L^N \to U^{[i]}$, *$i \in I$, are called* control functions. •

We will sometimes refer to a control and communication law as a *motion coordination algorithm.* Control and communication laws might have various properties.

Definition 2.9 (Properties of control and communication laws) *Let $\mathcal{S}$ be a robotic network and $\mathcal{CC}$ be a control and communication law for $\mathcal{S}$.*

(i) *If $\mathcal{S}$ is uniform and if $W^{[i]} = W$, $\mathrm{msg}^{[i]} = \mathrm{msg}$, $\mathrm{stf}^{[i]} = \mathrm{stf}$, $\mathrm{ctl}^{[i]} = \mathrm{ctl}$, for all $i \in I$, then $\mathcal{CC}$ is said to be* uniform *and is described by a tuple* $(\mathbb{T}, L, W, \{W_0^{[i]}\}_{i\in I}, \mathrm{msg}, \mathrm{stf}, \mathrm{ctl})$.

(ii) *If $W^{[i]} = W_0^{[i]} = \emptyset$ for all $i \in I$, then $\mathcal{CC}$ is said to be* static *and is described by a tuple* $(\mathbb{T}, L, \{\mathrm{msg}^{[i]}\}_{i\in I}, \{\mathrm{ctl}^{[i]}\}_{i\in I})$, *with* $\mathrm{msg}^{[i]} \colon \mathbb{T}\times X^{[i]} \times I \to L$, *and* $\mathrm{ctl}^{[i]} \colon \mathbb{T} \times X^{[i]} \times X^{[i]} \times L^N \to U^{[i]}$.

(iii) *$\mathcal{CC}$ is said to be* time-independent *if the message-generation, state-transition and control functions are of the form* $\mathrm{msg}^{[i]} \colon X^{[i]} \times W^{[i]} \times I \to L$, $\mathrm{stf}^{[i]} \colon W^{[i]} \times L^N \to W^{[i]}$, $\mathrm{ctl}^{[i]} \colon X^{[i]} \times X^{[i]} \times W^{[i]} \times L^N \to U^{[i]}$, *$i \in I$, respectively.* •

Roughly speaking this definition has the following meaning: for all $i \in I$, to the ith physical agent corresponds a logic process, labeled i, that performs the following actions. First, at each time instant $t_\ell \in \mathbb{T}$, the ith logic process sends to each of its neighbors in the communication graph a message (possibly the `null` message) computed by applying the message-generation function to the current values of $x^{[i]}$ and $w^{[i]}$. After a negligible period of time (therefore, still at time instant $t_\ell \in \mathbb{T}$), the ith logic process resets the value of its logic variables $w^{[i]}$ by applying the state-transition function to the current value of $w^{[i]}$, and to the messages received at time t_ℓ. Between communication instants, i.e., for $t \in [t_\ell, t_{\ell+1})$, the motion of the ith agent is determined by applying the control function to the current value of $x^{[i]}$, the value of $x^{[i]}$ at t_ℓ, and the current value of $w^{[i]}$. This idea is formalized as follows.

Definition 2.10 (Evolution of a robotic network subject to a control and communication law) *Let $\mathcal{S}$ be a robotic network and $\mathcal{CC}$ be a control and communication law for $\mathcal{S}$. The* evolution *of $(\mathcal{S}, \mathcal{CC})$ from initial conditions $x_0^{[i]} \in X_0{}^{[i]}$ and $w_0^{[i]} \in W_0^{[i]}$, $i \in I$, is the set of curves $x^{[i],\ell} \colon [t_\ell, t_{\ell+1}] \to X^{[i]}$, $i \in I$, $\ell \in \mathbb{N}_0$, and $w^{[i]} \colon \mathbb{T} \to W^{[i]}$, $i \in I$, satisfying*

$$\dot{x}^{[i],\ell}(t) = f\big(x^{[i],\ell}(t),\, \mathrm{ctl}^{[i]}(t, x^{[i],\ell}(t), x^{[i],\ell}(t_\ell), w^{[i]}(t_\ell), y^{[i]}(t_\ell))\big),$$

where, for $\ell \in \mathbb{N}_0$, and $i \in I$,

$$x^{[i],\ell}(t_\ell) = x^{[i],\ell-1}(t_\ell)\,, \quad w^{[i]}(t_\ell) = \mathrm{stf}^{[i]}(t_\ell, w^{[i]}(t_{\ell-1}), y^{[i]}(t_\ell))\,,$$

with the conventions that $x^{[i],-1}(t_0) = x_0^{[i]}$ *and* $w^{[i]}(t_{-1}) = w_0^{[i]}$, $i \in I$. *Here, the function* $y^{[i]} \colon \mathbb{T} \to L^N$ *(describing the messages received by agent* i*) has components* $y_j^{[i]}(t_\ell)$, *for* $j \in I$, *given by*

$$y_j^{[i]}(t_\ell) = \begin{cases} \mathrm{msg}^{[j]}(t_\ell, x^{[j],\ell-1}(t_\ell), w^{[j]}(t_{\ell-1}), i), & \textit{if } (i,j) \in E_{\mathrm{cmm}}(x^{[1],\ell-1}(t_\ell), \dots, x^{[N],\ell-1}(t_\ell)), \\ \texttt{null}, & \textit{otherwise.} \end{cases} \qquad \bullet$$

Let us emphasize two limitations regarding the proposed communication model.

Remarks 2.11 (Idealized aspects of communication model) (i) We refer to $\mathcal{CC}$ as a *synchronous* control and communication law, because the communications between all agents takes place always at the same time for all agents. We do not discuss here the important setting of asynchronous laws.

(ii) The set L is used to exchange information between two robotic agents. The message `null` indicates no communication. We assume that the messages in the communication language L allow us to encode logical expressions such as `true` and `false`, integers, and real numbers. A realistic assumption on the communication language would be to adopt a finite-precision representation for integers and real numbers in the messages. Instead, in what follows, we neglect any inaccuracies due to quantization and we discuss this topic as an open problem in Section 5. •

In the following remarks we introduce additional useful notations and emphasize an important assumption.

Remarks 2.12 (Related concepts and notations) (i) To distinguish between the `null` and the non-`null` messages received by an agent at a given time instant, it is convenient to define the *natural projection* $\pi_L \colon L^N \to 2^L$ that maps an array of messages y to the subset of L containing only the non-`null` messages in y.

(ii) In many uniform control and communication laws, the messages interchanged among the network agents are (quantized representations of) the agents' states and dynamic states. The corresponding communication language is $L = X \times W$ and message generation function $\mathrm{msg}_{\mathrm{std}} \colon \mathbb{T} \times X \times W \times I \to X \times W$ is referred to as the *standard message-generation function* and is defined by $\mathrm{msg}_{\mathrm{std}}(t, x, w, j) = (x, w)$.

(iii) By concatenating the curves $x^{[i],\ell}$ and $w^{[i],\ell}$, for $\ell \in \mathbb{N}_0$, we can define the evolution of the ith robotic agent $\overline{\mathbb{R}}_+ \ni t \mapsto (x^{[i]}(t), w^{[i]}(t)) \in X^{[i]} \times W^{[i]}$. Additionally we can define the curves

$$\overline{\mathbb{R}}_+ \ni t \mapsto x(t) = (x^{[1]}(t), \dots, x^{[N]}(t)) \in \prod_{i \in I} X^{[i]},$$
$$\overline{\mathbb{R}}_+ \ni t \mapsto w(t) = (w^{[1]}(t), \dots, w^{[N]}(t)) \in \prod_{i \in I} W^{[i]}.$$

(iv) In the proposed notion of synchronous robotic network, we assume that the states $(x^{[1]}, \dots, x^{[N]}) \in \prod_{i \in I} X^{[i]}$ are not reset after a communication round, i.e., we are setting $x^{[i],\ell}(t_\ell) = x^{[i],\ell-1}(t_\ell)$. This is reasonable for instance when $(x^{[1]}, \dots, x^{[N]})$ are positions or other physical variables. It would be possible, though, to allow for more general situations by defining state-transition functions of the form $\mathrm{stf}^{[i]} \colon \mathbb{T} \times X^{[i]} \times W^{[i]} \times L^N \to X^{[i]} \times W^{[i]}$, for $i \in I$, and setting $(x^{[i],\ell}, w^{[i],\ell})(t_\ell) =$

$\mathrm{stf}^{[i]}(t_\ell, x^{[i],\ell-1}(t_\ell), w^{[i],\ell-1}(t_\ell), y^{[i]}(t_\ell))$. Other possible extensions would be to consider (1) non-deterministic continuous-time evolutions of the state variables (by means of controlled differential inclusions, as opposed to the controlled differential equations) or (2) non-deterministic discrete-time updates of the dynamical variables (by means of set-valued state-transitions functions). Further generalizations are also possible, but we do not consider them here in the sake of simplicity. •

2.3 Example control and communication laws

It is now a good time to present various examples of control and communication laws. We start by considering a very simple averaging algorithm as a static control and communication law. We then present an interesting dynamic control and communication law on the circle.

Example 2.13 (The move-toward-average control and communication law) From Example 2.4, consider the uniform network $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$ of locally-connected first-order agents in $\mathbb{R}^d$. We now define a static, uniform and time-independent law that we refer to as the move-toward-average law and that we denote by $\mathcal{CC}_{\mathrm{avrg}}$. We loosely describe it as follows:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each communication round each agent transmits its position. Between communication rounds, each agent moves towards and reaches the point that is the average of its neighbors' positions; the average point is computed including the agent's own position.

Next, we *formally* define the law as follows. First, we take $\mathbb{T} = \mathbb{N}_0$ and we assume that each agent operates with the standard message-generation function, i.e., we set $L = \mathbb{R}^d$ and $\mathrm{msg}(x, j) = \mathrm{msg}_{\mathrm{std}}(x, j) = x$. Second, we define the control function $\mathrm{ctl}\colon \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\mathrm{ctl}(x, x_{\mathrm{smpld}}, y) = -k_{\mathrm{prop}} \,\mathrm{vers}\big(x - \mathrm{avrg}(y \cup \{x_{\mathrm{smpld}}\})\big),$$

where $k_{\mathrm{prop}} \geq r$, $\mathrm{vers}\colon \mathbb{R}^d \to \mathbb{R}^d$ is defined by $\mathrm{vers}(0) = 0$ and $\mathrm{vers}(v) = v/\|v\|_2$ for $v \neq 0$, and the map avrg computes the average of a finite point set in $\mathbb{R}^d$:

$$\mathrm{avrg}(S) = \frac{1}{\displaystyle\sum_{p\in\pi_{\mathbb{R}}(S)} 1} \sum_{p\in\pi_{\mathbb{R}}(S)} p\,.$$

In summary we set $\mathcal{CC}_{\mathrm{avrg}} = (\mathbb{N}_0, \mathbb{R}^d, \mathrm{msg}_{\mathrm{std}}, \mathrm{ctl})$. An implementation of this control and communication law is shown in Fig. 1 for $d = 1$. Note that, along the evolution, (1) several agents *rendezvous*, i.e., agree upon a common location, and (2) some agents are connected at the simulation's beginning and not connected at the simulation's end. Finally, we remark that this law is related to the Vicsek's model discussed in [17, 44]. •

Example 2.14 (The agree-and-pursue dynamic control and communication law) From Example 2.7, consider the uniform network $\mathcal{S}_{\mathbb{S}^1,r\text{-disk}}$ of locally-connected first-order agents in $\mathbb{S}^1$. We now define the agree-and-pursue law, denoted by $\mathcal{CC}_{\text{agr-pursuit}}$, as the uniform and time-independent law loosely described as follows:

> *[Informal description]* The dynamic variables are `drctn` taking values in $\{\mathtt{c}, \mathtt{cc}\}$ and `prior` taking values in I. At each communication round, each agent transmits its position and its dynamic variables and sets its dynamic variables to those of the incoming message

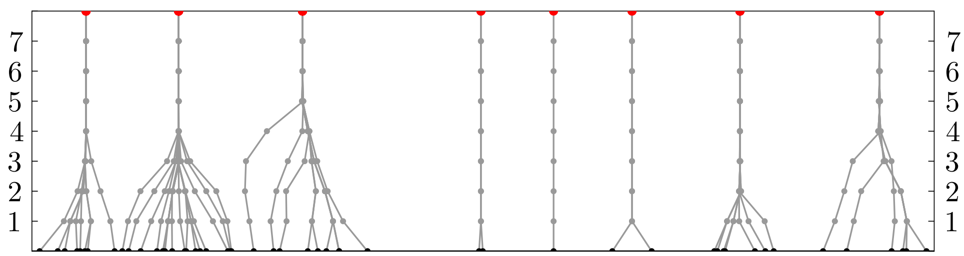

Figure 1: Evolution of a robotic network under the move-toward-average control and communication law in Example 2.13 implemented over the r-disk graph, with $r = 1.5$. The vertical axis corresponds to the elapsed time, and the horizontal axis to the positions of the agents in the real line. The 51 agents are initially randomly deployed over the interval $[-15, 15]$.

> with the largest value of `prior`. Between communication rounds, each agent moves in the counterclockwise or clockwise direction depending on whether its dynamic variable `drctn` is `cc` or `c`. For $k_{\text{prop}} \in]0, \frac{1}{2}[$, each agent moves k_{prop} times the distance to the immediately next neighbor in the chosen direction, or, if no neighbors are detected, k_{prop} times the communication range r.

Next, we *formally* define the law as follows. Each agent has logic variables $w = (\texttt{drctn}, \texttt{prior})$, where $w_1 = \texttt{drctn} \in \{\texttt{cc}, \texttt{c}\}$, with arbitrary initial value, and $w_2 = \texttt{prior} \in I$, with initial value equal to the agent's identifier i. In other words, we define $W = \{\texttt{cc}, \texttt{c}\} \times I$, and we set $W_0^{[i]} = \{\texttt{cc}, \texttt{c}\} \times \{i\}$. Each agent $i \in I$ operates with the standard message-generation function, i.e., we set $L = \mathbb{S}^1 \times W$ and $\text{msg}^{[i]} = \text{msg}_{\text{std}}$, where $\text{msg}_{\text{std}}(\theta, w, j) = (\theta, w)$. The state-transition function is defined by

$$\text{stf}(w, y) = \text{argmax}\{z_2 \mid z \in (\pi_L(y))_2 \cup \{w\}\}.$$

For $k_{\text{prop}} \in \mathbb{R}_+$, the control function is

$$\text{ctl}(\theta, \theta_{\text{smpld}}, w, y) = k_{\text{prop}} \begin{cases} \min\{r\} \cup \{\text{dist}_{\texttt{cc}}(\theta_{\text{smpld}}, \theta_{\text{rcvd}}) \mid \theta_{\text{rcvd}} \in (\pi_L(y))_1\}, & \text{if } \texttt{drctn} = \texttt{cc}, \\ -\min\{r\} \cup \{\text{dist}_{\texttt{c}}(\theta_{\text{smpld}}, \theta_{\text{rcvd}}) \mid \theta_{\text{rcvd}} \in (\pi_L(y))_1\}, & \text{if } \texttt{drctn} = \texttt{c}. \end{cases}$$

Finally, we sketch the control and communication in equivalent pseudocode language. This is possible for this example, and necessary for more complicated ones. For example, the state-transition function is written as:

```
function stf ((drctn, prior), y)
for each non-null message (θ_rcvd, (drctn_rcvd, prior_rcvd)) in y:
  if (prior_rcvd > prior), then
    drctn := drctn_rcvd
    prior := prior_rcvd
  endif
endfor
return (drctn,prior)
```

Similarly, the control function ctl is written as:

```
function ctl (θ, θ_smpld, (drctn, prior), y)
d_tmp := r
for each non-null message (θ_rcvd, (drctn_rcvd, prior_rcvd)) in y:
  if (drctn = cc) AND (dist_cc(θ_smpld, θ_rcvd) < d_tmp), then
    d_tmp := dist_cc(θ_smpld, θ_rcvd)
  else if (drctn = c) AND (dist_c(θ_smpld, θ_rcvd) < d_tmp), then
    d_tmp := dist_c(θ_smpld, θ_rcvd)
  endif
endfor
if (drctn = cc), then return k_prop d_tmp, else return −k_prop d_tmp
endif
```

An implementation of this control and communication law is shown in Fig. 2. Note that, along the evolution, all agents agree upon a common direction of motion and, after suitable time, they reach a uniform distribution. Finally, we remark that this law is related to the leader election algorithm discussed in [27]. •

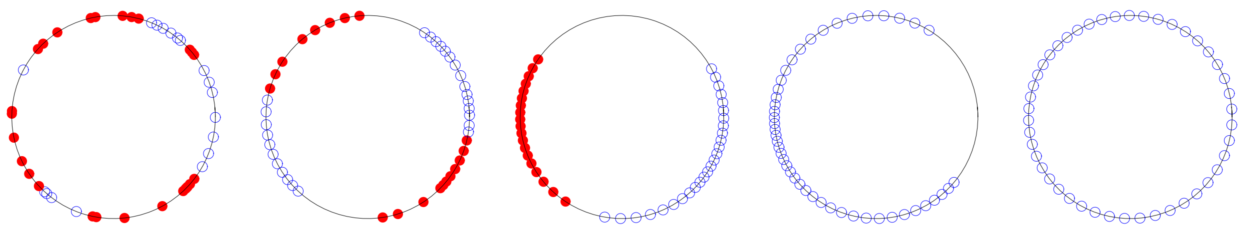

Figure 2: The agree-and-pursue control and communication law in Example 2.14 with $N = 45$, $r = 2\pi/40$, and $k_{\text{prop}} = 1/4$. Disks and circles correspond to agents moving counterclockwise and clockwise, respectively. The initial positions and the initial directions of motion are randomly generated. The five pictures depict the network state at times $0, 12, 37, 100, 400$.

2.4 Groups of robotic agents with relative-position sensing

In this last subsection on modeling, we discuss in some detail the Suzuki-Yamashita model mentioned in the Introduction, see [43]. This model consists of a group of identical mobile robots characterized as follows: no explicit communication takes place between the agents, at each instant of an "activation schedule," each robot measure the relative position of all other robots and moves according to a specified algorithm. In this model, robots are referred to as "anonymous" and "oblivious" in precisely the same way in which we defined control and communication laws to be uniform and static, respectively.

As compared with our notion of robotic network, the Suzuki-Yamashita model is more general in that the robots' activations schedules do not necessarily coincide (i.e., this model is asynchronous), and at the same time it is less general in that (1) robots cannot communicate any information other than their respective positions, and (2) each robot observes every other robot's position (i.e., the complete communication graph is adopted; this limitation is not present for example in [1]). Note that a control and communication law, as in our definition, can be implemented on a synchronous Suzuki-Yamashita model if the law (1) is static and uniform, (2) only relies on communicating the agents' positions (e.g., the message-generation function is the standard one), and (4) entails a control function that only depends on relative positions (as opposed to absolute positions).

3 Coordination tasks and complexity measures

In this section we introduce concepts and tools useful to analyze a communication and control law. We address the following issues: What is a coordination task for a robotic network? When does a control and communication law achieve a task? And with what time and communication complexity?

3.1 Coordination tasks

Our first analysis step is to characterize the correctness properties of a communication and control law. We do so by defining the notion of task and of task achievement by a robotic network.

Definition 3.1 (Coordination task) *Let $\mathcal{S}$ be a robotic network and let $\mathcal{W}$ be a set.*

(i) A coordination task *for $\mathcal{S}$ is a map $\mathcal{T}\colon \prod_{i\in I} X^{[i]} \times \mathcal{W}^N \to \texttt{BooleSet}$.*

(ii) If $\mathcal{W} = \emptyset$, then the coordination task is said to be static *and is described by a map $\mathcal{T}\colon \prod_{i\in I} X^{[i]} \to \texttt{BooleSet}$.*

Additionally, let $\mathcal{CC}$ a control and communication law for $\mathcal{S}$.

