Controlled Symmetries and Passive Walking

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Index Terms

Bipedal Locomotion, Passive Walking, Group Action, Symmetry, Limit Cycle, Potential Energy Shaping, Nonlinear Control.

Abstract

In this paper we investigate the relationship between nonlinear control and passive walking in bipedal locomotion for the general case of an n degree-of-freedom biped in three dimensional space. We introduce the notion of *Controlled Symmetry* to capture the effect of the control input on the invariance of the system Lagrangian under group action. We then show the existence of a controlled symmetry for general bipeds under the action of $SO(3)$ taking into account not only the kinetic energy but also the potential energy and impact dynamics. We use this result to show the existence of a nonlinear control law that reproduces so-called passive gaits independent of the particular ground slope. Our contribution in this paper is two-fold. First, our result contains the first rigorous proof of the existence of so-called passivity mimicking control laws that explicitly accounts for the impact dynamics. Second, whereas previous papers have studied only planar bipeds with and without knees, our result is completely general.

Our results can be viewed as direct extensions of several previous results, such as *passivity based control* [1], [2], *virtual gravity* [3] and *virtual passive dynamic walking* [4] from the planar case to general n-DOF robots in three dimensional space.

I. Introduction

The notion that it is possible to achieve walking gaits from mechanical bipeds powered only by gravity has intrigued robotics researchers since the pioneering work of McGeer more than

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a decade ago; see [5]. These so-called passive gaits may help to explain the efficiency of human and animal locomotion and provide insight into the development of walking robots. Several researchers have studied passive walking in planar mechanisms, with and without knees and analyzed their passive gaits [5]–[8]. The stable passive gaits found in these mechanisms typically exist only for very shallow slopes and exhibit extreme sensitivity to slope magnitude. For example, the compass gait biped studied in [8] exhibits period doubling bifurcations leading to chaos as the ground slope is changed from about 3° to about 5°.

The first results in active feedback control that exploit passive walking for planar bipeds appeared in [1], [2], [6], and later in [9] and [10]. Passive walking in three-dimensions was investigated by Kuo in [11]. Passive limit cycles were found in the lateral plane as well as in the sagittal plane. However, the lateral motion was unstable and had to be compensated by feedback control. More recently, true three dimensional passive walking has been achieved by Collins, et. al. in [12]. This remarkable biped has specially shaped feet to stabilize passively the lateral motion. It also has arms, whose motion is coupled to the leg motion in order to stabilize the yaw dynamics. The resulting gait is surprisingly anthropomorphic.

Motivated in part by the above work showing that passive walking can be achieved in three dimensions, we consider the general case of a three dimensional n degrees-of-freedom biped. We show that changing the ground slope defines a group action on the configuration manifold of the system and that both the kinetic energy and impact dynamics are invariant under this group action. Hence, to achieve invariance of the passive limit cycles, one need only compensate the potential energy of the system. We therefore introduce a potential energy shaping controller that ensures the closed-loop system is invariant under the slope-changing action. We refer to this as a *Controlled Symmetry* since the Lagrangian of the open-loop system is not invariant under this group action.

The idea of potential energy shaping in robotics goes back to the early work of Takegaki and Arimoto [13] and Koditschek [14]. More recently, potential energy shaping has been used in other classes of mechanical systems, for example in [15] and [16]. While the control algorithm that we derive in this paper is ultimately a potential energy shaping control of the type considered in these and other works, there are important differences especially with respect to the analysis methods used. First, we do not seek to stabilize an equilibrium configuration or relative equilibria but rather to create a stable limit cycle. Second, the analysis of walking must take into account the impact dynamics, which result in discontinuous changes in energy and which means that the system is fundamentally a hybrid dynamical system. Our analysis showing that the impact dynamics are invariant with respect to changes in the ground slope is an important part of the result, which distinguishes this work from previous work on biped control.

II. Mathematical Background

We first review some basic background from differential geometry. For more details we refer the reader to [17], [18].

A. Group Actions, Invariance and Equivariance

Definition 2.1: Let Q be a smooth manifold and G be a Lie group. A *left action* of G on Q is a map $\Phi\colon G \times Q \to Q$ taking a pair (g, q) to $\Phi(g, q) = \Phi_g(q) \in Q$ and satisfying for all $q \in Q$

(i) $\Phi_e(q) = q$, where e is the identity element of G, and

(ii) $\Phi_{g_1}(\Phi_{g_2}(q)) = \Phi_{g_1 g_2}(q)$.