(i) The law $\mathcal{CC}$ is compatible *with the task $\mathcal{T}\colon \prod_{i\in I} X^{[i]} \times \mathcal{W}^N \to \texttt{BooleSet}$ if its logic variables take values in $\mathcal{W}$, that is, if $W^{[i]} = \mathcal{W}$, for all $i \in I$.*

(ii) The law $\mathcal{CC}$ achieves *the task $\mathcal{T}$ if it is compatible with it and if, for all initial conditions $x_0^{[i]} \in X_0^{[i]}$ and $w_0^{[i]} \in W_0^{[i]}$, $i \in I$, the corresponding network evolution $t \mapsto (x(t), w(t))$ has the property that there exists $T \in \mathbb{R}_+$ such that $\mathcal{T}(x(t), w(t)) = \texttt{true}$ for all $t \geq T$.* •

Loosely speaking, achieving a task might mean obtaining a specified pattern in the position of the agents or of their dynamic variables. We now give some examples of interesting coordination tasks.

Example 3.2 (Rendezvous tasks) First, let $\mathcal{S} = (I, \mathcal{A}, E_{\mathrm{cmm}})$ be a uniform robotic network. The *(exact) rendezvous task* $\mathcal{T}_{\mathrm{rndzvs}}\colon X^N \to \texttt{BooleSet}$ for $\mathcal{S}$ is the static task defined by

$$\mathcal{T}_{\mathrm{rndzvs}}(x^{[1]}, \dots, x^{[N]}) = \begin{cases} \texttt{true}, & \text{if } x^{[i]} = x^{[j]}, \text{ for all } (i,j) \in E_{\mathrm{cmm}}(x^{[1]}, \dots, x^{[N]}), \\ \texttt{false}, & \text{otherwise.} \end{cases}$$

Second, let $\mathcal{S} = (I, \mathcal{A}, E_{\mathrm{cmm}})$ be a uniform robotic network with agents' state space $X \subset \mathbb{R}^d$. Examples networks of this form are $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$, see Examples 2.4 and 2.13, and $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$, see Examples 2.5. For $\varepsilon > 0$, the *ε-rendezvous task* $\mathcal{T}_{\varepsilon\text{-rndzvs}}\colon X^N \to \texttt{BooleSet}$ for $\mathcal{S}$ is defined by

$$\mathcal{T}_{\varepsilon\text{-rndzvs}}(x) = \begin{cases} \texttt{true}, & \text{if } \left\| x^{[i]} - \operatorname{avrg}\left(\{x^{[i]}\} \cup \{x^{[j]} \mid (i,j) \in E_{\mathrm{cmm}}(x)\}\right) \right\|_2 < \varepsilon, \text{ for all } i \in I, \\ \texttt{false}, & \text{otherwise,} \end{cases}$$

where $x = (x^{[1]}, \dots, x^{[N]}) \in X^N \subset (\mathbb{R}^d)^N$. In other words, $\mathcal{T}_{\varepsilon\text{-rndzvs}}$ is $\texttt{true}$ at $x \in (\mathbb{R}^d)^N$ if, for all $i \in I$, $x^{[i]}$ is at distance less than ε from the average of its own position with the position of its E_{cmm}-neighbors. •

Example 3.3 (Agreement and equidistance tasks) From Example 2.7, consider the uniform network $\mathcal{S}_{\mathbb{S}^1,r\text{-disk}}$ of locally-connected first-order agents in $\mathbb{S}^1$. From Example 2.14, recall the agree-and-pursue control and communication law $\mathcal{CC}_{\text{agr-pursuit}}$ with dynamic variables taking values in $W = \{\texttt{cc}, \texttt{c}\} \times I$. There are two tasks of interest. First, we define the *agreement task* $\mathcal{T}_{\texttt{drctn}} \colon (\mathbb{S}^1)^N \times W^N \to$ `BooleSet` by

$$\mathcal{T}_{\texttt{drctn}}(\theta, w) = \begin{cases} \texttt{true}, & \text{if } \texttt{drctn}^{[1]} = \dots = \texttt{drctn}^{[N]}, \\ \texttt{false}, & \text{otherwise,} \end{cases}$$

where $\theta = (\theta^{[1]}, \dots, \theta^{[N]})$, $w = (w^{[1]}, \dots, w^{[N]})$, and $w^{[i]} = (\texttt{drctn}^{[i]}, \texttt{prior}^{[i]})$, for $i \in I$. Furthermore, for $\varepsilon > 0$, we define the static ε-*equidistance task* $\mathcal{T}_{\text{eqdstnc}} \colon (\mathbb{S}^1)^N \to$ `BooleSet` by

$$\mathcal{T}_{\varepsilon\text{-eqdstnc}}(\theta) = \begin{cases} \texttt{true}, & \text{if } \left| \min_{j \neq i} \operatorname{dist}_{\texttt{c}}(\theta^{[i]}, \theta^{[j]}) - \min_{j \neq i} \operatorname{dist}_{\texttt{cc}}(\theta^{[i]}, \theta^{[j]}) \right| < \varepsilon, \text{ for all } i \in I, \\ \texttt{false}, & \text{otherwise.} \end{cases}$$

In other words, $\mathcal{T}_{\varepsilon\text{-eqdstnc}}$ is true when, for every agent, the clockwise distance to the closest clockwise neighbor and the counterclockwise distance to the closest counterclockwise neighbor are approximately equal. •

Example 3.4 (Deployment tasks) By optimal deployment on the convex simple polytope $Q \subset \mathbb{R}^d$ with density function $\phi \colon Q \to \mathbb{R}_+$, we mean the following objective: place the agents on Q so that the expected square Euclidean distance from any point in Q to one of the agents is minimized. To define this task formally, let us review some known preliminary notions; we will require some computational geometric notions from Appendix A. We consider the following network objective function $\mathcal{H}_{\text{deplmnt}} \colon Q^N \to \mathbb{R}$,

$$\mathcal{H}_{\text{deplmnt}}(x^{[1]}, \dots, x^{[N]}) = \int_Q \min_{i \in I} \|q - x^{[i]}\|_2^2 \, \phi(q) dq \,. \tag{1}$$

This function and variations of it are studied in the facility location and resource allocation research literature; see [7, 34]. It is convenient [6] to study a generalization of this function. For $r \in \mathbb{R}_+$, define the saturation function $\operatorname{sat}_r \colon \mathbb{R} \to \mathbb{R}$ by $\operatorname{sat}_r(x) = x$ if $x \leq r$ and $\operatorname{sat}_r(x) = r$ otherwise. For $r \in \mathbb{R}_+$, define the new objective function $\mathcal{H}_{r\text{-deplmnt}} \colon Q^N \to \mathbb{R}$ by

$$\mathcal{H}_{r\text{-deplmnt}}(x^{[1]}, \dots, x^{[N]}) = \int_Q \min_{i \in I} \operatorname{sat}_{\frac{r}{2}}(\|q - x^{[i]}\|_2^2) \, \phi(q) dq \,. \tag{2}$$

Note that if $r \geq 2 \operatorname{diam}(Q)$, then $\mathcal{H}_{\text{deplmnt}} = \mathcal{H}_{r\text{-deplmnt}}$. Let $\{V^{[1]}, \dots, V^{[N]}\}$ be the Voronoi partition of Q associated with $\{x^{[1]}, \dots, x^{[N]}\}$. The partial derivative of the cost function takes the following meaningful form

$$\frac{\partial \mathcal{H}_{r\text{-deplmnt}}}{\partial x^{[i]}}(x^{[1]}, \dots, x^{[N]}) = 2 \operatorname{Mass}(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})) \big(\operatorname{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \tfrac{r}{2})) - x^{[i]} \big) , \quad i \in I \,.$$

(Here, as in Appendix A, $\operatorname{Mass}(S)$ and $\operatorname{Centroid}(S)$ are, respectively, the mass and the centroid of $S \subset \mathbb{R}^d$.) Clearly, the critical points of $\mathcal{H}_{r\text{-deplmnt}}$ are network states where $x^{[i]} = \operatorname{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \frac{r}{2}))$. We call such configurations $\frac{r}{2}$-centroidal Voronoi configurations. For $r \geq 2 \operatorname{diam}(Q)$, they coincide with the standard centroidal Voronoi configurations on Q. Fig. 3 illustrates these notions.

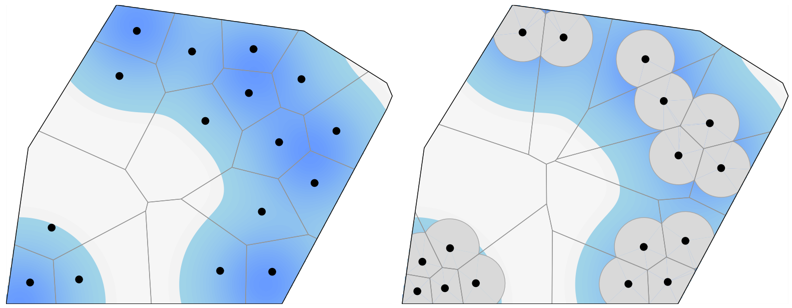

Figure 3: Centroidal and $\frac{r}{2}$-centroidal Voronoi configurations. The density function ϕ is depicted by a contour plot. For each agent i, the set $V^{[i]} \cap \overline{B}(p_i, \frac{r}{2})$ is plotted in light gray.

Motivated by these observations, we define the following deployment task. For $r, \varepsilon \in \mathbb{R}_+$, define the *$\varepsilon$-$r$-deployment task* $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}} \colon Q^N \to \texttt{BooleSet}$ by

$$\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}(x) = \begin{cases} \texttt{true}, & \text{if } \left\| x^{[i]} - \mathrm{Centroid}(V^{[i]} \cap \overline{B}(x^{[i]}, \frac{r}{2})) \right\|_2 \leq \varepsilon, \text{ for all } i \in I\,, \\ \texttt{false}, & \text{otherwise.} \end{cases}$$

Roughly speaking, $\mathcal{T}_{\varepsilon\text{-}r\text{deplmnt}}$ is $\texttt{true}$ for those network configurations where each agent is sufficiently close to the centroid of an appropriate region $V^{[i]} \cap \overline{B}(x^{[i]}, \frac{r}{2})$. Note that, with a natural definition of Voronoi partitions of the circle, the two tasks $\mathcal{T}_{\varepsilon\text{-eqdstnc}}$ and $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}$ are closely related. •

3.2 Complexity notions for control and communication laws and for coordination tasks

We are finally ready to define the key notions of time and communication complexity. These notions describe the cost that a certain control and communication law incurs while completing a certain coordination task. We also define the complexity of a task to be the infimum of the costs incurred by all laws that achieve that task.

First we define the time complexity of an achievable task as the minimum number of communication rounds needed by the agents to achieve the task $\mathcal{T}$.

Definition 3.5 (Time complexity) *Let $\mathcal{S}$ be a robotic network and let $\mathcal{T}$ be a coordination task for $\mathcal{S}$. Let $\mathcal{CC}$ be a control and communication law for $\mathcal{S}$ compatible with $\mathcal{T}$.*

(i) The time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ from $(x_0, w_0) \in \prod_{i \in I} X_0^{[i]} \times \prod_{i\in I} W_0^{[i]}$ *is*

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) = \inf\{\ell \mid \mathcal{T}(x(t_k), w(t_k)) = \texttt{true}\,, \textit{ for all } k \geq \ell\}\,,$$

where $t \mapsto (x(t), w(t))$ is the evolution of $(\mathcal{S}, \mathcal{CC})$ from the initial condition (x_0, w_0).

(ii) The time complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ *is*

$$\mathrm{TC}(\mathcal{T}, \mathcal{CC}) = \sup\left\{ \mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) \mid (x_0, w_0) \in \prod_{i\in I} X_0^{[i]} \times \prod_{i \in I} W_0^{[i]} \right\}.$$

(iii) The time complexity of $\mathcal{T}$ *is*

$$\mathrm{TC}(\mathcal{T}) = \inf\{\mathrm{TC}(\mathcal{T}, \mathcal{CC}) \mid \mathcal{CC} \textit{ is compatible with } \mathcal{T}\}\,.$$ •

Next, we define the notion of mean and total communication complexities for a task. As usual, we assume that the network $\mathcal{S}$ has a communication edge map E_{cmm} and that the control and communication law $\mathcal{CC}$ has language L and message-generation functions $\mathrm{msg}^{[i]}$, $i \in I$. With these data we can discuss the communication cost of realizing one communication round. At time $t \in \mathbb{T}$ from state $(x, w) \in \prod_{i\in I} X^{[i]} \times \prod_{i\in I} W^{[i]}$, an element of L needs to be transmitted for each edge of the directed graph $(I, E_{\mathrm{cmm}\setminus\emptyset}(t, x, w))$ defined by

$$(i, j) \in E_{\mathrm{cmm}\setminus\emptyset}(t, x, w) \quad \text{if and only if} \quad (i, j) \in E_{\mathrm{cmm}}(x) \ \text{ and } \ \mathrm{msg}^{[i]}(t, x^{[i]}, w^{[i]}, j) \neq \texttt{null}.$$

Next, we need a model for the cost of sending a message for each directed edge in $E_{\mathrm{cmm}\setminus\emptyset}$.

Definition 3.6 (One-round cost) *(i) For $I = \{1, \dots, N\}$, a function $\mathrm{C}_{\mathrm{rnd}} \colon 2^{I\times I} \to \overline{\mathbb{R}}_+$ is a* one-round cost function *if $\mathrm{C}_{\mathrm{rnd}}(\emptyset) = 0$, and $S_1 \subset S_2 \subset I \times I$ implies $\mathrm{C}_{\mathrm{rnd}}(S_1) \le \mathrm{C}_{\mathrm{rnd}}(S_2)$.*

(ii) A one-round cost function $\mathrm{C}_{\mathrm{rnd}}$ is additive *if, for all $S_1, S_2 \subset I \times I$, $S_1 \cap S_2 = \emptyset$ implies $\mathrm{C}_{\mathrm{rnd}}(S_1 \cup S_2) = \mathrm{C}_{\mathrm{rnd}}(S_1) + \mathrm{C}_{\mathrm{rnd}}(S_2)$.* •

We postpone our discussion about specific functions $\mathrm{C}_{\mathrm{rnd}}$ to the next subsection. Here we only emphasize that, for a given control and communication law $\mathcal{CC}$ with language L, the one-round cost depends on L; we therefore write it as $\mathrm{C}^L_{\mathrm{rnd}} \colon 2^{I\times I} \to \overline{\mathbb{R}}_+$.

Definition 3.7 (Communication complexity) *Let $\mathcal{S}$ be a robotic network and let $\mathcal{T}$ be a coordination task for $\mathcal{S}$. Let $\mathcal{CC}$ be a control and communication law for $\mathcal{S}$ compatible with $\mathcal{T}$, and let $\mathrm{C}^L_{\mathrm{rnd}} \colon 2^{I\times I} \to \overline{\mathbb{R}}_+$ be a one-round cost function.*

(i) Let $(x_0, w_0) \in \prod_{i\in I} X_0^{[i]} \times \prod_{i\in I} W_0^{[i]}$. The mean communication complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ from (x_0, w_0) *and the* total communication complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ from $(x_0, w_0) \in \prod_{i\in I} X_0^{[i]} \times \prod_{i\in I} W_0^{[i]}$ *are, respectively,*

$$\mathrm{MCC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) = \frac{1}{\mathrm{TC}(\mathcal{CC}, \mathcal{T}, x_0, w_0)} \sum_{\ell=0}^{\mathrm{TC}(\mathcal{T},\mathcal{CC},x_0,w_0)-1} \mathrm{C}^L_{\mathrm{rnd}} \circ E_{\mathrm{cmm}\setminus\emptyset}(t_\ell, x(t_\ell), w(t_\ell)),$$

$$\mathrm{TCC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) = \sum_{\ell=0}^{\mathrm{TC}(\mathcal{T},\mathcal{CC},x_0,w_0)-1} \mathrm{C}^L_{\mathrm{rnd}} \circ E_{\mathrm{cmm}\setminus\emptyset}(t_\ell, x(t_\ell), w(t_\ell)),$$

where $t \mapsto (x(t), w(t))$ is the evolution of $(\mathcal{S}, \mathcal{CC})$ from the initial condition (x_0, w_0). (Here MCC *is defined only for (x_0, w_0) with the property that $\mathcal{T}(x_0, w_0) = \texttt{false}$.)*

(ii) The mean communication complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ *and the* total communication complexity to achieve $\mathcal{T}$ with $\mathcal{CC}$ *are, respectively,*

$$\mathrm{MCC}(\mathcal{T}, \mathcal{CC}) = \sup \left\{ \mathrm{MCC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) \mid (x_0, w_0) \in \prod_{i\in I} X_0^{[i]} \times \prod_{i\in I} W_0^{[i]} \right\},$$

$$\mathrm{TCC}(\mathcal{T}, \mathcal{CC}) = \sup \left\{ \mathrm{TCC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) \mid (x_0, w_0) \in \prod_{i\in I} X_0^{[i]} \times \prod_{i\in I} W_0^{[i]} \right\}.$$

(iii) The mean communication complexity of $\mathcal{T}$ *and the* total communication complexity of $\mathcal{T}$ *are, respectively,*

$$\mathrm{MCC}(\mathcal{T}) = \inf\{\mathrm{MCC}(\mathcal{T}, \mathcal{CC}) \mid \mathcal{CC} \text{ is compatible with } \mathcal{T}\},$$

$$\mathrm{TCC}(\mathcal{T}) = \inf\{\mathrm{TCC}(\mathcal{T}, \mathcal{CC}) \mid \mathcal{CC} \text{ is compatible with } \mathcal{T}\}.$$ •

We conclude this subsection with some remarks.

Remarks 3.8 (i) The total communication complexity is equal to the average transmission cost during the execution multiplied by the number of rounds required to carry out the desired task. That is, for $(x_0, w_0) \in \prod_{i\in I} X_0^{[i]} \times \prod_{i\in I} W_0^{[i]}$,

$$\mathrm{TCC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) = \mathrm{MCC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) \cdot \mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0).$$

In turn this implies that $\mathrm{TCC}(\mathcal{T}, \mathcal{CC}) \leq \mathrm{MCC}(\mathcal{T}, \mathcal{CC}) \cdot \mathrm{TC}(\mathcal{T}, \mathcal{CC})$.

(ii) According to this notation, given a robotic network $\mathcal{S}$ and a control and communication law $\mathcal{CC}$, the time complexity of achieving a task $\mathcal{T}$ with $\mathcal{CC}$ is $\mathrm{TC}(\mathcal{T}, \mathcal{CC}) \in O(f)$ (resp. $\mathrm{TC}(\mathcal{T}, \mathcal{CC}) \in \Omega(f)$), if there exist $N_0 \in \mathbb{N}$ and $k \in \mathbb{R}_+$ such that $\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) \leq kf(N)$ for *all* initial conditions (x_0, w_0) for each $N \geq N_0$ (resp. if $\mathrm{TC}(\mathcal{T}, \mathcal{CC}, x_0, w_0) \geq kf(N)$ for at least *an* initial condition (x_0, w_0) for each $N \geq N_0$).

(iii) A different notion of communication complexity is defined in [19] for a different robotic network model. Transcribed to the current setting, this notion of communication complexity of the execution of a control and communication law $\mathcal{CC}$ from initial conditions (x_0, w_0) would read as

$$\mathrm{cc}(\mathcal{CC}, x_0, w_0) = \lim_{k\to+\infty} \frac{1}{k} \sum_{\ell=0}^{k} \mathrm{C}^L_{\mathrm{rnd}} \circ E_{\mathrm{cmm}\setminus\emptyset}(t_\ell, x(t_\ell), w(t_\ell)) \,. \tag{3}$$

Note that this definition does not make reference to the completion of a task. We will later come back to this notion in Section 3.4. •

3.3 Communication costs in unidirectional and omnidirectional wireless channels

In this subsection we discuss some modeling aspects of the one-round communication cost function $I \times I \supset E \mapsto \mathrm{C}_{\mathrm{rnd}}(E)$ described in Definition 3.6. First, let us mention that the definition is motivated by the assumptions that (1) the cost of exchanging any message is bounded, and that (2) this cost is zero only for the `null` message. More specific detail about the communication cost depends necessarily on the type of communication service available between the agents.

In *unidirectional* models of communication messages are sent in a point-to-point-wise fashion. Certain forms of communication, such as those based on the TCP-IP protocol, and certain technologies, such as wireless networks equipped with unidirectional antennas [22, 23], fall into this category. On the other hand, in an *omnidirectional* model of communication (e.g., wireless networks equipped with omnidirectional antennas), a single transmission made by a node can be heard by several other nodes at the same time. This has the advantage that, by choosing a sufficiently large transmission power, a signal can reach all the neighboring nodes in a single time instant.

Broadly speaking, it is very difficult to come up with an abstract model that captures adequately the cost of all possible communication technologies. For example, networking protocols for omnidirectional wireless networks rely on a many nested layers to handle, for example, media access, power control, congestion control, and routing. The presence of these layers and the non-trivial interactions between them make it difficult to assess communication costs of individual messages. Let us elaborate on this point in the following remark.

Remark 3.9 (Omnidirectional wireless communication) The *Minimum Power Broadcast* (MPB) problem, the *Medium Access Control* (MAC) problem, and their relationship are subjects of vigorous research in the wireless communications literature, see for instance [24] and references therein. Loosely speaking, the MPB problem consists of finding, for each agent i, the minimum broadcast radius $R^{[i]}$ such that if agent i sends a message with communication radius $R^{[i]}$, then its neighbors in a given graph E receive it. The MAC problem consists of determining a minimum number of broadcasting turns required for all agents to communicate their messages without interference. A schematic approach to these problems is as follows: first, from the communication graph (I, E), one constructs the *neighbor-induced* graph $(I, E_{\mathcal{N}})$ by

$$(i, j) \in E_{\mathcal{N}} \quad \text{if and only if} \quad (i, j) \in E \text{ or } (i, k), (j, k) \in E\,, \text{ for some } k \in I$$

In the new graph $(I, E_{\mathcal{N}})$, the set of neighbors of the agent i is composed by its neighbors in the graph (I, E), together with the their respective neighbors. As a second step, one has to compute the *chromatic number* of the graph, i.e., the minimum number of colors $\chi(E_{\mathcal{N}})$ needed to color the agents in such a way that there are no two neighboring agents with the same color. (This is also referred to as the *coloring-graph problem.*) Theorem 5.2.4 in [10] asserts that if a connected graph is neither complete, nor an odd cycle, then $\chi(E_{\mathcal{N}})$ is less than or equal to the maximum valency of the graph. Once the chromatic number has been determined, broadcasting turns can be established according to an ordered sequence of the agents' colors. Although this approach is clearly inadequate, it provides some basic pointers with regards to communication costs. •

Motivated by the difficulty of obtaining a detailed model, the rest of this paper relies on the following simplified models that capture some broad relevant aspects:

(i) For a unidirectional communication model, $E \mapsto \mathrm{C}_{\mathrm{rnd}}(E)$ is proportional to the total number of non-`null` messages sent over the directed edges in E, that is, $\mathrm{C}_{\mathrm{rnd}}(E) = c_0 \cdot \mathrm{cardinality}(E)$, where $c_0 \in \mathbb{R}_+$ is the cost of sending a single message. This one-round cost function is additive. This number is trivially bounded by twice the number of edges of the complete graph, which is $N(N-1)$. Therefore, for unidirectional models of communication, we have $\mathrm{MCC}(\mathcal{CC}, \mathcal{T}) \in O(N^2)$.

(ii) For an omnidirectional communication model, $E \mapsto \mathrm{C}_{\mathrm{rnd}}(E)$ is proportional to the number of turns employed to complete a communication round without interference between the agents (see Remark 3.9). This number is trivially upper bounded by N. Therefore, for omnidirectional models of communication, we have $\mathrm{MCC}(\mathcal{CC}, \mathcal{T}) \in O(N)$.

3.4 Invariance under rescheduling of control and communication laws

In this section, we discuss the invariance properties of the notions of time and communication complexity under the *rescheduling* of a control and communication law. The idea behind rescheduling is to "spread" the execution of the law over time without affecting the trajectories described by the robotic agents of the network. There are at least two natural ways of doing this. One possible way consists of the network slowing down its motion, and letting some communication rounds pass without effectively interchanging any messages. Another possible way is to schedule the messages originally sent at a single time instant to be sent over multiple consecutive time instants, and adapt the motion of the network accordingly. Our objective is here is to formalize these ideas and to examine the effect that these processes have on the notions of complexity introduced earlier. For simplicity we consider the setting of static laws; similar results can be obtained for the general setting.

Let $\mathcal{S} = (I, \mathcal{A}, E_{\mathrm{cmm}})$ be a robotic network with driftless physical agents, that is, a robotic network where each physical agent is a driftless control system. Let $\mathcal{CC} = (\mathbb{N}_0, L, \{\mathrm{ctl}^{[i]}\}_{i \in I}, \{\mathrm{msg}^{[i]}\}_{i \in I})$ be a static control and communication law. It is out intention to define a new control and communication law by modifying $\mathcal{CC}$; to do so we introduce some notation. Let $s \in \mathbb{N}$, with $s \leq N$, and let $\mathcal{P}_I = \{I_0, \dots, I_{s-1}\}$ be an *s-partition* of I, that is, $I_0, \dots, I_{s-1}$ are disjoint and nonempty subsets of I and $I = \cup_{k=0}^{s-1} I_k$.

For $i \in I$, define the message-generation functions $\mathrm{msg}^{[i]}_{(s,\mathcal{P}_I)} \colon \mathbb{N}_0 \times X^{[i]} \times I \to L$ by

$$\mathrm{msg}^{[i]}_{(s,\mathcal{P}_I)}(t_\ell, x, j) = \begin{cases} \mathrm{msg}^{[i]}(t_{\lfloor \ell/s \rfloor}, x, j), & \text{if } i \in I_k \text{ and } k = \ell (\mathrm{mod}\, s)\,, \\ \texttt{null}, & \text{otherwise}\,. \end{cases} \tag{4}$$

According to this new message-generation function, only the agents with unique identifier in I_k will send messages at time t_ℓ, with $\ell \in \{k + as \mid a \in \mathbb{N}_0\}$. Equivalently, this can be stated as follows. Define the increasing function $F \colon \mathbb{N}_0 \to \mathbb{N}_0$ by $F(\ell) = s(\ell+1) - 1$. According to the message-generation functions specified by (4), the messages originally sent at the time instant t_ℓ are now rescheduled to be sent at the time instants $t_{F(\ell)-s+1}, \dots, t_{F(\ell)}$. Fig. 4 illustrates this idea.

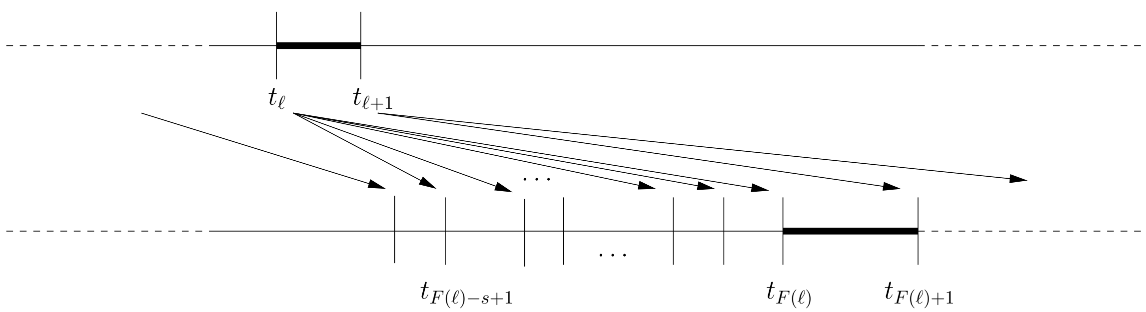

Figure 4: Under the rescheduling, the messages that are sent at the time instant t_ℓ under the control and communication law $\mathcal{CC}$ are rescheduled to be sent over the time instants $t_{F(\ell)-s+1}, \dots, t_{F(\ell)}$ under the control and communication law $\mathcal{CC}_{(s,\mathcal{P}_I)}$. Accordingly, the evolution of the robotic network under the original law during the time interval $[t_\ell, t_{\ell+1}]$ is now executed when all the corresponding messages have been transmitted, i.e., along the time interval $[t_{F(\ell)}, t_{F(\ell)+1}]$.

For $i \in I$, define the control functions $\mathrm{ctl}^{[i]} \colon \overline{\mathbb{R}}_+ \times X^{[i]} \times X^{[i]} \times L^N \to U^{[i]}$ by

$$\mathrm{ctl}^{[i]}_{(s,\mathcal{P}_I)}(t, x, x_{\mathrm{smpld}}, y) = \begin{cases} \frac{t_{F^{-1}(\ell)+1} - t_{F^{-1}(\ell)}}{t_{\ell+1} - t_\ell} \, \mathrm{ctl}^{[i]} \left(h_\ell(t), x, x_{\mathrm{smpld}}, y \right), & \text{if } t \in [t_\ell, t_{\ell+1}] \text{ and } \ell = -1 (\mathrm{mod}\, s)\,, \\ 0, & \text{otherwise}\,, \end{cases} \tag{5}$$

where $F^{-1} \colon \mathbb{N}_0 \to \mathbb{N}_0$ is the inverse of F, defined by $F^{-1}(\ell) = \frac{\ell+1}{s} - 1$, and for $\ell = -1 (\mathrm{mod}\, s)$, the function $h_\ell \colon [t_\ell, t_{\ell+1}] \to [t_{F^{-1}(\ell)}, t_{F^{-1}(\ell)+1}]$ is the time re-parameterization function defined by

$$h_\ell(t) = \frac{(t_{F^{-1}(\ell)+1} - t_{F^{-1}(\ell)})t + t_{\ell+1} t_{F^{-1}(\ell)} - t_\ell \, t_{F^{-1}(\ell)+1}}{t_{\ell+1} - t_\ell}\,, \quad t \in [t_\ell, t_{\ell+1}]\,.$$

Roughly speaking, the control law $\mathrm{ctl}^{[i]}_{(s,\mathcal{P}_I)}$ makes the agent i wait for the time intervals $[t_\ell, t_{\ell+1}]$, with $\ell \in \{as - 1 \mid a \in \mathbb{N}\}$, to execute any motion. Accordingly, the evolution of the robotic network under the original law $\mathcal{CC}$ during the time interval $[t_\ell, t_{\ell+1}]$ now takes place when all the corresponding messages have been transmitted, i.e., along the time interval $[t_{F(\ell)}, t_{F(\ell)+1}]$.

We gather the above construction in the following definition.

Definition 3.10 (Rescheduling of control and communication laws) *Let $\mathcal{S} = (I, \mathcal{A}, E_{\mathrm{cmm}})$ be a robotic network with driftless physical agents, and let $\mathcal{CC} = (\mathbb{N}_0, L, \{\mathrm{ctl}^{[i]}\}_{i\in I}, \{\mathrm{msg}^{[i]}\}_{i\in I})$ be a static control and communication law. Let $s \in \mathbb{N}$, with $s \leq N$, and let $\mathcal{P}_I$ be an s-partition of I. The control and communication law $\mathcal{CC}_{(s,\mathcal{P}_I)} = (\mathbb{N}_0, L, \{\mathrm{ctl}^{[i]}_{(s,\mathcal{P}_I)}\}_{i\in I}, \{\mathrm{msg}^{[i]}_{(s,\mathcal{P}_I)}\}_{i\in I})$ defined by equations (4) and (5) is called a $(s, \mathcal{P}_I)$-*rescheduling *of $\mathcal{CC}$.* •

Next, we examine the relation between the evolutions and the time and communication complexities of a control and communication law, and of those of its reschedulings.

Proposition 3.11 *With the same assumptions as in Definition 3.10, let $t \mapsto x(t)$ and $t \mapsto \tilde{x}(t)$ denote the network evolutions starting from $x_0 \in \prod_{i\in I} X_0^{[i]}$ under $\mathcal{CC}$ and $\mathcal{CC}_{(s,\mathcal{P}_I)}$, respectively, and let $\mathcal{T}\colon \prod_{i\in I} X^{[i]} \to$ `BooleSet` be a coordination task for $\mathcal{S}$.*

(i) For all $k \in \mathbb{N}_0$,

$$\tilde{x}^{[i]}(t) = \begin{cases} \tilde{x}^{[i]}(t_{F(k-1)+1}), & \text{for all } t \in \bigcup_{\ell=F(k-1)+1}^{F(k)-1} [t_\ell, t_{\ell+1}], \\ x^{[i]}(h_{F(k)}(t)), & \text{for all } t \in [t_{F(k)}, t_{F(k)+1}]. \end{cases} \tag{6}$$

(ii) For all $x_0 \in \prod_{i\in I} X_0^{[i]}$,

$$\mathrm{TC}(\mathcal{CC}_{(s,\mathcal{P}_I)}, \mathcal{T}, x_0) = s \cdot \mathrm{TC}(\mathcal{CC}, \mathcal{T}, x_0).$$

(iii) If $\mathrm{C}_{\mathrm{rnd}}$ is additive, then, for all $x_0 \in \prod_{i\in I} X_0^{[i]}$

$$\mathrm{MCC}(\mathcal{CC}_{(s,\mathcal{P}_I)}, \mathcal{T}, x_0) = \frac{1}{s} \cdot \mathrm{MCC}(\mathcal{CC}, \mathcal{T}, x_0),$$

and, therefore, the total communication cost of $\mathcal{CC}$ is invariant under rescheduling. •

Proof: The relationships (6) are direct consequences of the definition of rescheduling. We leave the bookkeeping to the interested reader. By definition of $\mathrm{TC}(\mathcal{CC}, \mathcal{T}, x_0)$, we have that $\mathcal{T}(x(t_k)) = \mathtt{true}$, for all $k \geq \mathrm{TC}(\mathcal{CC}, \mathcal{T}, x_0)$, and $\mathcal{T}(x(t_{\mathrm{TC}(\mathcal{CC},\mathcal{T},x_0)-1})) = \mathtt{false}$. Let us rewrite these equalities in terms of the trajectories corresponding to the rescheduled control and communication law. From equation (6), one can write $x^{[i]}(t_k) = x^{[i]}(h_{F(k)}(t_{F(k)})) = \tilde{x}^{[i]}(t_{F(k)})$, for all $i \in I$ and $k \in \mathbb{N}_0$. Therefore, we have

$$\begin{aligned} \mathcal{T}(\tilde{x}(t_{F(k)})) = \mathcal{T}(x(t_k)) = \mathtt{true}, \qquad & \text{for all } F(k) \geq F(\mathrm{TC}(\mathcal{CC}, \mathcal{T}, x_0)), \\ \mathcal{T}(\tilde{x}(t_{F(\mathrm{TC}(\mathcal{CC},\mathcal{T},x_0)-1)})) = \mathcal{T}(x(t_{\mathrm{TC}(\mathcal{CC},\mathcal{T},x_0)-1})) = \mathtt{false}, & \end{aligned}$$

where we have used the definition (4) of the rescheduled message-generation function. Now, note that by equation (6), one has

$$\tilde{x}^{[i]}(t_\ell) = \tilde{x}^{[i]}(t_{F(\lfloor \ell/s \rfloor -1)+1}), \quad \text{for all } \ell \in \mathbb{N}_0 \text{ and all } i \in I.$$

Therefore, $\mathcal{T}(\tilde{x}(t_{F(\mathrm{TC}(\mathcal{CC},\mathcal{T},x_0)-1)+1})) = \mathcal{T}(\tilde{x}(t_{F(\mathrm{TC}(\mathcal{CC},\mathcal{T},x_0))}))$ and we can rewrite the previous identities as

$$\begin{aligned} \mathcal{T}(\tilde{x}(t_k)) = \mathtt{true}, \quad & \text{for all } k \geq F(\mathrm{TC}(\mathcal{CC}, \mathcal{T}, x_0) - 1) + 1, \\ \mathcal{T}(\tilde{x}(t_{F(\mathrm{TC}(\mathcal{CC},\mathcal{T},x_0)-1)})) = \mathtt{false}, & \end{aligned}$$

which imply that

$$\mathrm{TC}(\mathcal{CC}_{(s,\mathcal{P}_I)}, \mathcal{T}, x_0) = F(\mathrm{TC}(\mathcal{CC}, \mathcal{T}, x_0) - 1) + 1 = s\,\mathrm{TC}(\mathcal{CC}, \mathcal{T}, x_0).$$

As for the mean communication complexity, additivity of $\mathrm{C}_{\mathrm{rnd}}$ implies

$$\mathrm{C}_{\mathrm{rnd}}^{L} \circ E_{\mathrm{cmm}\backslash\emptyset}(t_\ell, x(t_\ell)) = \mathrm{C}_{\mathrm{rnd}}^{L} \circ E_{\mathrm{cmm}\backslash\emptyset}(t_{F(\ell)-s+1}, \tilde{x}(t_{F(\ell)-s+1})) + \cdots + \mathrm{C}_{\mathrm{rnd}}^{L} \circ E_{\mathrm{cmm}\backslash\emptyset}(t_{F(\ell)}, \tilde{x}(t_{F(\ell)}))\,,$$

where we have used $F(\ell-1)+1 = F(\ell)-s+1$. Now, we compute

$$\begin{aligned}\sum_{\ell=0}^{\mathrm{TC}(\mathcal{CC}_{(s,\mathcal{P}_I)},\mathcal{T},x_0)-1} \mathrm{C}_{\mathrm{rnd}}^{L} \circ E_{\mathrm{cmm}\backslash\emptyset}(t_\ell, \tilde{x}(t_\ell)) &= \sum_{\ell=0}^{F(\mathrm{TC}(\mathcal{CC},\mathcal{T},x_0)-1)} \mathrm{C}_{\mathrm{rnd}}^{L} \circ E_{\mathrm{cmm}\backslash\emptyset}(t_\ell, \tilde{x}(t_\ell)) \\ = \sum_{\ell=0}^{\mathrm{TC}(\mathcal{CC},\mathcal{T},x_0)-1} \sum_{k=F(\ell)-s+1}^{F(\ell)} \mathrm{C}_{\mathrm{rnd}}^{L} \circ E_{\mathrm{cmm}\backslash\emptyset}(t_k, \tilde{x}(t_k)) &= \sum_{\ell=0}^{\mathrm{TC}(\mathcal{CC},\mathcal{T},x_0)-1} \mathrm{C}_{\mathrm{rnd}}^{L} \circ E_{\mathrm{cmm}\backslash\emptyset}(t_\ell, x(t_\ell))\,,\end{aligned}$$

which completes the proof of part (iii). ■

Remark 3.12 It is worth noting that the notion of communication complexity defined in (3) is not invariant under rescheduling. Indeed, reasoning as before, one computes

$$\begin{aligned}\lim_{k\to+\infty} \frac{1}{k}\sum_{\ell=0}^{k} \mathrm{C}_{\mathrm{rnd}}^{L} \circ E_{\mathrm{cmm}\backslash\emptyset}(t_\ell, \tilde{x}(t_\ell)) &= \lim_{\tilde{k}\to+\infty} \frac{1}{\tilde{k}s-1}\sum_{\ell=0}^{\tilde{k}s-1} \mathrm{C}_{\mathrm{rnd}}^{L} \circ E_{\mathrm{cmm}\backslash\emptyset}(t_\ell, \tilde{x}(t_\ell)) \\ &= \lim_{\tilde{k}\to+\infty} \frac{1}{\tilde{k}s-1}\sum_{\tilde{\ell}=0}^{\tilde{k}-1} \sum_{\ell=F(\tilde{\ell}-1)+1}^{F(\tilde{\ell})} \mathrm{C}_{\mathrm{rnd}}^{L} \circ E_{\mathrm{cmm}\backslash\emptyset}(t_\ell, \tilde{x}(t_\ell)) \\ &= \lim_{\tilde{k}\to+\infty} \frac{1}{\tilde{k}s-1}\sum_{\tilde{\ell}=0}^{\tilde{k}-1} \mathrm{C}_{\mathrm{rnd}}^{L} \circ E_{\mathrm{cmm}\backslash\emptyset}(t_\ell, x(t_{\tilde{\ell}}))\,.\end{aligned}$$

Therefore, $\mathrm{cc}(\mathcal{CC}_{(s,\mathcal{P}_I)}, x_0) = \frac{1}{s}\,\mathrm{cc}(\mathcal{CC}, x_0)$. This means that, by performing a rescheduling of the control and communication law, one can indeed lower the measure of communication complexity cc, although the trajectory described by the robotic network will continue to be the same. •

4 Motion coordination algorithms and their time complexity

In this section we provide examples of motion coordination algorithms for robotic networks performing a variety of distributed tasks. For each algorithm and task, we present some results on the corresponding time and communication complexity.