Definition 2.2: Let T_qQ be the linear space of tangent vectors at $q \in Q$, and let $TQ = \bigcup_q T_qQ$ be the tangent bundle of Q. If Φ is a group action on Q, we let $T_q\Phi_g$ denote the tangent function to Φ_g mapping T_qQ onto $T_{\Phi_g(q)}Q$. This defines a mapping $T\Phi : G \times TQ \to TQ$ which is called the *Lifted Action*.

A group action on Q thus induces, in a natural way, a corresponding action on TQ. Likewise a group action induces corresponding maps on other quantities, such as scalar functions over Q, one-forms, and covector fields. Such induced maps are important in order to determine how vector fields and their associated flows are affected by the group action.

Definition 2.3: Let $F : M \to N$ be a smooth mapping between manifolds M and N and let $\Phi : G \times M \to M$ be an action of the Lie Group G on M. Then we say that

(i) F is *Invariant* under the group action if $F \circ \Phi = F$, i.e., if, for all $g \in G$ and $m \in M$

$$(F \circ \Phi_g)(m) = F(m).$$

(ii) F is *Equivariant* if there exists an associated group action $\tilde{\Phi} : G \times N \to N$ such that $F \circ \Phi = \tilde{\Phi} \circ F$ in the sense that for all $g \in G$ there exists $\tilde{g} \in \tilde{G}$ such that

$$(F \circ \Phi_g)(m) = (\tilde{\Phi}_{\tilde{g}} \circ F)(m) \text{ for all } m \in M$$

Invariance is then seen to be a special case of equivariance corresponding to the choice $\tilde{\Phi} = I$, the identity transformation. For a vector field X, considered as a mapping from $Q \to TQ$, equivariance means that for all $g \in G$ and $q \in Q$

$$X(\Phi_g(q)) = T_q\Phi_g(X(q)). \tag{1}$$

We note that it is more common to refer to such a vector field as *invariant*, since Equation (1) can be equivalently expressed as

$$\Phi_g^* X = X.$$

where Φ_g^* denotes the pullback map. Henceforth we will use the term invariant when referring to group actions on vector fields. Similarly, a covector field α on Q is invariant if, for all $g \in G$ and $q \in Q$

$$\alpha(\Phi_g(q)) = T_q^*\Phi_g(\alpha(q)).$$

In addition, one can show that if a function $h \colon Q \to \mathbb{R}$ is invariant (respectively, equivariant), then so is its differential dh [18]. The importance of these definitions for us is the following result.

Lemma 2.4: Let the vector field X be Φ-invariant, and let $\gamma : [0, T] \to Q$ be an integral curve of X, i.e., the solution of the differential equation defined by X with initial condition $\gamma(0)$. Then, for all $g \in G$, the map $\Phi_g \circ \gamma : [0, T] \to Q$ is an integral curve for X.

III. Dynamics of Bipedal Locomotion

The act of walking involves both a swing phase and a stance phase for each leg as well as impacts between the swing leg and ground, and possibly "internal" impacts, such as a knee-strike, which are due to mechanical constraints on the joints.

Consider an n degree-of-freedom biped during the single-support phase as shown in Figure 1. Each joint of the robot is assumed to be revolute and to allow a single degree-of-freedom rotation. Multi-degree-of-freedom joints, such as ball and socket joints, can be represented as multiple single degree of freedom joints with zero link lengths in between. The stance leg, which is in contact with the ground, has three degrees-of-freedom relative to an inertial frame (assuming no slipping). We can therefore use $Q = \mathrm{SO}(3) \times \mathbb{T}^{n-3}$ to represent the configuration space of the biped, where $\mathrm{SO}(3)$ is the Rotation Group in $\mathbb{R}^3$ and $\mathbb{T}^{n-3}$ is the $(n-3)$-torus.

In what follows, we shall consider the standard coordinate chart on $\mathbb{T}$ that identifies points with angles in the interval $[0, 2\pi)$. A configuration is then characterized by an ordered pair $q = (R, r)$ where $R \in \mathrm{SO}(3)$ is the orientation of the first link, and $r \in \mathbb{T}^{n-3}$ is the shape of the multi-body chain, for example the angle of each joint referenced to the previous joint. Given a