4.1 Agreement on direction of motion and equidistance

From Examples 2.7, 2.14 and 3.3, recall the definition of uniform network $\mathcal{S}_{\mathbb{S}^1,r\text{-disk}}$ of locally-connected first-order agents in $\mathbb{S}^1$, the agree-and-pursue control and communication law $\mathcal{CC}_{\text{agr-pursuit}}$, and the two coordination tasks $\mathcal{T}_{\mathtt{drctn}}$ and $\mathcal{T}_{\varepsilon\text{-eqdstnc}}$.

Theorem 4.1 *For $k_{prop} \in]0, \frac{1}{2}[$, $r \in]0, 2\pi]$, $\alpha = Nr - 2\pi$ and $\varepsilon \in]0,1[$, the network $\mathcal{S}_{\mathbb{S}^1,r\text{-disk}}$, the law $\mathcal{CC}_{\text{agr-pursuit}}$, and the tasks $\mathcal{T}_{\mathtt{drctn}}$ and $\mathcal{T}_{\varepsilon\text{-eqdstnc}}$ together satisfy:*

(i) the upper bound $\mathrm{TC}(\mathcal{T}_{\mathtt{drctn}}, \mathcal{CC}_{\text{agr-pursuit}}) \in O(Nr^{-1})$ and the lower bound

$$\mathrm{TC}(\mathcal{T}_{\mathtt{drctn}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \begin{cases}\Omega(r^{-1}) & \text{if } \alpha \geq 0, \\ \Omega(N) & \text{if } \alpha \leq 0;\end{cases}$$

(ii) *if $\alpha > 0$, then the upper bound* $\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-eqdstnc}}, \mathcal{CC}_{\text{agr-pursuit}}) \in O(N^2 \log(N\varepsilon^{-1}) + N \log \alpha^{-1})$ *and the lower bound* $\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-eqdstnc}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \Omega(N^2 \log(\varepsilon^{-1}))$. *If $\alpha \leq 0$, then $\mathcal{CC}_{\text{agr-pursuit}}$ does not achieve $\mathcal{T}_{\varepsilon\text{-eqdstnc}}$ in general.*

Proof: Let us start by proving fact (i). Without loss of generality, assume $\mathtt{drctn}^{[N]}(0) = \mathtt{c}$, and that $\mathcal{T}_{\mathtt{drctn}}$ is `false` at time 0. Therefore, at least one agent is moving counterclockwise at time 0, and we can define $k = \max\{i \in I \mid \mathtt{drctn}^{[i]}(0) = \mathtt{cc}\}$. Define $t_k = \inf(\{\ell \in \mathbb{N}_0 \mid \mathtt{drctn}^{[k]}(\ell) = \mathtt{c}\} \cup \{+\infty\})$. In what follows we provide an upper bound on t_k.

For $\ell < t_k$, define

$$j(\ell) = \operatorname{argmin}\{\operatorname{dist}_{\mathtt{c}}(\theta^{[i]}(\ell), \theta^{[k]}(\ell)) \mid \mathtt{prior}^{[i]} = N,\, i \in I\}.$$

In other words, for all instants of time when agent k is moving counterclockwise, the agent $j(l)$ has `prior` equal to N, is moving clockwise, and is the agent closest to agent k with these two properties. Clearly,

$$2\pi > \operatorname{dist}_{\mathtt{c}}(\theta^{[N]}(0), \theta^{[k]}(0)) = \operatorname{dist}_{\mathtt{c}}(\theta^{[j(0)]}(0), \theta^{[k]}(0)).$$

Additionally, for $\ell < t_k - 1$, we claim that

$$\operatorname{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[j(\ell+1)]}(\ell+1)) > k_{\text{prop}} r.$$

This happens because either (1) there is no agent clockwise-ahead of $\theta^{[j(\ell)]}(\ell)$ within clockwise distance r and, therefore, the claim is obvious, or (2) there are such agents. In case (2), let m denote the agent whose clockwise distance to agent $j(\ell)$ is maximal within the set of agents with clockwise distance r from $\theta^{[j(\ell)]}(\ell)$. Then,

$$\begin{aligned}
\operatorname{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[j(\ell+1)]}(\ell+1)) &= \operatorname{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell+1)) \\
&= \operatorname{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell)) + \operatorname{dist}_{\mathtt{c}}(\theta^{[m]}(\ell), \theta^{[m]}(\ell+1)) \\
&\geq \operatorname{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell)) + k_{\text{prop}}\big(r - \operatorname{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell))\big) \\
&= k_{\text{prop}} r + (1 - k_{\text{prop}}) \operatorname{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell)) \ \geq k_{\text{prop}} r,
\end{aligned}$$

where the first inequality follows from the fact that at time ℓ there can be no agent whose clockwise distance to agent m is less than $(r - \operatorname{dist}_{\mathtt{c}}(\theta^{[j(\ell)]}(\ell), \theta^{[m]}(\ell)))$.

In summary, either agent k changes direction of motion or at each instant of time its distance to the closest agent with `prior` equal to N decreases by a constant $k_{\text{prop}} r$. This shows that

$$t_k \in O(r^{-1}).$$

Because at least one more agent moves in the clockwise direction after $O(r^{-1})$ time, it follows that all agents will move clockwise after $O(Nr^{-1})$ time. This completes the proof of the upper bound in (i). Let us prove the lower bound in (i). Note that the number of agents that can fit into the circle $\mathbb{S}^1$, spaced a distance r apart one from each other, is at most $\lfloor \frac{2\pi}{r} \rfloor$. Consider an initial configuration where $\mathtt{drctn}^{[i]}(0) = \mathtt{cc}$ for $i \in \{1, \ldots, N-1\}$, $\mathtt{drctn}^{[N]}(0) = \mathtt{c}$, and

(i) for $\alpha > 0$, $\lfloor \frac{2\pi}{r} \rfloor$ agents (including the agent N) lie at a distance r one from each other in $\mathbb{S}^1$, and the remaining $N - \lfloor \frac{2\pi}{r} \rfloor$ agents are located within a clockwise distance $\frac{r}{2}$ of the agent i^* lying in the position in $\mathbb{S}^1$ symmetric to the location of agent N;

(ii) for $\alpha \leq 0$, all agents lie at a distance r in $\mathbb{S}^1$ in counterclockwise order incrementally according to their unique identifier (note that they might not cover the whole circle).

Note that the displacement of each agent is upper bounded by $k_{\text{prop}} r \leq \frac{r}{2}$. When $\alpha \geq 0$, the number of time steps that takes agent i^* to receive the message with priority N is lower bounded by $\frac{1}{2}\left\lfloor \frac{2\pi}{r} \right\rfloor$. Therefore, $\mathrm{TC}(\mathcal{T}_{\texttt{drctn}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \Omega(r^{-1})$. When $\alpha \leq 0$, the analysis of the network evolution is parallel to the one discussed in [27, Chapter 1] for the leader election algorithm in static networks with the ring topology. The number of time steps that takes agent 1 to receive the message with priority N is given by $N-1$. Therefore, $\mathrm{TC}(\mathcal{T}_{\texttt{drctn}}, \mathcal{CC}_{\text{agr-pursuit}}) \in \Omega(N)$.

To prove fact (ii), we assume that $\mathcal{T}_{\texttt{drctn}}$ has been achieved (so that all agents are moving clockwise), and we first prove a fact regarding connectivity. At time $\ell \in \mathbb{N}_0$, define

$$H(\ell) = \{x \in \mathbb{S}^1 \mid \min_{i \in I} \mathrm{dist}_{\mathtt{c}}(x, \theta^{[i]}(\ell)) + \min_{j \in I} \mathrm{dist}_{\mathtt{cc}}(x, \theta^{[j]}(\ell)) > r\}.$$

In other words, any point in $H(\ell)$ is at least a distance r, clockwise or counterclockwise, from an agent. Therefore, $H(\ell)$ does not contain any point between two agents separated by a distance less than r, and each connected component of $H(\ell)$ has length at least r. Let $n_H(\ell)$ be the number of connected components of $H(\ell)$, if $H(\ell)$ is empty, then we take the convention that $n_H(\ell) = 0$. Clearly, $n_H(\ell) \leq N$. We claim that, if $n_H(\ell) > 0$, then $t \mapsto n_H(\ell + t)$ is non-increasing. Let $d(\ell) < r$ be the distance between any two consecutive agents at time ℓ. Because both agents move in the same direction, a simple calculation shows that

$$d(\ell+1) \leq d(\ell) + k_{\text{prop}}(r - d(\ell)) = (1 - k_{\text{prop}}) d(\ell) + k_{\text{prop}} r < (1 - k_{\text{prop}}) r + k_{\text{prop}} r = r.$$

This means that the two agents remain within distance r and, therefore connected, at the following time instant. Because the number of connected components of $E_r(\theta^{[1]}, \dots, \theta^{[N]})$ does not increase, it follows that the number of connected components of H cannot increase.

Next we claim that, if $n_H(\ell) > 0$, then there exists $t > \ell$ such that $n_H(t) < n_H(\ell)$. By contradiction, assume $n_H(\ell) = n_H(t)$ for all $t > \ell$. Without loss of generality, let $\{1, \dots, m\}$ be a set of agents with the properties that $\mathrm{dist}_{\mathtt{cc}}\left(\theta^{[i]}(\ell), \theta^{[i+1]}(\ell)\right) \leq r$, for $i \in \{1, \dots, m\}$, that $\theta^{[1]}(\ell)$ and $\theta^{[m]}(\ell)$ belong to the boundary of $H(\ell)$, and that there is no other set with the same properties and more agents. One can show that, for $t \geq \ell$,

$$\begin{aligned}
\theta^{[1]}(t+1) &= \theta^{[1]}(t) - k_{\text{prop}} r, \\
\theta^{[i]}(t+1) &= \theta^{[i]}(t) - k_{\text{prop}} \,\mathrm{dist}_{\mathtt{c}}(\theta^{[i]}(t), \theta^{[i-1]}(t)), \quad i \in \{2, \dots, m\}.
\end{aligned}$$

If we define $d(t) = \left(\mathrm{dist}_{\mathtt{cc}}(\theta^{[1]}(t), \theta^{[2]}(t)), \dots, \mathrm{dist}_{\mathtt{cc}}(\theta^{[m-1]}(t), \theta^{[m]}(t))\right) \in \mathbb{R}_+^{m-1}$, then one can show that

$$d(t+1) = \mathrm{Trid}_{m-1}(k_{\text{prop}}, 1 - k_{\text{prop}}, 0)\, d(t) - r \begin{bmatrix} k_{\text{prop}} \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where the linear map $(a, b, c) \mapsto \mathrm{Trid}_{m-1}(a, b, c) \in \mathbb{R}^{(m-1)\times(m-1)}$ is defined in Appendix B. This is a discrete-time linear time-invariant dynamical system with unique equilibrium point $r(1, \dots, 1)$. By Theorem B.3(ii) in Appendix B, for $\eta \in]0, 1[$, the solution $t \mapsto d(t)$ to this system reaches a ball of radius η centered at the equilibrium point in time $O(m \log m + \log \eta^{-1})$. (Here we used the fact that the initial condition of this system is bounded.) In turn, this implies that $t \mapsto \sum_{i=1}^{m} d_i(t)$ is larger than $(m-1)(r-\eta)$ in time $O(m \log m + \log \eta^{-1})$.

We are now ready to find the contradiction and show that $n_H(t)$ cannot remain equal to $n_H(\ell)$ for all time t. After time $O(m \log m + \log \eta^{-1}) = O(N \log N + \log \eta^{-1})$, we have:

$$2\pi \geq n_H(\ell) r + \sum_{j=1}^{n_H(\ell)} (r - \eta)(m_j - 1) = n_H(\ell) r + (N - n_H(\ell))(r - \eta) = n_H(\ell)\eta + N(r - \eta).$$

Here $m_1, \dots, m_{n_H(\ell)}$ are the number of agents in each isolated group, and each connected component of $H(\ell)$ has length at least r. Now, take $\eta = \frac{Nr-2\pi}{N} = \frac{\alpha}{N}$, and the contradiction follows from

$$2\pi \geq n_H(\ell)\eta + Nr - N\eta = n_H(\ell)\eta + Nr + 2\pi - Nr = n_H(\ell)\eta + 2\pi.$$

In summary this shows that, in time $O(N \log N + \log \eta^{-1}) = O(N \log N + \log \alpha^{-1})$, the number of connected components of H will decrease by one. Therefore, in time $O(N^2 \log N + N \log \alpha^{-1})$ the set H will become empty. At that time, the resulting network will obey the discrete-time linear time-invariant dynamical system:

$$d(t+1) = \mathrm{Circ}_N(k_{\mathrm{prop}}, 1 - k_{\mathrm{prop}}, 0)\, d(t).$$

Here $d(t) = \big(\mathrm{dist}_{\mathtt{cc}}(\theta^{[1]}(t), \theta^{[2]}(t)), \dots, \mathrm{dist}_{\mathtt{cc}}(\theta^{[N]}(t), \theta^{[N+1]}(t))\big) \in \mathbb{R}_+^N$, with the convention $\theta^{[N+1]} = \theta^{[1]}$. By Theorem B.3(iii) in Appendix B, the solution $t \mapsto d(t)$ reaches the desired configuration in time $O(N^2 \log \varepsilon^{-1})$ with an error whose 2-norm, and therefore, its ∞-norm is of order ε. In summary, the desired configuration is achieved in time $O(N^2 \log(N\varepsilon^{-1}) + N \log \alpha^{-1})$.

For the lower bound, consider an initial configuration with the properties that (i) agents are counterclockwise-ordered according to their unique identifier, (ii) the set H is empty, and (iii) the inter-agent distances $d(0) = \big(\mathrm{dist}_{\mathtt{cc}}(\theta^{[1]}(0), \theta^{[2]}(0)), \dots, \mathrm{dist}_{\mathtt{cc}}(\theta^{[N]}(0), \theta^{[1]}(0))\big)$ are given by

$$d(0) = \begin{bmatrix} \frac{2\pi}{N} \\ \vdots \\ \frac{2\pi}{N} \end{bmatrix} + k(\mathbf{v}_N + \overline{\mathbf{v}}_N),$$

where $\mathbf{v}_N$ is the eigenvector of $\mathrm{Circ}_N(k_{\mathrm{prop}}, 1 - k_{\mathrm{prop}}, 0)$ corresponding to the eigenvalue $1 - k_{\mathrm{prop}} + k_{\mathrm{prop}} \cos\left(\frac{2\pi}{N}\right) - k_{\mathrm{prop}}\sqrt{-1} \sin\left(\frac{2\pi}{N}\right)$ (see Appendix B), and $k > 0$ is chosen sufficiently small so that $d(0) \in \mathbb{R}_+^N$. By Theorem B.3(iii) in Appendix B, the solution $t \mapsto d(t)$ reaches the desired configuration in time $\Theta(N^2 \log \varepsilon^{-1})$ with an error whose 2-norm, and therefore, its ∞-norm is of order ε. This concludes the result. ■

4.2 Rendezvous without connectivity constraint

From Examples 2.4, 2.13 and 3.2, recall the definition of uniform network $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$ of locally-connected first-order agents in $\mathbb{R}^d$, the move-toward-average control and communication law $\mathcal{CC}_{\mathrm{avrg}}$, and the coordination task $\mathcal{T}_{\mathrm{rndzvs}}$.

Theorem 4.2 *For $d = 1$, the network $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$, the law $\mathcal{CC}_{\mathrm{avrg}}$, and the task $\mathcal{T}_{\mathrm{rndzvs}}$ together satisfy* $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{avrg}}) \in O(N^5)$ *and* $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{avrg}}) \in \Omega(N)$.

Proof: One can easily prove that, along the evolution of the network, the ordering of the agents is preserved, i.e., if $x^{[i]}(\ell) \leq x^{[j]}(\ell)$, then $x^{[i]}(\ell+1) \leq x^{[j]}(\ell+1)$. However, links between agents are not necessarily preserved (see e.g. Figure 1). Indeed, connected components may split along the evolution. However, mergings are not possible. Consider two contiguous connected components C_1 and C_2, with C_1 to the left of C_2. By definition, the rightmost agent of C_1 and the leftmost agent of C_2 are at a distance strictly bigger than r. Now, by executing the algorithm, they can only but increase that distance, since the rightmost agent of C_1 will move to the left, and the leftmost agent of C_2 will move to the right. Therefore, connected components do not merge.

Consider first the case of an initial configuration of the network for which the communication graph remains connected throughout the evolution. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \leq \cdots \leq x^{[N]}(0) = (x_0)_N$. Let $\alpha \in \{3, \dots, N\}$ have the property that agents $\{2, \dots, \alpha - 1\}$ are neighbors of agent 1, and agent α is not. (If instead all agents are within an interval of length r, then rendezvous is achieved in 1 time instant, and the statement in theorem is easily seen to be true.) Note that we can assume that agents $\{2, \dots, \alpha - 1\}$ are also neighbors of agent α. If this is not the case, then those agents that are neighbors of agent 1 and not of agent α, rendezvous with agent 1 at the next time instant. At the time instant $\ell = 1$, the new updated positions satisfy

$$x^{[1]}(1) = \frac{1}{\alpha - 1} \sum_{k=1}^{\alpha-1} x^{[k]}(0), \qquad x^{[\gamma]}(1) \in \Big[\frac{1}{\alpha} \sum_{k=1}^{\alpha} x^{[k]}(0), *\Big], \ \gamma \in \{2, \dots, \alpha - 1\},$$

where $*$ denotes certain unimportant point.

Now, we show that

$$x^{[1]}(\alpha - 1) - x^{[1]}(0) \geq \frac{r}{\alpha(\alpha - 1)}. \tag{7}$$

Let us first show the inequality for $\alpha = 3$. Note that the fact that the communication graph remains connected implies that agent 2 is still a neighbor of agent 1 at the time instant $\ell = 1$. Therefore $x^{[1]}(2) \geq \frac{1}{2}(x^{[1]}(1) + x^{[2]}(1))$, and from here we deduce

$$\begin{aligned} x^{[1]}(2) - x^{[1]}(0) &\geq \frac{1}{2}\big(x^{[2]}(1) - x^{[1]}(0)\big) \\ &\geq \frac{1}{2}\Big(\frac{1}{3}\big(x^{[1]}(0) + x^{[2]}(0) + x^{[3]}(0)\big) - x^{[1]}(0)\Big) \geq \frac{1}{6}\big(x^{[3]}(0) - x^{[1]}(0)\big) \geq \frac{r}{6}. \end{aligned}$$

Let us now proceed by induction. Assume that inequality (7) is valid for $\alpha - 1$, and let us prove it for α. Consider first the possibility when at the time instant $\ell = 1$, the agent $\alpha - 1$ is still a neighbor of agent 1. In this case, $x^{[1]}(2) \geq \frac{1}{\alpha-1} \sum_{k=1}^{\alpha-1} x^{[k]}(1)$, and from here we deduce

$$\begin{aligned} x^{[1]}(2) - x^{[1]}(0) &\geq \frac{1}{\alpha - 1}\Big(x^{[\alpha-1]}(1) - x^{[1]}(0)\Big) \geq \frac{1}{\alpha - 1}\Big(\frac{1}{\alpha}\sum_{k=1}^{\alpha} x^{[k]}(0) - x^{[1]}(0)\Big) \\ &\geq \frac{1}{\alpha(\alpha - 1)}\Big(x^{[\alpha]}(0) - x^{[1]}(0)\Big) \geq \frac{r}{\alpha(\alpha - 1)}, \end{aligned}$$

which in particular implies (7). Consider then the case when agent $\alpha - 1$ is not a neighbor of agent 1 at the time instant $\ell = 1$. Let $\beta < \alpha$ such that agent $\beta - 1$ is a neighbor of agent 1 at $\ell = 1$, but agent β is not. Since $\beta < \alpha$, we have by induction $x^{[1]}(\beta) - x^{[1]}(1) \geq \frac{r}{\beta(\beta-1)}$. From here, we deduce that $x^{[1]}(\alpha - 1) - x^{[1]}(0) \geq \frac{r}{\alpha(\alpha-1)}$.

Inequality (7) implies that, at most in $\alpha - 1 \leq N - 1$ time instants, the leftmost agent traverses a distance greater than or equal to $\frac{r}{N(N-1)}$ (provided that at each step there exists at least another agent which is not its neighbor). Since $\operatorname{diam}(x_0, I) \leq (N - 1)r$, we deduce that in $N(N - 1)^3$ time instants there cannot be any agent which is not a neighbor of the agent 1. Hence, all agents rendezvous at the next time instant. Consequently,

$$\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{avrg}}, x_0) \leq N(N - 1)^3 + 1.$$

Finally, for a general initial configuration x_0, because there are a finite number of agents, only a finite number of splittings (at most $N - 1$) of the connected components of the communication graph can take place along the evolution. Therefore, we conclude $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{avrg}}) = O(N^5)$.

Let us now prove the lower bound. Consider an initial configuration $x_0 \in \mathbb{R}^N$ where all agents are positioned in increasing order according to their identity, and exactly at a distance r apart, say $(x_0)_{i+1} - (x_0)_i = r$, $i \in \{1, \dots, N-1\}$. Assume for simplicity that N is odd - when N is even, one can reason in an analogous way. Because of the symmetry of the initial condition, in the first time step, only agents 1 and N move. All the remaining agents remain in their position because it coincides with the average of its neighbors' position and its own. At the second time step, only agents 1, 2, $N-1$ and N move, and the others remain still because of the symmetry. Applying this idea iteratively, one deduces the time step when agents $\frac{N-1}{2}$ and $\frac{N+3}{2}$ move for the first time is lower bounded by $\frac{N-1}{2}$. Since both agents have still at least a neighbor (agent $\frac{N+1}{2}$), the task $\mathcal{T}_{\mathrm{rndzvs}}$ has not been achieved yet at this time step. Therefore, $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{avrg}}, x_0) \geq \frac{N-1}{2}$, and the result follows. ∎

4.3 Rendezvous with connectivity constraint

In this section we shall consider both networks $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$ and $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$ presented in Examples 2.4 and 2.5.

Circumcenter control and communication law

Here we define the *circumcenter* control and communication law $\mathcal{CC}_{\mathrm{crcmcntr}}$ for both networks $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$ and $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$. This is a uniform, static, time-independent law originally introduced by [1] and later studied in [5, 25]. Loosely speaking, the evolution of the network under the circumcenter control and communication law can be described as follows:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each communication round each agent performs the following tasks: (i) it transmits its position and receives its neighbors' positions; (ii) it computes the circumcenter of the point set comprised of its neighbors and of itself, and (iii) it moves toward this circumcenter while maintaining connectivity with its neighbors.

Let us present this description in more formal terms. We set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\mathrm{msg}^{[i]} = \mathrm{msg}_{\mathrm{std}}$, $i \in I$. In order to define the control function, we need to introduce some preliminary constructions. First, connectivity is maintained by restricting the allowable motion of each agent in the following appropriate manner. If agents i and j are neighbors at time $\ell \in \mathbb{N}_0$, then we require their subsequent positions to belong to $\overline{B}(\frac{x^{[i]}(\ell)+x^{[j]}(\ell)}{2}, \frac{r}{2})$. If an agent i has its neighbors at locations $\{q_1, \dots, q_l\}$ at time ℓ, then its *constraint set* $\mathcal{D}_{x^{[i]}(\ell),r}(\{q_1, \dots, q_l\})$ is

$$\mathcal{D}_{x^{[i]}(\ell),r}(\{q_1, \dots, q_l\}) = \bigcap_{q \in \{q_1, \dots, q_l\}} \overline{B}\Big(\frac{x^{[i]}(\ell) + q}{2}, \frac{r}{2}\Big).$$

Second, in order to maximize the displacement toward the circumcenter of the point set comprised of its neighbors and of itself, each agent solves a convex optimization problem that can be stated in general as follows. For q_0 and q_1 in $\mathbb{R}^d$, and for a convex closed set $Q \subset \mathbb{R}^d$ with $q_0 \in Q$, let $\lambda(q_0, q_1, Q)$ denote the solution to the strictly convex problem:

$$\begin{aligned} &\text{maximize } \lambda \\ &\text{subject to } \lambda \leq 1, \ (1-\lambda)q_0 + \lambda q_1 \in Q. \end{aligned}$$

Under the stated assumptions the solution exists and is unique. Third, note that since the agents operate with the standard message-generation function, it is clear that the natural projection $\pi_{\mathbb{R}^d}$

maps the messages $y^{[i]}(\ell)$ received at time $\ell \in \mathbb{N}_0$ by the agent $i \in I$ onto the positions of its neighbors. We are now ready to define the last constitutive element of $\mathcal{CC}_{\text{crcmcntr}}$. Define the control function $\text{ctl}\colon \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\text{ctl}(x, x_{\text{smpld}}, y) = \lambda_* \cdot (\text{Circum}(\pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}) - x_{\text{smpld}}) \,, \tag{8}$$

where $\lambda_* = \lambda(x_{\text{smpld}}, (\text{Circum}(\pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}), \mathcal{D}_{x_{\text{smpld}},r}(\pi_{\mathbb{R}^d}(y)))$. Evolving under this control law, it is clear that, at time $\lfloor t \rfloor + 1$, each agent i reaches the point $(1-\lambda_*)x^{[i]}(\lfloor t \rfloor) + \lambda_* \, \text{Circum}(\pi_{\mathbb{R}^d}(y^{[i]}(\lfloor t \rfloor)) \cup \{x^{[i]}(\lfloor t \rfloor)\})$.

Next, we consider the network $\mathcal{S}_{r\text{-}\infty\text{-disk}}$ of locally-∞-connected first-order agents in $\mathbb{R}^d$, see Example 2.6. For this network we define the *parallel circumcenter law*, $\mathcal{CC}_{\text{pll-crcmcntr}}$, by designing d decoupled circumcenter laws running in parallel on each coordinate axis of $\mathbb{R}^d$. As before, this law is uniform, static and time-independent. As before, we set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\text{msg}^{[i]} = \text{msg}_{\text{std}}$, $i \in I$. We define the control function $\text{ctl}\colon \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\text{ctl}(x, x_{\text{smpld}}, y) = \big(\,\text{Circum}(\tau_1(\mathcal{M})) - \tau_1(x_{\text{smpld}}), \ldots, \,\text{Circum}(\tau_d(\mathcal{M})) - \tau_d(x_{\text{smpld}})\big) \,, \tag{9}$$

where $\mathcal{M} = \pi_{\mathbb{R}^d}(y) \cup \{x_{\text{smpld}}\}$, and $\tau_1, \ldots, \tau_d \colon \mathbb{R}^d \to \mathbb{R}$ denote the canonical projections of $\mathbb{R}^d$ onto $\mathbb{R}$. See Fig. 5 for an illustration of this law in $\mathbb{R}^2$.

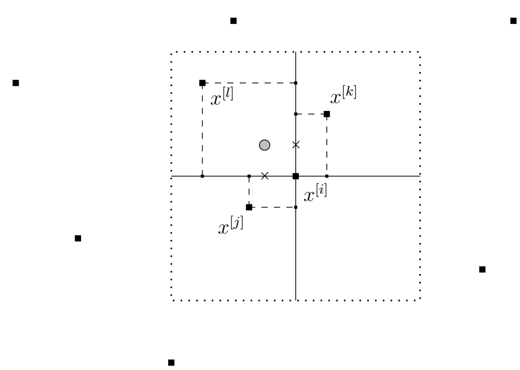

Figure 5: Parallel circumcenter control and communication law in $\mathbb{R}^2$. The target point for the agent i is plotted in light gray and has coordinates $(\text{Circum}(\tau_1(\mathcal{M}^{[i]})), \text{Circum}(\tau_2(\mathcal{M}^{[i]})))$.

Asymptotic behavior and complexity analysis

The following theorem summarizes the results known in the literature about the asymptotic properties of the circumcenter control and communication law.

Theorem 4.3 (Correctness of the circumcenter law) *For $d \in \mathbb{N}$, $r \in \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_+$, the following statements hold:*

(i) on the network $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$, the law $\mathcal{CC}_{\text{crcmcntr}}$ achieves the exact rendezvous task $\mathcal{T}_{\text{rndzvs}}$;

(ii) on the network $\mathcal{S}_{\mathbb{R}^d,r\text{-LD}}$, the law $\mathcal{CC}_{\text{crcmcntr}}$ achieves the ε-rendezvous task $\mathcal{T}_{\varepsilon\text{-rndzvs}}$;

(iii) on the network $\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}$, the law $\mathcal{CC}_{\text{pll-crcmcntr}}$ achieves the exact rendezvous task $\mathcal{T}_{\text{rndzvs}}$;

(iv) *the evolutions of* $(\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}, \mathcal{CC}_{\text{crcmcntr}})$*, of* $(\mathcal{S}_{\mathbb{R}^d,r\text{-LD}}, \mathcal{CC}_{\text{crcmcntr}})$*, and of* $(\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}, \mathcal{CC}_{\text{pll-crcmcntr}})$ *have the property that, if two agents belong to the same connected component of the communication graph at some time* $\ell \in \mathbb{N}_0$*, then they continue to belong to the same connected component of the communication graph for all subsequent times* $k \geq \ell$. •

Proof: The results on $\mathcal{S}_{r\text{-disk}}$ appeared originally in [1]. The proof for the results on $\mathcal{S}_{r\text{-LD}}$ is provided in [5]. We postpone the proof for $\mathcal{S}_{r\text{-}\infty\text{-disk}}$ to the proof of Theorem 4.4 below. ■

Next we analyze the time complexity of $\mathcal{CC}_{\text{crcmcntr}}$. We provide complete results only for the case $d = 1$. As we see next, the complexity properties of $\mathcal{CC}_{\text{crcmcntr}}$ differ dramatically when applied to the two robotic networks with different communication graphs.

Theorem 4.4 (Time complexity of circumcenter law) *For* $r \in \mathbb{R}_+$ *and* $\varepsilon \in]0, 1[$*, the following statements hold:*

(i) *for* $d = 1$*, on the network* $\mathcal{S}_{\mathbb{R},r\text{-disk}}$*,* $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(N)$*;*

(ii) *for* $d = 1$*, on the network* $\mathcal{S}_{\mathbb{R},r\text{-LD}}$*,* $\text{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\text{crcmcntr}}) \in \Theta(N^2 \log(N\varepsilon^{-1}))$*;*

(iii) *for* $d \in \mathbb{N}$*, on the network* $\mathcal{S}_{\mathbb{R}^d,r\text{-}\infty\text{-disk}}$*,* $\text{TC}(\mathcal{T}_{\text{rndzvs}}, \mathcal{CC}_{\text{pll-crcmcntr}}) \in \Theta(N)$. •

Proof: Let $x_0 \in \mathbb{R}^N$. Throughout the proof, neighboring relationships are understood with respect to the r-disk graph. First of all, let us show that, for $n = 1$, the connectivity constraints on each agent $i \in I$ imposed by the constraint set $\mathcal{D}_{x^{[i]},r}(\pi_{\mathbb{R}}(y))$ are superfluous, i.e., the solution of the convex optimization problem is $\lambda_* = 1$ (cf. equation (8)). To see this, assume that agents i and j are neighbors at time instant ℓ, define $\mathcal{M}^{[i]}$ as $\pi_{\mathbb{R}^d}(y^{[i]}(\ell)) \cup \{x^{[i]}(\ell)\}$, and let us show that $\text{Circum}(\mathcal{M}^{[i]})$ belongs to $\overline{B}(\frac{x^{[i]}(\ell)+x^{[j]}(\ell)}{2}, \frac{r}{2})$. Without loss of generality, let $x^{[i]}(\ell) \leq x^{[j]}(\ell)$. Let $x^{[i]}_-(\ell)$, $x^{[i]}_+(\ell)$ denote the positions of the leftmost and rightmost agents among the neighbors of agent i. Note that $x^{[i]}(\ell) \leq x^{[j]}(\ell) \leq x^{[i]}_+(\ell)$ and $\text{Circum}(\mathcal{M}^{[i]}) = \frac{1}{2}(x^{[i]}_-(\ell) + x^{[i]}_+(\ell))$. Then,

$$\begin{aligned}\big|\,\text{Circum}(\mathcal{M}^{[i]}) - \frac{1}{2}(x^{[i]}(\ell) + x^{[j]}(\ell))\big| &= \frac{1}{2}\big|x^{[i]}_-(\ell) - x^{[i]}(\ell) + x^{[i]}_+(\ell) - x^{[j]}(\ell)\big| \\ &\leq \frac{1}{2}\max\{|x^{[i]}_-(\ell) - x^{[i]}(\ell)|, |x^{[i]}_+(\ell) - x^{[j]}(\ell)|\} \leq \frac{r}{2}\,,\end{aligned}$$

as claimed. Therefore, we have that $x^{[i]}(\ell{+}1) = \text{Circum}(\mathcal{M}^{[i]})$. Likewise, one can deduce $\text{Circum}(\mathcal{M}^{[i]}) \leq \text{Circum}(\mathcal{M}^{[j]})$, and therefore, the order of the agents is preserved.

Consider first the case when $E_{r\text{-disk}}(x_0)$ is connected. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \leq \cdots \leq x^{[N]}(0) = (x_0)_N$. Let $\alpha \in \{3, \dots, N\}$ have the property that agents $\{2, \dots, \alpha - 1\}$ are neighbors of agent 1, and agent α is not. (If instead all agents are within an interval of length r, then rendezvous is achieved in 1 time instant, and the statement in theorem is easily seen to be true.) See Fig. 6 for an illustration of these definitions. Note that we can assume that agents $\{2, \dots, \alpha-1\}$ are also neighbors of agent α. If this is not the case, then those agents that are neighbors of agent 1 and not of agent α, rendezvous with agent 1 at the next time instant. At the time instant $\ell = 1$, the new updated positions satisfy

$$x^{[1]}(1) = \frac{x^{[1]}(0) + x^{[\alpha-1]}(0)}{2}\,, \quad x^{[\gamma]}(1) \in \left[\frac{x^{[1]}(0) + x^{[\alpha]}(0)}{2}, \frac{x^{[1]}(0) + x^{[\gamma]}(0) + r}{2}\right],\ \gamma \in \{2, \dots, \alpha - 1\}\,.$$

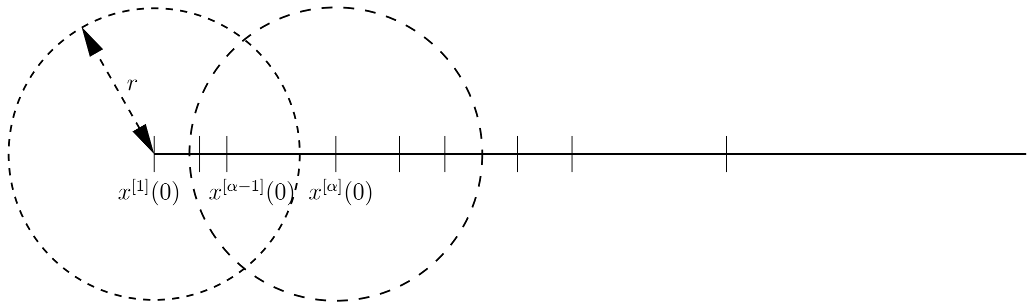

Figure 6: Definition of $\alpha \in \{3, \dots, N\}$ for an initial network configuration.

These equalities imply that $x^{[1]}(1) - x^{[1]}(0) = \frac{1}{2}\big(x^{[\alpha-1]}(0) - x^{[1]}(0)\big) \leq \frac{1}{2}r$. Analogously, we deduce $x^{[1]}(2) - x^{[1]}(1) \leq \frac{1}{2}r$, and therefore

$$x^{[1]}(2) - x^{[1]}(0) \leq r\,. \tag{10}$$

On the other hand, from $x^{[1]}(2) \in \big[\frac{1}{2}\big(x^{[1]}(1) + x^{[\alpha-1]}(1)\big), *\big]$ (where the symbol $*$ represents a certain unimportant point in $\mathbb{R}$), we deduce that

$$\begin{aligned} x^{[1]}(2) - x^{[1]}(0) &\geq \frac{1}{2}\big(x^{[1]}(1) + x^{[\alpha-1]}(1)\big) - x^{[1]}(0) \geq \frac{1}{2}\big(x^{[\alpha-1]}(1) - x^{[1]}(0)\big) \\ &\geq \frac{1}{2}\Big(\frac{x^{[1]}(0) + x^{[\alpha]}(0)}{2} - x^{[1]}(0)\Big) = \frac{1}{4}\big(x^{[\alpha]}(0) - x^{[1]}(0)\big) \geq \frac{1}{4}r\,. \end{aligned} \tag{11}$$

Inequalities (10) and (11) mean that, after at most two time instants, agent 1 has traveled an amount larger than $r/4$. In turn this implies that

$$\frac{\operatorname{diam}(x_0, I)}{r} \leq \operatorname{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}, x_0) \leq \frac{4\operatorname{diam}(x_0, I)}{r}\,.$$

If $E_{r\text{-disk}}(x_0)$ is not connected, note that along the network evolution, the connected components of the r-disk graph do not change. Therefore, using the previous characterization on the amount traveled by the leftmost agent of each connected component in at most two time instants, we deduce that

$$\frac{1}{r} \max_{C \in \mathcal{C}_{E_{r\text{-disk}}}(x_0)} \operatorname{diam}(x_0, C) \leq \operatorname{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}, x_0) \leq \frac{4}{r} \max_{C \in \mathcal{C}_{E_{r\text{-disk}}}(x_0)} \operatorname{diam}(x_0, C)\,.$$

Note that the connectedness of each $C \in \mathcal{C}_{E_{r\text{-disk}}}(x_0)$ implies that $\operatorname{diam}(x_0, C) \leq (N-1)r$, and therefore $\operatorname{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in O(N)$. Moreover, for $x_0 \in \mathbb{R}^N$ such that $(x_0)_{i+1} - (x_0)_i = r$, $i \in \{1, \dots, N-1\}$, we have $\operatorname{diam}(x_0, I) = (N-1)r$, and therefore $\operatorname{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}, x_0) \geq N - 1$. This concludes the proof of fact (i):

$$\operatorname{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Theta(N)\,.$$

Next we prove fact (ii). In the r-limited Delaunay graph, two agents on the line that are at most at a distance r from each other are neighbors if and only if there are no other agents between them. Also, note that the r-limited Delaunay graph and the r-disk graph have the same connected components (cf. [6]). Using an argument similar to the one above, one can show that the connectivity constraints imposed by the constraint sets set $\mathcal{D}_{x^{[i]}(\lfloor t \rfloor), r}(\pi_{\mathbb{R}}(y)))$ are again superfluous.