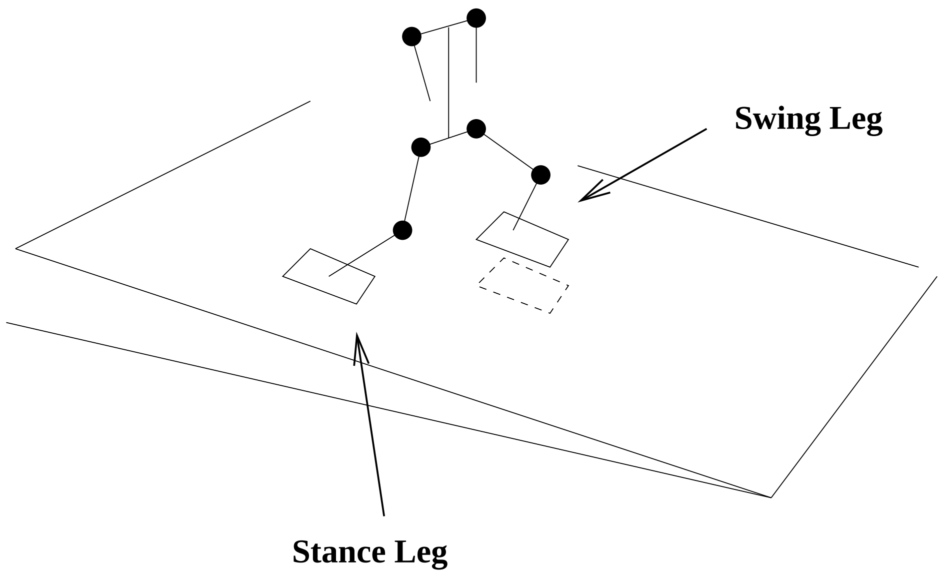

Fig. 1. A General 3-D Biped in the single-support phase showing the stance leg (right leg) and swing leg (left leg).

configuration, $q = (R, r) \in \mathrm{SO}(3) \times \mathbb{T}^{n-3}$, we represent a velocity vector in T_qQ via the pair $(R^{-1}\dot{R}, \dot{r}) \in \mathfrak{so}(3) \times \mathbb{R}^3$, where $\mathfrak{so}(3)$ is the Lie Algebra of 3×3 skew-symmetric matrices.

The advantage of this formalism is that only the first degree-of-freedom is referenced to an absolute or world frame. The remaining joint variables, called the *shape variables*, are then invariant under a change of basis of the world frame. Configuration spaces that can be written as the Cartesian product of a Lie group and a shape space are referred to as principal bundles; see [17].

Remark 3.1: In the case of an n degrees-of-freedom planar mechanism, $Q = \mathrm{SO}(2) \times S$ and we may identify Q with $\mathbb{T}^n$ since elements of $\mathrm{SO}(2)$ can be represented by scalars (angles). In the case of a serial link mechanism we may again identify Q with $\mathbb{T}^n$ using the familiar Denavit-Hartenberg variables to define the configuration q.

A. Lagrangian Dynamics

In order to write the equations of motion for the walking machine during the single-support phase, we introduce a parametrization of the configuration space $\mathrm{SO}(3) \times \mathbb{T}^{n-3}$ which is equivalent to $q = (R, r)$ but *minimal,* in the sense that only n coordinates are required. For example, we shall let $(q^1, \ldots, q^n)$ be a coordinate chart where (q^1, q^2, q^3) are Euler angles for $\mathrm{SO}(3)$ and $(q^4, \ldots, q^n)$ are angles in $[0, 2\pi)$ for $\mathbb{T}^{n-3}$. Accordingly, we can write the Euler-Lagrange equations of motion as

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = \sum_{j=1}^{n} B_{i,j}(q), u_j, \qquad i = 1, \ldots, n$$

where $\mathcal{L}(q, \dot{q}) = \mathcal{K}(q, \dot{q}) - \mathcal{V}(q)$ is the difference of the kinetic energy $\mathcal{K}\colon TQ \to \mathbb{R}$ and the potential energy due to gravity, $\mathcal{V}\colon Q \to \mathbb{R}$, $B_{i,j}$ is the i-th component of the j-th force which has magnitude u_j. If we express the kinetic energy in the usual fashion as $\mathcal{K}(q, \dot{q}) = \frac{1}{2}\dot{q}M(q)\dot{q}$, where $M(q)$ is the symmetric, positive definite $n \times n$ inertia matrix, the controlled Euler-Lagrange equations can be written in matrix form as [19]

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = B(q)u, \tag{2}$$

where $\dot{M} - 2C$ is skew symmetric and $g(q) = d\mathcal{V}(q)$ is the vector of gravitational torques. We assume the walking biped is fully actuated so that the $n \times n$ matrix B in (2) is full rank for all q.