Consider first the case when $E_{r\text{-LD}}(x_0)$ is connected. Note that this is equivalent to $E_{r\text{-disk}}(x_0)$ being connected. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \leq \cdots \leq x^{[N]}(0) = (x_0)_N$. The evolution of the network under $\mathcal{CC}_{\text{crcmcntr}}$ can then be described as the discrete-time dynamical system

$$x^{[1]}(\ell+1) = \frac{1}{2}(x^{[1]}(\ell) + x^{[2]}(\ell)), \quad x^{[2]}(\ell+1) = \frac{1}{2}(x^{[1]}(\ell) + x^{[3]}(\ell)), \ \dots,$$
$$, \ \dots, \ x^{[N-1]}(\ell+1) = \frac{1}{2}(x^{[N-2]}(\ell) + x^{[N]}(\ell)), \quad x^{[N]}(\ell+1) = \frac{1}{2}(x^{[N-1]}(\ell) + x^{[N]}(\ell)).$$

Note that this evolution respects the ordering of the agents. Equivalently, we can write $x(\ell+1) = A\,x(\ell)$, where A is the $N \times N$ matrix given by

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & \dots & \dots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \dots & \dots & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \dots & \dots & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

Now, note that $A = \mathrm{ATrid}_N^+\left(\frac{1}{2}, 0\right)$ as defined in Appendix B. Theorem B.4(i) implies that, for $x_{\text{ave}} = \frac{1}{N}\mathbf{1}^T x(0)$, we have that $\lim_{\ell\to+\infty} x(\ell) = x_{\text{ave}}\mathbf{1}$, and that the maximum time required for $\left\|x(\ell) - x_{\text{ave}}\mathbf{1}\right\|_2 \leq \eta \|x(0) - x_{\text{ave}}\mathbf{1}\|_2$ (over all initial conditions $x(0) \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \eta^{-1}\big)$. (As an aside, this also implies that the agents rendezvous at the location given by the average of their initial positions. In other words, we can forecast the asymptotic rendezvous position for this case, as opposed to the case with the r-disk communication graph.)

Next, let us convert the contraction inequality on 2-norms into an appropriate inequality on ∞-norms. Note that $\mathrm{diam}(x_0, I) \leq (N-1)r$ because $E_{r\text{-LD}}(x_0)$ is connected. Therefore

$$\|x(0) - x_{\text{ave}}\mathbf{1}\|_\infty = \max_{i \in I} |x^{[i]}(0) - x_{\text{ave}}| \leq |x_0^{[1]} - x_0^{[N]}| \leq (N-1)r.$$

For ℓ of order $N^2 \log \eta^{-1}$, we use this bound on $\|x(0) - x_{\text{ave}}\mathbf{1}\|_\infty$ and the basic inequalities $\|v\|_\infty \leq \|v\|_2 \leq \sqrt{N}\|v\|_\infty$ for all $v \in \mathbb{R}^N$, to obtain:

$$\|x(\ell) - x_{\text{ave}}\mathbf{1}\|_\infty \leq \|x(\ell) - x_{\text{ave}}\mathbf{1}\|_2 \leq \eta\|x(0) - x_{\text{ave}}\mathbf{1}\|_2 \leq \eta\sqrt{N}\|x(0) - x_{\text{ave}}\mathbf{1}\|_\infty \leq \eta\sqrt{N}(N-1)r.$$

This means that $(r\varepsilon)$-rendezvous is achieved for $\eta\sqrt{N}(N-1)r = r\varepsilon$, that is, in time $O(N^2 \log \eta^{-1}) = O(N^2 \log(N\varepsilon^{-1}))$.

Next, we show the lower bound. From equation (15) in the proof of Theorem B.3, we recall the unit-length vector $\mathbf{v}_{N-1} \in \mathbb{R}^{N-1}$. This vector is an eigenvector of $\mathrm{Trid}_{N-1}(\frac{1}{2}, 0, \frac{1}{2})$ corresponding to the largest singular value $\cos(\frac{\pi}{N})$. For $\mu = \frac{-1}{10\sqrt{2}} r N^{5/2}$, we then define the initial condition $x_0 = \mu P_+ \begin{bmatrix} 0 \\ \mathbf{v}_{N-1} \end{bmatrix} \in \mathbb{R}^N$. One can show that $(x_0)_i < (x_0)_{i+1}$ for $i \in \{1, \dots, N-1\}$, that $(x_0)_{\text{ave}} = 0$, and that $\max\{(x_0)_{i+1} - (x_0)_i \mid i \in \{1, \dots, N-1\}\} \leq r$. By Lemma B.5 and because $\|w\|_\infty \leq \|w\|_2 \leq \sqrt{N}\|w\|_\infty$ for all $w \in \mathbb{R}^N$, we compute

$$\|x_0\|_\infty = \frac{rN^{5/2}}{10\sqrt{2}} \Big\| P_+ \begin{bmatrix} 0 \\ \mathbf{v}_{N-1} \end{bmatrix} \Big\|_\infty \geq \frac{rN^2}{10\sqrt{2}} \Big\| P_+ \begin{bmatrix} 0 \\ \mathbf{v}_{N-1} \end{bmatrix} \Big\|_2 \geq \frac{rN}{10\sqrt{2}} \|\mathbf{v}_{N-1}\|_2 = \frac{rN}{10\sqrt{2}}.$$

The trajectory $x(\ell) = (\cos(\frac{\pi}{N}))^\ell x_0$ therefore satisfies

$$\|x(\ell)\|_\infty = \Big(\cos\Big(\frac{\pi}{N}\Big)\Big)^\ell \|x_0\|_\infty \geq \frac{rN}{10\sqrt{2}}\Big(\cos\Big(\frac{\pi}{N}\Big)\Big)^\ell .$$

Therefore, $\|x(\ell)\|_\infty$ is larger than $\frac{1}{2}r\varepsilon$ so long as $\frac{1}{10\sqrt{2}}N(\cos(\frac{\pi}{N}))^\ell > \frac{1}{2}\varepsilon$, that is, so long as

$$\ell < \frac{\log(\varepsilon^{-1}N) - \log(5\sqrt{2})}{-\log\big(\cos(\frac{\pi}{N})\big)} .$$

The rest of the proof is the same as in Theorem B.3(i) for the lower bound result.

If $E_{r\text{-LD}}(x_0)$ is not connected, note that along the network evolution, the connected components do not change. Therefore, the previous reasoning can be applied to each connected component. Since the number of agents in each connected component is strictly less that N, the time complexity can only but improve. Therefore, we conclude that

$$\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Theta(N^2 \log(N\varepsilon^{-1})) .$$

This completes the proof of fact (ii).

Finally, we prove the statements regarding $\mathcal{S}_{\mathbb{R}^d, r\text{-}\infty\text{-disk}}$ and $\mathcal{CC}_{\text{pll-crcmcntr}}$ in fact (iii) and in the previous Theorem 4.3. By definition, agents i and j are neighbors at time $\ell \in \mathbb{N}_0$ if and only if $\|x^{[i]}(\ell) - x^{[j]}(\ell)\|_\infty \leq r$, which is equivalent to

$$|\tau_k(x^{[i]}(\ell)) - \tau_k(x^{[j]}(\ell))| \leq r , \quad k \in \{1, \dots, d\} .$$

Recall from the proof of fact (i) that the connectivity constraints of $\mathcal{CC}_{\mathrm{crcmcntr}}$ on each agent are trivially satisfied in the 1-dimensional case. This fact has the following important consequence: from the expression for the control function in $\mathcal{CC}_{\text{pll-crcmcntr}}$, we deduce that the evolution under $\mathcal{CC}_{\text{pll-crcmcntr}}$ of the robotic network $\mathcal{S}_{\mathbb{R}^d, r\text{-}\infty\text{-disk}}$ (in d dimensions) can be alternatively described as the evolution under $\mathcal{CC}_{\mathrm{crcmcntr}}$ of d robotic networks $\mathcal{S}_{\mathbb{R}, r\text{-disk}}$ in $\mathbb{R}$. The correctness and the time complexity results now follows from the analysis of $\mathcal{CC}_{\mathrm{crcmcntr}}$ at $d = 1$. ■

Finally we proceed to characterize the mean communication complexity of the circumcenter control and communication law. We consider the case of a unidirectional communication model with one-round cost function depending linearly on the cardinality of the communication graph.

Theorem 4.5 (Mean communication complexity of circumcenter law for unidirectional communicatio *For $r \in \mathbb{R}_+$ and $\varepsilon \in]0, 1[$, the following statements hold:*

(i) for $d = 1$, on the network $\mathcal{S}_{\mathbb{R}^d, r\text{-disk}}$, $\mathrm{MCC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Theta(N^2)$;

(ii) for $d = 1$, on the network $\mathcal{S}_{\mathbb{R}^d, r\text{-LD}}$, $\mathrm{MCC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Theta(N)$;

(iii) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{\mathbb{R}^d, r\text{-}\infty\text{-disk}}$, $\mathrm{MCC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\text{pll-crcmcntr}}) \in \Theta(N^2)$. •

Proof: We start by proving fact (i). Let $x_0 \in (\mathbb{R}^d)^N$ be such that all $(x_0)_i$, $i \in I$, belong to a closed ball of radius $\sqrt{2}r/4$. In such a case, one can deduce that (i) $E_{r\text{-disk}}(x_0)$ is the complete graph, and therefore all agents compute the same goal point Circum, and (ii) this circumcenter belongs to the constrained set of each agent $i \in I$. As a consequence, all mobile agents rendezvous at the same location

Circum in one time instant. For such initial conditions, we have $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}, x_0) = 1$, and therefore

$$\begin{aligned}\mathrm{MCC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}, x_0) &= \frac{1}{\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}, x_0)} \sum_{\ell=0}^{\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}, x_0)-1} \mathrm{C}^L_{\mathrm{rnd}} \circ E_{\mathrm{cmm}\setminus\emptyset}(\ell, x(\ell)) \\ &= N(N-1)\,.\end{aligned}$$

This proves that $\mathrm{MCC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Theta(N^2)$, that is, fact (i). Additionally, fact (iii) is proved by analyzing the parallel circumcenter law as d decoupled versions of the circumcenter law. Next we show fact (ii). Given that the r-limited Delaunay graph is a subgraph of the Delaunay graph [6], and the number of edges of the latter is bounded by $3N - 6$ (see, for instance, [34]), we deduce that $\mathrm{MCC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in O(N)$. On the other hand, for any initial configuration x_0 where all the agents are aligned and non-coincident, there are $N-1$ neighboring relationships which are preserved throughout the network evolution. In this case, $\mathrm{MCC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}, x_0) = 2(N-1)$, and the result in fact (ii) follows. ■

Remark 4.6 Theorems 4.4 and 4.5 induce lower bounds on the time and mean communication complexity of the circumcenter law for the higher-dimensional case. Indeed, as a consequence of these results, we have

(i) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{\mathbb{R},r\text{-disk}}$, $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Omega(N)$ and $\mathrm{MCC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Omega(N^2)$;

(ii) for $d \in \mathbb{N}$, on the network $\mathcal{S}_{\mathbb{R},r\text{-LD}}$, $\mathrm{TC}(\mathcal{T}_{(r\varepsilon)\text{-rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Omega(N^2 \log(N\varepsilon^{-1}))$ and $\mathrm{MCC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) \in \Omega(N)$.

We have performed extensive numerical simulations for the case $d = 2$ and the network $\mathcal{S}_{\mathbb{R}^d,r\text{-disk}}$. We have ran the algorithm starting from generic initial configurations (where, in particular, agents' positions are not aligned) contained in a bounded region of $\mathbb{R}^2$. We have consistently obtained that the time complexity to achieve $\mathcal{T}_{\mathrm{rndzvs}}$ with $\mathcal{CC}_{\mathrm{crcmcntr}}$ starting from these initial configurations is independent of the number of agents. This leads us to conjecture that, in fact, initial configurations where all agents are aligned (i.e., the 1-dimensional case) give rise to the worst possible performance of the algorithm. In more formal terms, we conjecture that, for $d \geq 2$, $\mathrm{TC}(\mathcal{T}_{\mathrm{rndzvs}}, \mathcal{CC}_{\mathrm{crcmcntr}}) = \Theta(N)$. •

4.4 Deployment

In this section we consider the uniform robotic network $\mathcal{S}_{\mathbb{R}^d,r\text{-LD}}$ presented in Example 2.5 with parameter $r \in \mathbb{R}_+$. We assume we are given a convex simple polytope $Q \subset \mathbb{R}^d$, with an integrable density function $\phi\colon Q \to \mathbb{R}_+$. We assume that the initial positions of the agents belong to Q and we intend to design a control law that keeps them in Q for subsequent times. To achieve the ε-r-deployment task discussed in Example 3.4, we define the *centroid* control and communication law $\mathcal{CC}_{\mathrm{crcmcntr}}$. This is a uniform, static, time-independent law studied in [6, 7]. Loosely speaking, the evolution of the network under the centroid control and communication law can be described as follows:

> *[Informal description]* Communication rounds take place at each natural instant of time. At each communication round each agent performs the following tasks: (i) it transmits its position and receives its neighbors' positions; (ii) it computes the centroid of an appropriate region (the region is the intersection between the agent's Voronoi cell and a closed ball centered at its position and of radius $\frac{r}{2}$), and (iii) it moves toward this centroid.

Let us present this description in more formal terms. We set $\mathbb{T} = \mathbb{N}_0$, $L = \mathbb{R}^d$, and $\mathrm{msg}^{[i]} = \mathrm{msg}_{\mathrm{std}}$, $i \in I$. We define the control function $\mathrm{ctl}\colon \mathbb{R}^d \times \mathbb{R}^d \times L^N \to \mathbb{R}^d$ by

$$\mathrm{ctl}(x, x_{\mathrm{smpld}}, y) = \mathrm{Centroid}(\mathcal{X}) - x_{\mathrm{smpld}}\,,$$

where $\mathcal{X} = Q \cap \overline{B}(x_{\mathrm{smpld}}, \frac{r}{2}) \cap \big(\cap_{p \in \pi_L(y)} H_{x_{\mathrm{smpld}},p}\big)$ and $H_{x_{\mathrm{smpld}},p}$ is the half-space $\{q \in \mathbb{R}^d \mid \|q - x_{\mathrm{smpld}}\|_2 \leq \|q-p\|_2\}$. One can show that Q^N is a positively-invariant set for this control law.

The following theorem on the centroid control and communication law summarizes the known results about the asymptotic properties and the novel results on the complexity of this law. In characterizing complexity, we assume $\mathrm{diam}(Q)$ is independent of N, r and ε, and we do not calculate how the bounds depend on r. As for the circumcenter law, we provide complete time-complexity results only for the case $d = 1$.

Theorem 4.7 (Time and mean communication complexity of centroid law) *For $r \in \mathbb{R}_+$ and $\varepsilon \in \mathbb{R}_+$, consider the network $\mathcal{S}_{\mathbb{R}^d,r\text{-LD}}$ with initial conditions in Q. The following statements hold:*

(i) for $d \in \mathbb{N}$, the law $\mathcal{CC}_{\mathrm{centrd}}$ achieves the ε-r-deployment task $\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}$;

(ii) for $d = 1$ and $\phi = 1$, $\mathrm{TC}(\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}, \mathcal{CC}_{\mathrm{centrd}}) \in O(N^3 \log(N\varepsilon^{-1}))$;

(iii) for $d = 1$, $\phi = 1$ and for unidirectional communication, $\mathrm{MCC}(\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}, \mathcal{CC}_{\mathrm{centrd}}) \in \Theta(N)$. •

Proof: Fact (i) is proved in [6] for $d \in \{1, 2\}$ and it is clear that the same proof technique can be generalized to any dimension. Fact (iii) is proved in an analogous way to that of $\mathcal{CC}_{\mathrm{crcmcntr}}$ in Theorem 4.5. In what follows we sketch the proof of fact (ii). For $d = 1$, Q is a compact interval on $\mathbb{R}$, say $Q = [q_-, q_+]$.

We start with a brief discussion about connectivity. Note that in the r-limited Delaunay graph, two agents on the line that are at most at a distance r from each other are neighbors if and only if there are no other agents between them. Additionally we claim that, if agents i and j are neighbors at time instant ℓ, then $|\,\mathrm{Centroid}(\mathcal{X}^{[i]}(\ell)) - \mathrm{Centroid}(\mathcal{X}^{[j]}(\ell))| \leq r$. To see this, assume without loss of generality that $x^{[i]}(\ell) \leq x^{[j]}(\ell)$. Let us consider the case where the agents have neighbors on both sides (the other cases can be treated analogously). Let $x^{[i]}_-(\ell)$ (respectively, $x^{[j]}_+(\ell)$) denote the position of the neighbor of agent i to the left (respectively, of agent j to the right). Now, we have

$$\mathrm{Centroid}(\mathcal{X}^{[i]}(\ell)) = \frac{1}{4}(x^{[i]}_-(\ell) + 2x^{[i]}(\ell) + x^{[j]}(\ell))\,, \quad \mathrm{Centroid}(\mathcal{X}^{[j]}(\ell)) = \frac{1}{4}(x^{[i]}(\ell) + 2x^{[j]}(\ell) + x^{[j]}_+(\ell))\,.$$

Therefore, $|\,\mathrm{Centroid}(\mathcal{X}^{[i]}(\ell)) - \mathrm{Centroid}(\mathcal{X}^{[j]}(\ell))| \leq \frac{1}{4}\big(|x^{[i]}_-(\ell) - x^{[i]}(\ell)| + 2|x^{[i]}(\ell) - x^{[j]}(\ell)| + |x^{[j]}(\ell) - x^{[j]}_+(\ell)|\big) \leq r$. This implies that agents i and j are in the same connected component of the r-limited Delaunay graph at time instant $\ell + 1$.

Next, let us consider the case that $E_{r\text{-LD}}(x_0)$ is connected. Without loss of generality, assume that the agents are ordered from left to right according to their identifier, that is, $x^{[1]}(0) = (x_0)_1 \leq \cdots \leq x^{[N]}(0) = (x_0)_N$. We distinguish three cases depending on the proximity of the leftmost and rightmost agents 1 and N, respectively, to the boundary of the environment: ($\mathfrak{a}$) both agents are within a distance $\frac{r}{2}$ of ∂Q; ($\mathfrak{b}$) none of the two is within a distance $\frac{r}{2}$ of ∂Q; and ($\mathfrak{c}$) only one of the agents is within a distance $\frac{r}{2}$ of ∂Q. Here is an important observation: from one time instant to the next one, the network configuration can fall into any of the cases described above. However, because of the discussion on connectivity, transitions can only occur from case ($\mathfrak{b}$) to either case ($\mathfrak{a}$) or ($\mathfrak{c}$); and from case ($\mathfrak{c}$) to case ($\mathfrak{a}$). As we show in the following, for each of these cases, the network evolution under $\mathcal{CC}_{\mathrm{centrd}}$ can be described as a discrete-time linear dynamical system which respects agents' ordering.