B. Impact Dynamics

Impacts arise in two ways: from the foot/ground contact and from internal constraints such as mechanical stops designed to prevent hyperextension of the knees. For space reasons, we analyze here only the impacts resulting from the foot/ground contact. Since the impacts resulting from internal constraints are dependent only on the shape variables and their velocities, it follows immediately that they are independent of the ground slope and we will omit the details. With regard to the foot/ground impact, we make the standard assumptions, namely,

(i) impacts are perfectly inelastic (no bounce),

(ii) transfer of support between swing and stance legs is instantaneous, i.e. the double support phase is negligible,

(iii) there is no slipping at the stance leg ground contact.

Under these assumptions each impact results in an instantaneous jump in velocities, hence a discontinuity in kinetic energy, whereas the position variables are continuous through the impact; see [20].

Let $h\colon Q \to \mathbb{R}$ be the smooth function defining the foot height and assume that foot/ground impacts take place precisely when $h(q) = 0$. Assuming an impact has taken place at q_0, the foot/ground contact imposes a number of holonomic constraints on the translational and, possibly, rotational motion of the foot. There constraints can be written as $h_{\text{foot}}(q) = h_{\text{foot}}(q_0)$ for an appropriate function $h_{\text{foot}}\colon Q \to \mathbb{R}^\nu$. For bipeds with point foot contact, the dimension ν is two in the planar $2D$ case and three in the general $3D$ case. For bipeds with extended feet, ν is three for planar bipeds and six in the most general case. The two functions h and h_{foot} can be computed using the forward kinematic equations of the robot.

The change in velocity at impact is found by integrating the Euler-Lagrange equations over the (infinitesimally small) duration of the impact:

$$\left.\frac{\partial \mathcal{L}}{\partial \dot{q}}\right|_{t^-}^{t^+} = \int_{t^-}^{t^+} F(q,t)dt \tag{3}$$

where $F(q,t)$ represents the contact force over the impact event $[t^-, t^+]$. Because

$$\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{K}}{\partial \dot{q}} = M(q)\dot{q}$$

we conclude $\dot{q}(t^+) - \dot{q}(t^-) = M(q)^{-1} \int_{t^-}^{t^+} F(q,t)dt$. Second, we note that the contact force F is aligned with the constraint directions $d(h_{\text{foot}})_1, \dots, d(h_{\text{foot}})_\nu$ so that there exist[1] a function $f(t) = (f_1(t), \dots, f_\nu(t))$ such that $F(q,t) = \sum_{i=1}^{\nu} f_i(t) d(h_{\text{foot}})_i(q)$. Thus

$$\dot{q}(t^-) = \dot{q}(t^+) - M(q)^{-1} \int_{t^-}^{t^+} \left(\sum_{i=1}^{\nu} f_i(t) d(h_{\text{foot}})_i(q) \right) dt. \tag{4}$$

Third, after the impact the quantities $t \mapsto (h_{\text{foot}})_i(q(t))$ are constant and therefore

$$0 = \frac{d}{dt}(h_{\text{foot}})_i(q(t)) = d(h_{\text{foot}})_i \cdot \dot{q}(t^+), \qquad \text{for } i = 1, \dots, \nu.$$

The geometric interpretation of this fact is that $\dot{q}(t^+)$ is perpendicular with respect to the M-inner product to the vectors $M(q)^{-1} d(h_{\text{foot}})_i$. This fact shows that the right-hand side of equation (4) is an orthogonal sum and therefore $\dot{q}(t^+)$ equals the $M(q)$-orthogonal projection of $\dot{q}(t^-)$ onto the feasible space $\{v \in T_q Q |\ d(h_{\text{foot}})_i(q) \cdot v = 0,\ i = 1, \dots, \nu\}$. We refer the reader to [21] for

[1] This is a standard fact from constrained Lagrangian systems [17].

more details on this characterization of ideal impact dynamics. In summary, the impact dynamics may be represented as

$$\dot{q}(t^+) = P_q(\dot{q}(t^-)), \tag{5}$$

where the *plastic projection* P_q for the impact occurring at $h(q) = 0$ is the $M(q)$-orthogonal projection of $\dot{q}(t^-)$ onto $\{v \in T_qQ|\ d(h_{\text{foot}})_i(q) \cdot v = 0,\ i = 1, \ldots, \nu\}$. Putting these previous notions together leads to a hybrid dynamical system

$$\begin{array}{rcll} \dfrac{d}{dt}\dfrac{\partial \mathcal{L}}{\partial \dot{q}} - \dfrac{\partial \mathcal{L}}{\partial q} & = & B(q)u, & \text{for } h(q) \neq 0 \\ q(t^+) & = & q(t^-) & \\ & & & \text{for } h(q) = 0 \\ \dot{q}(t^+) & = & P_q(\dot{q}(t^-)) & \end{array} \tag{6}$$

IV. Main Results

A. Slope Changing Symmetry

Let us now consider the effect of symmetries on Lagrangian dynamics. Let $\Phi : G \times Q \to Q$ be a group action and suppose that for all $g \in G$

$$\mathcal{L}(q, \dot{q}) = \mathcal{L}(\Phi_g(q), T_q\Phi_g(\dot{q})) \tag{7}$$

i.e., the Lagrangian is invariant under the group action Φ. Such a Lagrangian system is said to possess a *Symmetry* with respect to Φ.