Let us consider case ($\mathfrak{a}$). In this case, we have

$$x^{[1]}(\ell+1)=\frac{1}{4}(x^{[1]}(\ell)+x^{[2]}(\ell))+\frac{1}{2}q_-\,,\quad x^{[2]}(\ell+1)=\frac{1}{4}(x^{[1]}(\ell)+2x^{[2]}(\ell)+x^{[3]}(\ell))\,,\ \dots,$$
$$\dots,\ x^{[N-1]}(\ell+1)=\frac{1}{4}(x^{[N-2]}(\ell)+2x^{[N-1]}(\ell)+x^{[N]}(\ell))\,,\quad x^{[N]}(\ell+1)=\frac{1}{4}(x^{[N-1]}(\ell)+x^{[N]}(\ell))+\frac{1}{2}q_+\,.$$

Equivalently, we can write $x(\ell+1)=A_{\mathfrak{a}}\cdot x(\ell)+b_{\mathfrak{a}}$, where the $N\times N$-matrix $A_{\mathfrak{a}}$ and the vector $b_{\mathfrak{a}}$ are given by

$$A_{\mathfrak{a}}=\begin{bmatrix}\frac{1}{4}&\frac{1}{4}&0&\dots&\dots&0\\ \frac{1}{4}&\frac{1}{2}&\frac{1}{4}&\dots&\dots&0\\ 0&\frac{1}{4}&\frac{1}{2}&\frac{1}{4}&\dots&0\\ \vdots&&\ddots&\ddots&\ddots&\vdots\\ 0&\dots&\dots&\frac{1}{4}&\frac{1}{2}&\frac{1}{4}\\ 0&\dots&\dots&0&\frac{1}{4}&\frac{1}{4}\end{bmatrix},\quad b_{\mathfrak{a}}=\begin{bmatrix}\frac{1}{2}q_-\\ 0\\ \vdots\\ 0\\ \frac{1}{2}q_+\end{bmatrix}.$$

Note that the only equilibrium network configuration x_* respecting the ordering of the agents is given by

$$x_*^{[i]}=q_-+\frac{1}{2N}(1+2(i-1))(q_+-q_-)\,,\quad i\in I\,,$$

and note that this is a $\frac{r}{2}$-centroidal Voronoi configuration (under the assumption of case ($\mathfrak{a}$)). We can therefore write $(x(\ell)-x_*)=A_{\mathfrak{a}}(x(\ell-1)-x_*)$. Now, note that $A_{\mathfrak{a}}=\mathrm{ATrid}_N^-\left(\frac{1}{4},\frac{1}{2}\right)$ as defined in Appendix B. Theorem B.4(ii) implies that $\lim_{\ell\to+\infty}\big(x(\ell)-x_*\big)=\mathbf{0}$, and that the maximum time required for $\|x(\ell)-x_*\|_2\leq\varepsilon\|x(0)-x_*\|_2$ (over all initial conditions $x(0)\in\mathbb{R}^N$) is $\Theta\big(N^2\log\varepsilon^{-1}\big)$. It is not obvious, but it can be verified, that the initial condition providing the lower bound in the time complexity estimate does indeed have the property of respecting the agents' ordering; this fact holds for all three cases ($\mathfrak{a}$), ($\mathfrak{b}$) and ($\mathfrak{c}$).

The case ($\mathfrak{b}$) can be treated in the same way. The network evolution takes now the form $x(\ell+1)=A_{\mathfrak{b}}\cdot x(\ell)+b_{\mathfrak{b}}$, where the $N\times N$-matrix $A_{\mathfrak{b}}$ and the vector $b_{\mathfrak{b}}$ are given by

$$A_{\mathfrak{b}}=\begin{bmatrix}\frac{3}{4}&\frac{1}{4}&0&\dots&\dots&0\\ \frac{1}{4}&\frac{1}{2}&\frac{1}{4}&\dots&\dots&0\\ 0&\frac{1}{4}&\frac{1}{2}&\frac{1}{4}&\dots&0\\ \vdots&&\ddots&\ddots&\ddots&\vdots\\ 0&\dots&\dots&\frac{1}{4}&\frac{1}{2}&\frac{1}{4}\\ 0&\dots&\dots&0&\frac{1}{4}&\frac{3}{4}\end{bmatrix},\quad b_{\mathfrak{b}}=\begin{bmatrix}-\frac{1}{4}r\\ 0\\ \vdots\\ 0\\ \frac{1}{4}r\end{bmatrix}.$$

In this case, a (non-unique) equilibrium network configuration respecting the ordering of the agents is of the form

$$x_*^{[i]}=ir-\frac{1+N}{2}r\,,\quad i\in I\,.$$

Note that this is a $\frac{r}{2}$-centroidal Voronoi configuration (under the assumption of case ($\mathfrak{b}$)). We can therefore write $(x(\ell)-x_*)=A_{\mathfrak{b}}(x(\ell-1)-x_*)$. Now, note that $A_{\mathfrak{b}}=\mathrm{ATrid}_N^+\left(\frac{1}{4},\frac{1}{2}\right)$ as defined in Appendix B. As in the notation of Theorem B.4(i) we compute $x_{\mathrm{ave}}=\frac{1}{N}\mathbf{1}^T(x_0-x_*)=\frac{1}{N}\mathbf{1}^Tx_0$. With this calculation, Theorem B.4(i) implies that $\lim_{\ell\to+\infty}\big(x(\ell)-x_*-x_{\mathrm{ave}}\mathbf{1}\big)=\mathbf{0}$, and that the maximum time required for $\|x(\ell) - x_* - x_{\text{ave}}\mathbf{1}\|_2 \leq \varepsilon \|x(0) - x_* - x_{\text{ave}}\mathbf{1}\|_2$ (over all initial conditions $x(0) \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \varepsilon^{-1}\big)$.

Case ($\mathfrak{c}$) needs to be handled differently. Without loss of generality, assume that agent 1 is within distance $\frac{r}{2}$ of ∂Q and agent N is not (the other case is treated analogously). Then, the network evolution takes now the form $x(\ell+1) = A_{\mathfrak{c}} \cdot x(\ell) + b_{\mathfrak{c}}$, where the $N \times N$-matrix $A_{\mathfrak{c}}$ and the vector $b_{\mathfrak{c}}$ are given by

$$A_{\mathfrak{c}} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & 0 & \dots & \dots & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \dots & \dots & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & \dots & \dots & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix}, \quad b_{\mathfrak{c}} = \begin{bmatrix} \frac{1}{2}q_- \\ 0 \\ \vdots \\ 0 \\ \frac{1}{4}r \end{bmatrix}.$$

Note that the only equilibrium network configuration x_* respecting the ordering of the agents is given by

$$x_*^{[i]} = q_- + \frac{1}{2}(2i-1)r\,, \quad i \in I\,,$$

and note that this is a $\frac{r}{2}$-centroidal Voronoi configuration (under the assumption of case ($\mathfrak{c}$)). In order to analyze $A_{\mathfrak{c}}$, we recast the N-dimensional discrete-time dynamical system as a $2N$-dimensional one. To do this, we define a $2N$-dimensional vector y by

$$y^{[i]} = x^{[i]}, i \in I, \quad \text{and} \quad y^{[N+i]} = x^{[N-i+1]}, i \in I, \tag{12}$$

Now, one can see that the network evolution can be alternatively described in the variables $(y^{[1]}, \dots, y^{[2N]})$ as a linear dynamical system determined by the $2N \times 2N$ matrix $\text{ATrid}^-_{2N}(\frac{1}{4}, \frac{1}{2})$. Using analogous arguments to the ones used before in Theorems B.3 and B.4 and exploiting the chain of equalities (12), we can characterize the eigenvalues and eigenvectors of $\text{Trid}_{2N-1}(\frac{1}{4}, \frac{1}{2}, \frac{1}{4})$, and infer that, even for case ($\mathfrak{c}$), the maximum time required for $\|x(\ell) - x_*\|_2 \leq \varepsilon \|x(0) - x_*\|_2$ (over all initial conditions $x(0) \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \varepsilon^{-1}\big)$.

In summary, for all three cases ($\mathfrak{a}$), ($\mathfrak{b}$) and ($\mathfrak{c}$), our calculations show that, in time $O\big(N^2 \log \varepsilon^{-1}\big)$, the error 2-norm satisfies the contraction inequality $\|x(\ell) - x_*\|_2 \leq \varepsilon \|x(0) - x_*\|_2$. We convert this inequality on 2-norms into an appropriate inequality on ∞-norms as follows. Note that $\|x(0) - x_*\|_\infty = \max_{i \in I} |x^{[i]}(0) - x_*^{[i]}| \leq (q_+ - q_-)$. For ℓ of order $N^2 \log \eta^{-1}$, we have:

$$\|x(\ell) - x_*\|_\infty \leq \|x(\ell) - x_*\|_2 \leq \eta \|x(0) - x_*\|_2 \leq \eta\sqrt{N} \|x(0) - x_*\|_\infty \leq \eta\sqrt{N}(q_+ - q_-).$$

This means that ε-r-deployment is achieved for $\eta\sqrt{N}(q_+ - q_-) = \varepsilon$, that is, in time $O(N^2 \log \eta^{-1}) = O(N^2 \log(N\varepsilon^{-1}))$.

Up to here we have proved that, if the graph $(I, E_{r\text{-LD}}(x_0))$ is connected, then $\text{TC}(\mathcal{T}_{\varepsilon\text{-}r\text{-deplmnt}}, \mathcal{CC}_{\text{centrd}}) \in O(N^2 \log(N\varepsilon^{-1}))$. If $(I, E_{r\text{-LD}}(x_0))$ is not connected, note that along the network evolution there can only be a finite number of time instants, at most $N-1$ where a merging of two connected components occurs. Therefore, the time complexity is at most $O(N^3 \log(N\varepsilon^{-1}))$. ■

5 Conclusions

We have introduced a formal model for the design and analysis of coordination algorithms executed by networks composed of robotic agents. In our discrete-time communication, continuous-time motion model, the robotic agents evolve in the physical domain in continuous-time and have the ability to exchange information (position and/or dynamic variables) that affect their motion at discrete-time instants. Under this framework, motion coordination algorithms are formalized as feedback control and communication laws. Drawing analogies with the classical theory on distributed algorithms, we have defined two measures of complexity for this formal notion of execution: the time and the mean communication complexity of achieving a specific task. We have defined the notion of re-scheduling of a control and communication law and analyzed the invariance of the proposed complexity measures under this operation. These concepts and results have been illustrated with various examples of different types of robotic networks and different classes of feedback control and communication laws. Finally, we have computed the proposed complexity measures for a variety of algorithms performing spatially-distributed tasks such as rendezvous, pursuit, and deployment.

Throughout the paper, we have made compromises in order to arrive at a simple yet useful model that is able to capture a wide class of algorithms. However, it is important to acknowledge that our results on network, task and complexity modeling, and our results on algorithm analysis have numerous limitations and open up numerous avenues for future research. Let us at provide a concise list here: (1) modeling of asynchronous as opposed to only synchronous networks (see however [7, 9, 14, 25]); (2) models and analysis of failures in the agents (arrivals/departures) and in the communication links (see however [5, 32, 35, 39]); (3) probabilistic versions of the complexity measures, as opposed to only worst-case analysis (see however [19]); (4) quantization and delays in the communication channels (see however [12] and the literature on quantized control); (5) parallel, sequential and hierarchical composition of control and communication laws. Additionally, our analysis results essentially consist of a time-complexity analysis of some basic algorithms, but many more open algorithmic questions remain unresolved including (6) communication complexity for omnidirectional communication models; (7) analysis of known algorithms for flocking and cohesion; (8) complexity analysis of tasks as opposed to algorithms.

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A Basic geometric notions

Let $S \subset \mathbb{R}^d$, $d \in \mathbb{N}$, be compact. The *circumcenter of* S, denoted by $\text{Circum}(S)$, is the center of the smallest-radius sphere in $\mathbb{R}^d$ enclosing S. Given an integrable function $\phi\colon S \to \mathbb{R}_+$, the *mass* of S is $\text{Mass}(S) = \int_S \phi(q)dq$, and the *centroid* of S is

$$\text{Centroid}(S) = \frac{1}{\text{Mass}(S)} \int_S q\phi(q)dq\,.$$

A *partition* of S is a collection of subsets of S with disjoint interiors and whose union is S. Given a set of N distinct points $\mathcal{P} = \{p_i\}_{i\in\{1,\dots,N\}}$ in S, the *Voronoi partition of* S generated by $\mathcal{P}$ (with respect to the Euclidean norm) is the collection of sets $\{V_i(\mathcal{P})\}_{i\in\{1,\dots,N\}}$ defined by $V_i(\mathcal{P}) = \{q \in S \mid \|q-p_i\|_2 \leq \|q-p_j\|_2\,,\text{ for all } p_j \in \mathcal{P}\}$. We usually refer to $V_i(\mathcal{P})$ as V_i. For a detailed treatment of Voronoi partitions we refer to [8, 34].

For $I = \{1,\dots,N\}$ and $S \subset \mathbb{R}^d$, a *proximity edge map* is a map of the form $E\colon S^N \to 2^{I\times I\setminus \text{diag}(I\times I)}$. For $r \in \mathbb{R}_+$, we define the r-disk proximity edge map $E_{r\text{-disk}}\colon (\mathbb{R}^d)^N \to 2^{I\times I}$ and the r-limited Delaunay proximity edge map $E_{r\text{-LD}}\colon (\mathbb{R}^d)^N \to 2^{I\times I}$ by

$$E_{r\text{-disk}}(x_1,\dots,x_N) = \{(i,j) \in I\times I \setminus \text{diag}(I\times I) \mid \|x_i - x_j\|_2 \leq r\}\,, \tag{13a}$$

$$E_{\text{LD},\text{r}}(x_1,\dots,x_N) = \{(i,j) \in I\times I \setminus \text{diag}(I\times I) \mid \big(V_i \cap \overline{B}(x_i,\tfrac{r}{2})\big) \cap \big(V_j \cap \overline{B}(x_j,\tfrac{r}{2})\big) \neq \emptyset\}\,, \tag{13b}$$

where $\{V_1,\dots,V_N\}$ is the Voronoi partition of $\mathbb{R}^d$ generated by $\{x_1,\dots,x_N\}$. Illustrations of these concepts are given in Fig. 7. For $d = 1$, the r-limited Delaunay proximity edge map has the following intuitive characterization: two points are neighbors if and only if they are within distance r and no other point is between them.

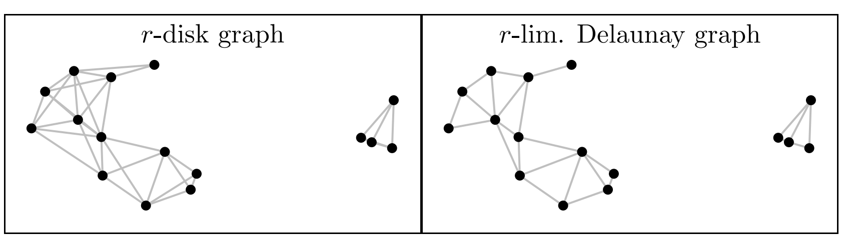

Figure 7: The r-disk and r-limited Delaunay graphs in $\mathbb{R}^2$

As proved in [6], the r-limited Delaunay graph and the r-disk graph have the same connected components. Additionally, the r-limited Delaunay graph is "computable" on the r-disk graph in the following sense: any node in the network can compute the set of its neighbors in the r-limited Delaunay graph if it is given the set of its neighbors in the r-disk graph. This implies that any control and communication law for a network with communication graph $E_{r\text{-LD}}$ can be implemented on a analogous network with communication graph $E_{r\text{-disk}}$.

B Tridiagonal Toeplitz and circulant matrices

We briefly review some basic facts about certain classes of Toeplitz matrices, see [31]. For $N \geq 2$ and $a, b, c \in \mathbb{R}$, define the $N \times N$ Toeplitz matrices $\mathrm{Trid}_N(a, b, c)$ and $\mathrm{Circ}_N(a, b, c)$ by

$$\mathrm{Trid}_N(a,b,c) = \begin{bmatrix} b & c & 0 & \dots & 0 \\ a & b & c & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & a & b & c \\ 0 & \dots & 0 & a & b \end{bmatrix}, \quad \text{and} \quad \mathrm{Circ}_N(a,b,c) = \mathrm{Trid}_N(a,b,c) + \begin{bmatrix} 0 & \dots & \dots & 0 & a \\ 0 & \dots & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ c & 0 & \dots & 0 & 0 \end{bmatrix}.$$

The matrices Trid_N and Circ_N are tridiagonal and circulant, respectively. The two matrices only differ in their $(1, N)$ and $(N, 1)$ entries. Note our convention that $C_2(a, b, c) = \begin{bmatrix} b & a+c \\ a+c & b \end{bmatrix}$.

Lemma B.1 (Eigenvalues of tridiagonal Toeplitz and circulant matrices) *For $N \geq 2$ and $a, b, c \in \mathbb{R}$, the following statements hold:*

(i) for $ac \neq 0$, the eigenvalues and eigenvectors of $\mathrm{Trid}_N(a, b, c)$ are

$$b + 2c\sqrt{\frac{a}{c}} \cos\left(\frac{i\pi}{N+1}\right), \qquad \begin{bmatrix} \left(\frac{a}{c}\right)^{1/2} \sin\left(\frac{i\pi}{N+1}\right) \\ \left(\frac{a}{c}\right)^{2/2} \sin\left(\frac{2i\pi}{N+1}\right) \\ \vdots \\ \left(\frac{a}{c}\right)^{N/2} \sin\left(\frac{Ni\pi}{N+1}\right) \end{bmatrix}, \quad i \in \{1, \dots, N\};$$

(ii) the eigenvalues and eigenvectors of $\mathrm{Circ}_N(a, b, c)$ are (note $\omega = \exp(\frac{2\pi\sqrt{-1}}{N})$)

$$b + (a+c)\cos\left(\frac{i2\pi}{N}\right) + \sqrt{-1}(c-a)\sin\left(\frac{i2\pi}{N}\right), \qquad \begin{bmatrix} 1 \\ \omega^i \\ \vdots \\ \omega^{(N-1)i} \end{bmatrix}, \quad i \in \{1, \dots, N\}. \qquad \bullet$$

Proof: Both facts are discussed, for example, in [31, Example 7.2.5 and Exercise 7.2.20]. Fact (ii) requires some straightforward algebraic manipulations. ∎

Remarks B.2 (i) The set of eigenvalues of $\mathrm{Trid}_N(a, b, c)$ is contained in the real interval $[b-2\sqrt{ac}, b+2\sqrt{ac}]$, if $ac \geq 0$, and in the interval in the complex plane $[b - 2\sqrt{-1}\sqrt{|ac|}, b + 2\sqrt{-1}\sqrt{|ac|}]$, if $ac \leq 0$.

(ii) The set of eigenvalues of $\mathrm{Circ}_N(a, b, c)$ is contained in the ellipse on the complex plane with center b and horizontal axis $2|a + c|$ and vertical axis $2|c - a|$.

(iii) Recall from [31] that (1) a square matrix is normal if it has a complete orthonormal set of eigenvectors, (2) circulant matrices and real-symmetric matrices are normal, and (3) if a normal matrix has eigenvalues $\{\lambda_1, \dots, \lambda_n\}$, then its singular values are $\{|\lambda_1|, \dots, |\lambda_n|\}$. •

We can now state the main result of this section.