A consequence of symmetry of the Lagrangian is that the vector field X associated with the Lagrangian dynamics is invariant with respect to Φ and hence its integral curves, i.e. solutions of the Euler-Lagrange equations of motion are preserved according to Lemma (2.4). See [17].

We are interested in deriving control laws that preserve or create symmetries with respect to group actions. For this reason we introduce the notion of *Controlled Symmetry* as follows.

Definition 4.1: The Lagrangian system (2) is said to possess a *Controlled Symmetry* with respect to a group action Φ if, for each $g \in G$, there exists a control input $u = u_g(q, \dot{q})$, which depends on g, such that

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} - B(q)u_g = \frac{d}{dt}\frac{\partial \mathcal{L}_g}{\partial \dot{q}} - \frac{\partial \mathcal{L}_g}{\partial q} \tag{8}$$

where $\mathcal{L}_g(q, \dot{q}) = \mathcal{L}(\Phi_g(q), T_q\Phi_g(\dot{q}))$.

It immediately follows that there is a one-one correspondence between solutions of the closed loop dynamics

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} - \frac{\partial \mathcal{L}}{\partial q} = B(q)u_g \tag{9}$$

and solutions of

$$\frac{d}{dt}\frac{\partial \mathcal{L}_g}{\partial \dot{q}} - \frac{\partial \mathcal{L}_g}{\partial q} = 0. \tag{10}$$

We show next how to create a controlled symmetry with $G = SO(3)$ representing changing ground slopes. Including the analysis of the impact dynamics will complete the main result. Let $\Sigma = \{O, \{e_1, e_2, e_3\}\}$ be an inertial reference frame. Assume the point O is fixed on the ground and assume the ground is defined by a plane in $\mathbb{R}^3$. Given the coordinates $x \in \mathbb{R}^3$ of a point on the ground, changing the ground slope is an $\mathrm{SO}(3)$-group action $(A, x) \mapsto Ax$. Assuming that the contact point for the stance leg is at the origin O of Σ, we define a corresponding action Φ of $SO(3)$ on the configuration space $Q = \mathrm{SO}(3) \times \mathbb{T}^{n-3}$ that maps $(A, q) = (A, (R, r)) \in \mathrm{SO}(3) \times Q$ into Q by

$$\Phi(A, (R, r)) = \Phi_A(R, r) = (AR, r)\,. \tag{11}$$

This group action is illustrated in Figure 2.

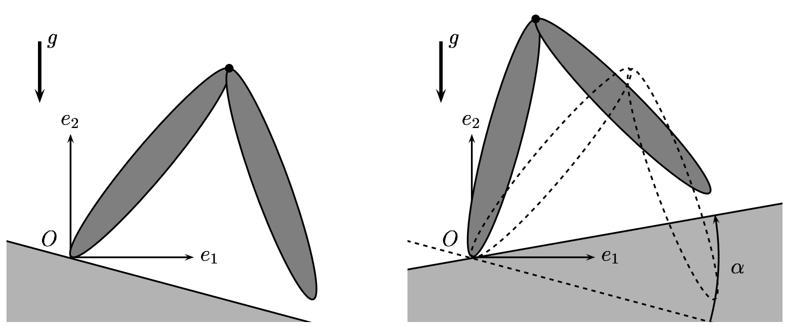

Fig. 2. A planar illustration of the slope-changing action: slope and walking biped are affected, whereas the inertial reference frame and gravity remain unchanged.

Next, let us write coordinate expressions for the lifted action $T\Phi\colon \mathrm{SO}(3) \times TQ \to TQ$. Given $A \in SO(3)$, one can easily see that $(AR)^{-1}(\dot{AR}) = R^{-1}\dot{R}$. Therefore, for all $\dot{q} = (R^{-1}\dot{R}, \dot{r})$ the lifted action satisfies

$$T\Phi_A(q, \dot{q}) = (\Phi_A(q), T_q\Phi_A(\dot{q})) = (\Phi_A(q), \dot{q}). \tag{12}$$

Putting these ideas together, we can state the following proposition.