Theorem B.3 (Tridiagonal Toeplitz and circulant dynamical systems) *Let $N \geq 2$, $\varepsilon \in]0,1[$, and $a, b, c \in \mathbb{R}$. Let $x\colon \mathbb{N}_0 \to \mathbb{R}^N$ and $y\colon \mathbb{N}_0 \to \mathbb{R}^N$ be solutions to*

$$x(\ell+1) = \mathrm{Trid}_N(a,b,c)\, x(\ell), \quad \textit{and} \quad y(\ell+1) = \mathrm{Circ}_N(a,b,c)\, y(\ell),$$

with initial conditions $x(0) = x_0$ and $y(0) = y_0$, respectively. The following statements hold:

(i) if $a = c \neq 0$ and $|b| + 2|a| = 1$, then $\lim_{\ell\to+\infty} x(\ell) = \mathbf{0}$, and the maximum time required for $\|x(\ell)\|_2 \leq \varepsilon\|x_0\|_2$ (over all initial conditions $x_0 \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \varepsilon^{-1}\big)$;

(ii) if $a \neq 0$, $c = 0$ and $0 < |b| < 1$, then $\lim_{\ell\to+\infty} x(\ell) = \mathbf{0}$, and the maximum time required for $\|x(\ell)\|_2 \leq \varepsilon\|x_0\|_2$ (over all initial conditions $x_0 \in \mathbb{R}^N$) is $O\big(N \log N + \log \varepsilon^{-1}\big)$;

(iii) if $a \geq 0$, $c \geq 0$, $b > 0$, and $a + b + c = 1$, then $\lim_{\ell\to+\infty} y(\ell) = y_{\mathrm{ave}}\mathbf{1}$, where $y_{\mathrm{ave}} = \frac{1}{N}\mathbf{1}^T y_0$, and the maximum time required for $\|y(\ell) - y_{\mathrm{ave}}\mathbf{1}\|_2 \leq \varepsilon\|y_0 - y_{\mathrm{ave}}\mathbf{1}\|_2$ (over all initial conditions $y_0 \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \varepsilon^{-1}\big)$. •

Proof: Let us prove fact (i). We start by bounding from above the eigenvalue with largest absolute value, that is, the largest singular value, of $\mathrm{Trid}_N(a,b,a)$:

$$\max_{i\in\{1,\dots,N\}} \left| b + 2a\cos\left(\frac{i\pi}{N+1}\right)\right| \leq |b| + 2|a| \max_{i\in\{1,\dots,N\}} \left|\cos\left(\frac{i\pi}{N+1}\right)\right| \leq |b| + 2|a|\cos\left(\frac{\pi}{N+1}\right).$$

Because $\cos(\frac{\pi}{N+1}) < 1$ for any $N \geq 2$, the matrix $\mathrm{Trid}_N(a,b,a)$ is stable. Additionally, for $\ell > 0$, we bound from above the magnitude of the curve x as

$$\|x(\ell)\|_2 = \|\,\mathrm{Trid}_N(a,b,a)^\ell x_0\|_2 \leq \left(|b| + 2|a|\cos\left(\frac{\pi}{N+1}\right)\right)^\ell \|x_0\|_2.$$

In order to have $\|x(\ell)\|_2 < \varepsilon\|x_0\|_2$, it is sufficient that $\ell \log\left(|b| + 2|a|\cos\left(\frac{\pi}{N+1}\right)\right) < \log\varepsilon$, that is

$$\ell > \frac{\log \varepsilon^{-1}}{-\log\left(|b| + 2|a|\cos\left(\frac{\pi}{N+1}\right)\right)}. \tag{14}$$

To show the upper bound, note that as $t \to 0$ we have

$$-\frac{1}{\log(1 - 2|a|(1-\cos t))} = \frac{1}{|a|t^2} + O(1).$$

Now, assume without loss of generality that $ab > 0$ and consider the eigenvalue $b + 2a\cos(\frac{\pi}{N+1})$ of $\mathrm{Trid}_N(a,b,a)$. Note that $|b + 2a\cos(\frac{\pi}{N+1})| = |b| + 2|a|\cos(\frac{\pi}{N+1})$. (If $ab < 0$, then consider the eigenvalue $b + 2a\cos(\frac{N\pi}{N+1})$.) For $N > 2$, define the unit-length vector

$$\mathbf{v}_N = \sqrt{\frac{2}{N+1}} \begin{bmatrix} \sin\frac{\pi}{N+1} \\ \vdots \\ \sin\frac{N\pi}{N+1} \end{bmatrix} \in \mathbb{R}^N, \tag{15}$$

and note that, by Lemma B.1(i), $\mathbf{v}_N$ is an eigenvector of $\mathrm{Trid}_N(a,b,a)$ with eigenvalue $b + 2a\cos(\frac{\pi}{N+1})$. The trajectory x with initial condition $\mathbf{v}_N$ satisfies $\|x(\ell)\|_2 = \left(|b| + 2|a|\cos\left(\frac{\pi}{N+1}\right)\right)^\ell \|\mathbf{v}_N\|_2$ and, therefore, it will enter $B(\mathbf{0}, \varepsilon\|\mathbf{v}_N\|_2)$ only when ℓ satisfies (14). This completes the proof of fact (i).

Next we consider statement (ii). Clearly, $\mathrm{Trid}_N(a,b,0)$ is stable. For $\ell > 0$, we compute

$$\mathrm{Trid}_N(a,b,0)^\ell = b^\ell\Big(I_N + \frac{a}{b}\,\mathrm{Trid}_N(1,0,0)\Big)^\ell = b^\ell \sum_{j=0}^{N-1} \frac{\ell!}{j!(\ell-j)!}\Big(\frac{a}{b}\Big)^j\, \mathrm{Trid}_N(1,0,0)^j$$

because of the nilpotency of $\mathrm{Trid}_N(1,0,0)$. Now we can bound from above the magnitude of the curve x as

$$\begin{aligned}\|x(\ell)\|_2 = \|\,\mathrm{Trid}_N(a,b,0)^\ell x_0\|_2 &\le |b|^\ell \sum_{j=0}^{N-1} \frac{\ell!}{j!(\ell-j)!}\Big(\frac{a}{b}\Big)^j \left\|\,\mathrm{Trid}_N(1,0,0)^j x_0\right\|_2 \\ &\le \mathrm{e}^{a/b}\ell^{N-1}\,|b|^\ell\,\|x_0\|_2.\end{aligned}$$

Here we used $\|\,\mathrm{Trid}_N(1,0,0)^j x_0\|_2 \le \|x_0\|_2$ and $\max\{\frac{\ell!}{(\ell-j)!} \mid j \in \{0,\dots,N-1\}\} \le \ell^{N-1}$. Therefore, in order to have $\|x(\ell)\|_2 < \varepsilon\|x_0\|_2$, it suffices that $\log(\mathrm{e}^{a/b}) + (N-1)\log\ell + \ell\log|b| \le \log\varepsilon$, that is

$$\ell - \frac{N-1}{-\log|b|}\log\ell > \frac{\frac{a}{b}-\log\varepsilon}{-\log|b|}.$$

A sufficient condition for $\ell - \alpha\log\ell > \beta$, for $\alpha,\beta > 0$, is that $\ell \ge 2\beta + 2\alpha\max\{1,\log\alpha\}$. For, if $\ell \ge 2\alpha$, then $\log\ell$ is bounded from above by the line $\ell/2\alpha + \log\alpha$. Furthermore, the line $\ell/2\alpha+\log\alpha$ is a lower bound for the line $(\ell-\beta)/\alpha$ if $\ell \ge 2\beta + 2\alpha\log\alpha$. In summary, it is true that $\|x(\ell)\|_2 \le \varepsilon\|x(0)\|_2$ whenever

$$\ell \ge 2\frac{\frac{a}{b}-\log\varepsilon}{-\log|b|} + 2\frac{N-1}{-\log|b|}\max\left\{1, \log\frac{N-1}{-\log|b|}\right\}.$$

This completes the proof of the upper bound, that is, fact (ii).

The proof of fact (iii) is similar to that of fact (i). We analyze the singular values of $\mathrm{Circ}_N(a,b,c)$. It is clear that the eigenvalue corresponding to $i=N$ is equal to 1; this is the largest singular value of $\mathrm{Circ}_N(a,b,c)$ and the corresponding eigenvector is $\mathbf{1}$. In the orthogonal decomposition induced by the eigenvectors of $\mathrm{Circ}_N(a,b,c)$, the vector y_0 has a component y_{ave} along the eigenvector $\mathbf{1}$. We now compute the second largest singular value:

$$\begin{aligned}\max_{i\in\{1,\dots,N-1\}} &\left| b + (a+c)\cos\left(\frac{i2\pi}{N}\right) + \sqrt{-1}(c-a)\sin\left(\frac{i2\pi}{N}\right)\right| \\ &= \left|1-(a+c)\Big(1-\cos\big(\frac{2\pi}{N}\big)\Big) + \sqrt{-1}(c-a)\sin\left(\frac{2\pi}{N}\right)\right|.\end{aligned}$$

Here $|\cdot|$ is the norm in $\mathbb{C}$. Because of the assumptions on a,b,c, the second largest singular value is strictly less than 1. For $\ell > 0$, we bound the distance of the curve $y(\ell)$ from $y_{\mathrm{ave}}\mathbf{1}$ as

$$\begin{aligned}\|y(\ell) - y_{\mathrm{ave}}\mathbf{1}\|_2 &= \|\,\mathrm{Circ}_N(a,b,c)^\ell y_0 - y_{\mathrm{ave}}\mathbf{1}\|_2 = \|\,\mathrm{Circ}_N(a,b,c)^\ell\big(y_0 - y_{\mathrm{ave}}\mathbf{1}\big)\|_2 \\ &\le \left|1-(a+c)\Big(1-\cos\big(\frac{2\pi}{N}\big)\Big) + \sqrt{-1}(c-a)\sin\left(\frac{2\pi}{N}\right)\right|^\ell \|y_0 - y_{\mathrm{ave}}\mathbf{1}\|_2.\end{aligned}$$

This proves that $\lim_{\ell\to+\infty} y(\ell) = y_{\mathrm{ave}}\mathbf{1}$. Also, for $\alpha = a+c, \beta = c-a$ and as $t \to 0$, we have

$$-\frac{1}{\log\Big(\big(1-\alpha(1-\cos t)\big)^2 + \beta^2\sin^2 t\Big)^{1/2}} = \frac{2}{(\alpha-\beta^2)t^2} + O(1).$$

Here $\beta^2 < \alpha$ because $a, c \in]0,1[$.

Now, consider the eigenvalues $\lambda_N = b + (a+c)\cos\big(\frac{2\pi}{N}\big) + \sqrt{-1}(c-a)\sin\big(\frac{2\pi}{N}\big)$ and $\overline{\lambda}_N = b + (a+c)\cos\left(\frac{(N-1)2\pi}{N}\right) + \sqrt{-1}(c-a)\sin\left(\frac{(N-1)2\pi}{N}\right)$ of $\mathrm{Circ}_N(a,b,c)$, and its associated eigenvectors (cf. Lemma B.1(ii))

$$\mathbf{v}_N = \begin{bmatrix} 1 \\ w \\ \vdots \\ w^{N-1}\end{bmatrix} \in \mathbb{C}^N, \quad \overline{\mathbf{v}}_N = \begin{bmatrix} 1 \\ w^{N-1} \\ \vdots \\ w\end{bmatrix} \in \mathbb{C}^N. \tag{16}$$

Note that the vector $\mathbf{v}_N + \overline{\mathbf{v}}_N$ belongs to $\mathbb{R}^N$. Moreover, its component y_{ave} along the eigenvector $\mathbf{1}$ is 0. The trajectory y with initial condition $\mathbf{v}_N + \overline{\mathbf{v}}_N$ satisfies $\|y(\ell)\|_2 = \|\lambda_N^\ell \mathbf{v}_N + \overline{\lambda}_N^\ell \overline{\mathbf{v}}_N\|_2 = |\lambda_N|^\ell \|\mathbf{v}_N + \overline{\mathbf{v}}_N\|_2$ and, therefore, it will enter $B(\mathbf{0}, \varepsilon\|\mathbf{v}_N + \overline{\mathbf{v}}_N\|_2)$ only when

$$\ell > \frac{\log \varepsilon^{-1}}{-\log \left|1 - (a+c)\Big(1 - \cos\left(\frac{2\pi}{N}\right)\Big) + \sqrt{-1}(c-a)\sin\left(\frac{2\pi}{N}\right)\right|}.$$

This completes the proof of fact (iii). ■

Next, we extend these results to another interesting set of matrices. For $N \geq 2$ and $a, b \in \mathbb{R}$, define the $N \times N$ matrices $\text{ATrid}_N^+(a,b)$ and $\text{ATrid}_N^-(a,b)$ by

$$\text{ATrid}_N^\pm(a,b) = \text{Trid}_N(a,b,a) \pm \begin{bmatrix} a & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 & 0 \\ 0 & \dots & \dots & 0 & a \end{bmatrix}.$$

If we define

$$P_+ = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ 1 & -1 & 1 & 0 & \dots & 0 \\ 1 & 0 & -1 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ 1 & 0 & \dots & 0 & -1 & 1 \\ 1 & 0 & \dots & 0 & 0 & -1 \end{bmatrix}, \text{ and } P_- = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 \\ -1 & 1 & 1 & 0 & \dots & 0 \\ 1 & 0 & 1 & 1 & \dots & 0 \\ \vdots & & \ddots & \ddots & \ddots & \\ (-1)^{N-2} & 0 & \dots & 0 & 1 & 1 \\ (-1)^{N-1} & 0 & \dots & 0 & 0 & 1 \end{bmatrix},$$

then the following similarity transforms are satisfied:

$$\begin{aligned} \text{ATrid}_N^+(a,b) &= P_+ \begin{bmatrix} b+2a & 0 \\ 0 & \text{Trid}_{N-1}(a,b,a) \end{bmatrix} P_+^{-1}, \\ \text{ATrid}_N^-(a,b) &= P_- \begin{bmatrix} b-2a & 0 \\ 0 & \text{Trid}_{N-1}(a,b,a) \end{bmatrix} P_-^{-1}. \end{aligned} \tag{17}$$

To analyze the convergence properties of the dynamical systems determined by $\text{ATrid}_N^+(a,b)$ and $\text{ATrid}_N^-(a,b)$, we recall that $\mathbf{1} = (1,\dots,1) \in \mathbb{R}^d$, and we define $\mathbf{1}_- = (1,-1,1,\dots,(-1)^{N-2},(-1)^{N-1}) \in \mathbb{R}^N$.

Theorem B.4 *Let $N \geq 2$, $\varepsilon \in]0,1[$, and $a, b \in \mathbb{R}$ with $a \neq 0$ and $|b| + 2|a| = 1$. Let $x\colon \mathbb{N}_0 \to \mathbb{R}^N$ and $z\colon \mathbb{N}_0 \to \mathbb{R}^N$ be solutions to*

$$x(\ell+1) = \text{ATrid}_N^+(a,b)\, x(\ell), \quad \text{and} \quad z(\ell+1) = \text{ATrid}_N^-(a,b)\, z(\ell),$$

with initial conditions $x(0) = x_0$ and $z(0) = z_0$, respectively. The following statements hold:

(i) $\lim_{\ell\to+\infty}\big(x(\ell) - x_{\text{ave}}(\ell)\mathbf{1}\big) = \mathbf{0}$, where $x_{\text{ave}}(\ell) = (\frac{1}{N}\mathbf{1}^T x_0)(b+2a)^\ell$, and the maximum time required for $\|x(\ell) - x_{\text{ave}}(\ell)\mathbf{1}\|_2 \leq \varepsilon\|x_0 - x_{\text{ave}}(0)\mathbf{1}\|_2$ (over all initial conditions $x_0 \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \varepsilon^{-1}\big)$;

(ii) $\lim_{\ell\to+\infty}\big(z(\ell) - z_{\text{ave}}(\ell)\mathbf{1}_-\big) = \mathbf{0}$, where $z_{\text{ave}}(\ell) = (\frac{1}{N}\mathbf{1}_-^T z_0)(b-2a)^\ell$, and the maximum time required for $\|z(\ell) - z_{\text{ave}}(\ell)\mathbf{1}_-\|_2 \leq \varepsilon\|z_0 - z_{\text{ave}}(0)\mathbf{1}_-\|_2$ (over all initial conditions $z_0 \in \mathbb{R}^N$) is $\Theta\big(N^2 \log \varepsilon^{-1}\big)$.

Proof: We prove fact (i) and remark that the proof of fact (ii) is analogous. Consider the change of coordinates

$$x(\ell) = P_+ \begin{bmatrix} x'_{\text{ave}}(\ell) \\ y(\ell) \end{bmatrix} = x'_{\text{ave}}(\ell)\mathbf{1} + P_+ \begin{bmatrix} 0 \\ y(\ell) \end{bmatrix},$$

where $x'_{\text{ave}}(\ell) \in \mathbb{R}$ and $y(\ell) \in \mathbb{R}^{N-1}$. A quick calculation shows that $x'_{\text{ave}}(\ell) = \frac{1}{N}\mathbf{1}^T x(\ell)$, and the similarity transformation described in equation (17) implies

$$y(\ell+1) = \text{Trid}_{N-1}(a,b,a)\, y(\ell), \text{ and } x'_{\text{ave}}(\ell+1) = (b+2a)x'_{\text{ave}}(\ell).$$

Therefore, $x_{\text{ave}} = x'_{\text{ave}}$. It is also clear that

$$x(\ell+1) - x_{\text{ave}}(\ell+1)\mathbf{1} = P_+ \begin{bmatrix} 0 \\ y(\ell+1) \end{bmatrix} = \Big(P_+ \begin{bmatrix} 0 & 0 \\ 0 & \text{Trid}_{N-1}(a,b,a) \end{bmatrix} P_+^{-1}\Big)(x(\ell) - x_{\text{ave}}(\ell)\mathbf{1}).$$

Consider the matrix in parenthesis determining the trajectory $\ell \mapsto (x(\ell) - x_{\text{ave}}(\ell)\mathbf{1})$. This matrix is symmetric, its singular values are 0 and the singular values of $\text{Trid}_{N-1}(a,b,a)$, and its eigenvectors are $\mathbf{1}$ and the eigenvectors of $\text{Trid}_{N-1}(a,b,a)$ (padded with an extra zero). These facts are sufficient to duplicate, step by step, the proof of fact (i) in Theorem B.3. Therefore, the trajectory $\ell \mapsto (x(\ell) - x_{\text{ave}}(\ell)\mathbf{1})$ satisfies the same properties as those stated in Theorem B.3(i). ■

We conclude this section with some useful bounds.

Lemma B.5 *Assume $x \in \mathbb{R}^N$, $y \in \mathbb{R}^{N-1}$ and $z \in \mathbb{R}^{N-1}$ jointly satisfy*

$$x = P_+ \begin{bmatrix} 0 \\ y \end{bmatrix}, \qquad x = P_- \begin{bmatrix} 0 \\ z \end{bmatrix}.$$

Then $\frac{1}{2}\|x\|_2 \le \|y\|_2 \le (N-1)\|x\|_2$ and $\frac{1}{2}\|x\|_2 \le \|z\|_2 \le (N-1)\|x\|_2$. •

Proof: The proof is based on the coordinate expressions:

$$x_1 = y_1,\ x_2 = y_2 - y_1,\ \dots\ x_{N-1} = y_{N-1} - y_{N-2},\ x_N = -y_{N-1},$$
$$y_1 = x_1,\ y_2 = x_2 + x_1,\ y_3 = x_3 + x_2 + x_1,\ \dots\ y_{N-1} = x_{N-1} + \cdots + x_1,$$

and

$$x_1 = z_1,\ x_2 = z_2 + z_1,\ \dots\ x_{N-1} = z_{N-1} + z_{N-2},\ x_N = z_{N-1},$$
$$z_1 = x_1,\ z_2 = x_2 - x_1,\ z_3 = x_3 - x_2 + x_1,\ \dots\ z_{N-1} = x_{N-1} + \cdots + (-1)^{N-1}x_1.$$

■