Proposition 4.2: The kinetic energy $\mathcal{K}\colon TQ \to \mathbb{R}$ is invariant under the lifted slope changing action, that is, for all $A \in \mathrm{SO}(3)$,

$$\mathcal{K} \circ T\Phi_A = \mathcal{K}. \tag{13}$$

In terms of the generalized coordinates $(q, \dot{q})$, this means $\mathcal{K}(q, \dot{q}) = \mathcal{K}(\Phi_A(q), \dot{q})$.

Proof: The kinetic energy $\mathcal{K}$ of a single rigid body with center of mass at position $P \in \mathbb{R}^3$ is the sum of its translational and rotational kinetic energies as

$$\mathcal{K} = \frac{1}{2} m \dot{P}^T \dot{P} + \frac{1}{2} \omega^T I \omega \tag{14}$$

where $\omega \in \mathbb{R}^3$ is the body angular velocity, and I is the inertia matrix. It is easily shown that both inner products $\dot{P}^T \dot{P}$ and $\omega^T I \omega$ are independent of the world coordinate system, i.e., are invariant under a rotation of the world frame (see [19] for the details). In the general case of an n degrees-of-freedom biped with configuration $q = (R, r)$, only the first degree-of-freedom is referenced to the world frame. Since its kinetic energy is invariant under rotations of the world frame, it follows that the kinetic energy of the entire system is invariant. ■

B. Equivariance of the kinematics and impacts

Let

$$f(q) = \begin{bmatrix} f_p(q) \\ f_O(q) \end{bmatrix}$$

be the forward kinematics map [19] that associates to each configuration $q \in Q$ of the biped the position and orientation of the tip of the swing leg. In most cases the map $h(q)$ defining the foot/ground contact will be one or more components of $f(q)$. It follows, for $A \in SO(3)$, that

$$A f_p(q) = A f_p(R, r) = f_p(AR, r) = f_p(\Phi_A(q)). \tag{15}$$

hence the forward position kinematics map f_p is equivariant with respect to $SO(3)$. A similar relation can be shown for the forward orientation kinematics but is omitted here for space reasons. An important consequence is that the function $h(q)$ defining the foot/ground impact constraint, that we used previously to determine the impact equations, is likewise equivariant with respect to $SO(3)$. Because the forward kinematic map as well as the ground surface are equivariant, the distance of the swing leg to the ground is *invariant*. This concept is illustrated in Figure 3.

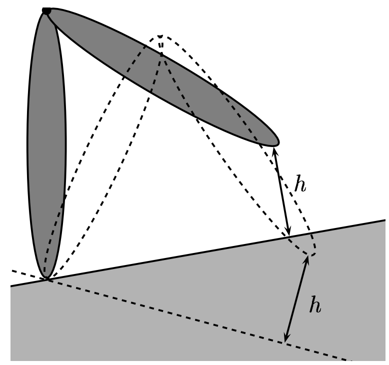

Fig. 3. Illustrating invariance of the distance of the tip of the swing leg to the ground under the slope-changing action.

Next, recall that the *plastic projection* P_q for the impact $h(q) = 0$ is the orthogonal projection with respect to M onto $\{v \in T_qQ| \; dh_i(q) \cdot v = 0 \;\; i = 1, \ldots, \nu\}$. Because h has been shown to be equivariant with respect to $SO(3)$ and M was previously shown to be invariant under this action, we can immediately state

Lemma 4.3: The projection operator P_q defining the velocity change at impact is equivariant with respect to $SO(3)$, i.e.

$$T\Phi_A \left(P_q(v)\right) = P_{\Phi_A(q)} \left(T\Phi_A(v)\right),$$

for all $v \in T_qQ$.

As a consequence we have

Corollary 4.4: The velocity change, $\dot{q}^+ - \dot{q}^-$, due to the ideal foot/ground impact is invariant under the above slope changing action.

C. Potential Energy Shaping

We consider the mechanical control system governed by the controlled Euler-Lagrange equations (2) and by the plastic impact dynamics in equation (5). We have shown invariance of the kinetic energy and equivariance of the impact dynamics with respect to the action of $SO(3)$. Therefore, in order to generate a controlled symmetry and preserve the solutions of the system under impacts, we need only compensate the potential energy. Our main result is thus expressed as

Theorem 4.5: Let $\eta : [0, T] \to Q$ be a solution trajectory to equation (2) at $u = 0$ undergoing impacts according to equation (5). Let $A \in \mathrm{SO}(3)$ and define

$$u_A(q) = B^{-1}(q)\frac{\partial}{\partial q}\Big(\mathcal{V}(q) - \mathcal{V}(\Phi_A(q))\Big). \tag{16}$$

Then the trajectory $\Phi_A \circ \eta : [0, T] \to Q$ is a solution for the closed-loop system, that is, for the controlled walking machine characterized by (2) and (5).

Proof: Substituting the control law (16) into (2) and using invariance of the kinetic energy under the group action we know that during the smooth evolution (single-support phase)

$$\frac{d}{dt}\frac{\partial \mathcal{L}_A}{\partial \dot{q}} - \frac{\partial \mathcal{L}_A}{\partial q} = 0 \tag{17}$$

where $\mathcal{L}_A(q, \dot{q}) = \mathcal{L}(\Phi_A(q), T_q\Phi_A(\dot{q}))$. Thus, if η is a solution of (2) with $u = 0$ in the absence of impacts, then $\Phi_A \circ \eta$ is a solution of (17) in the absence of impacts. Furthermore, the impact dynamics (5) being equivariant implies that $\Phi_A \circ \eta$ is a solution for the closed-loop system even through impacts. ■

In particular, Theorem 4.5 tells us that any limit cycle that exists for the passive walker for one ground slope can be reproduced by the active control law (16) for any other ground slope. Also, if $(q_0, \dot{q}_0)$ lies in the basin of attraction of the passive limit cycle, then $(\Phi_A(q_0), T_{q_0}\Phi_A(\dot{q}_0))$ lies in the basin of attraction of the closed loop system. Thus, we are able to determine the appropriate initial conditions on any slope given one initial condition that leads to a passive gait on one particular slope.

V. Example and Discussion

Space limitations preclude the inclusion of detailed simulation results. We present here some simulations of the compass gait biped from [6], whose dynamics are sufficiently well known that we omit them here. The compass gait biped studied is equivalent to a double pendulum with point masses concentrated at the hip and legs. The compass gait biped of [6] exhibits a passive limit cycle for a three degree ground slope. Figure 4 shows limit cycles generated using the above potential energy shaping control strategy on level ground and on slopes of ± 10-degrees, for which no passive limit cycles exist. As expected, the limit cycles are shifted in position but otherwise identical.

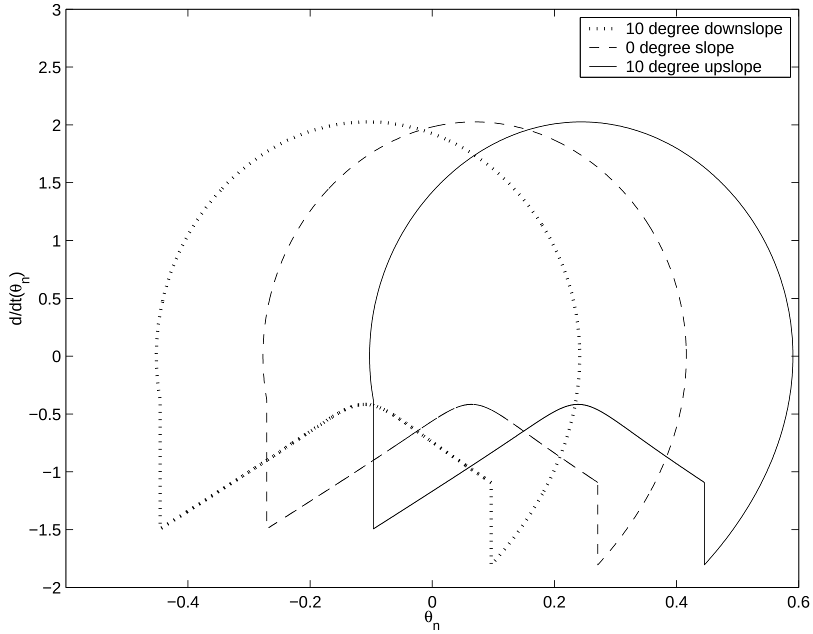

Fig. 4. Limit Cycle (velocity vs. position for one leg) for three distinct slopes - level ground, $+10$-degrees and -10-degrees.

A. Practical Considerations and Conclusions

This paper shows how active feedback control can completely remove the sensitivity of passive limit cycles to ground slope. As with all theoretical results, practical implementation on real bipeds requires consideration of several factors, such as friction, actuator saturation, sensing of the slope angle, parametric uncertainty, and a host of other effects. Some of these effects, for example saturation, will reduce the range of slopes for which our control can be used. We discuss here only the practical considerations that arise from the fact that the foot/ground constraint is unilateral, i.e. the foot can push but not pull on the ground, namely constraints on the magnitude and direction of the ground reaction forces and constraints on the Zero Moment Point (ZMP)[2].

First, the reaction force normal to the slope should be always directed downward in order to maintain the foot/ground contact. Figure 5 shows the ground reaction forces for a ten degree slope showing that the normal force applied to the ground is positive and bounded away from zero. The forces are computed using the recursive Newton-Euler formula from [19]. Also, the ground reaction force tangential to the slope is balanced by the friction force. Insufficient friction will cause the foot to slip. Figure 5 also shows the tangential foot/ground force. These constraints cannot be guaranteed a priori for an arbitrary ground surface and slope, but can be checked via

[2] The Zero Moment Point is commonly called the Center-of-Pressure (CoP) in the biomechanics literature

simulation to determine a possible range of allowable slopes given the material properties of the foot/ground contact.

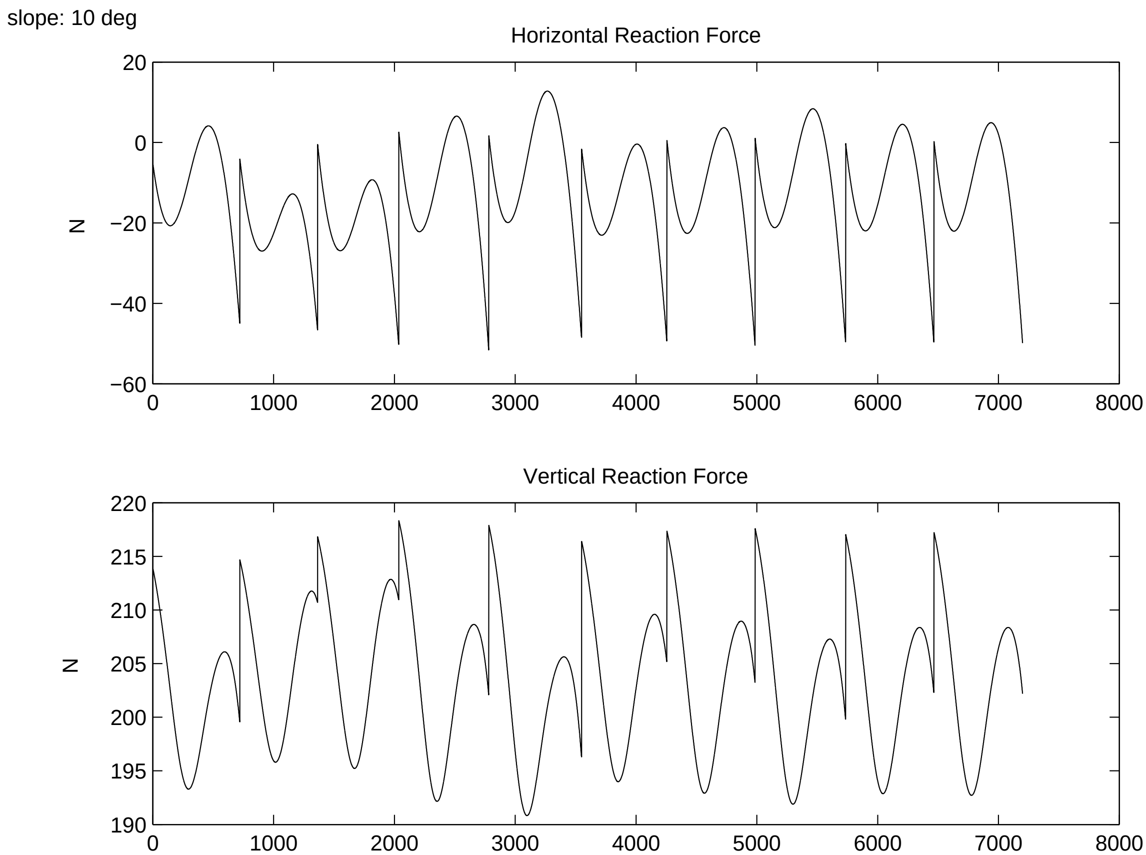

Fig. 5. Ground Reaction Forces and Joint Torques for a 10-degree Slope

The second constraint deals with the Zero Moment Point (ZMP), which is the resultant of the normal forces acting on the foot during contact with the ground. If the ZMP reaches the edge of the foot support polygon during the step, the foot will begin to rotate off the ground before the swing leg impacts (see [22] for details). Such foot rotation may or may not be part of the passive limit cycle. However, this phenomenon obviously depends on the size and shape of the foot and is therefore outside the scope of the present article.

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