On quantized control and geometric optimization

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***Abstract*— This paper studies state quantization schemes for feedback stabilization of linear control systems with limited information. The focus is on designing the least destabilizing quantizer subject to a given information constraint. We explore several ways of measuring the destabilizing effect of a quantizer on the closed-loop system, including (but not limited to) the worst-case quantization error. In each case, we show how quantizer design can be naturally reduced to a version of the so-called multicenter problem from locational optimization. Algorithms for obtaining solutions to such problems, all in terms of suitable Voronoi quantizers, are discussed. In particular, an iterative solver is developed for a novel weighted multicenter problem which most accurately represents the least destabilizing quantizer design.**

I. Introduction

In this paper we study linear control systems whose state variables are quantized. We think of a quantizer as a device that converts a real-valued signal into a piecewise constant one taking a finite set of values. The recent papers [2], [9], [12] discuss various situations where this type of quantization arises and provide references to the literature. Mathematically, a quantizer can be described by a piecewise constant function $q : \mathcal{D} \subset \mathbb{R}^n \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite subset of $\mathbb{R}^n$ with a fixed number of elements N. Here n is the state dimension of a given system and $\mathcal{D}$ is a domain of interest in the state space. We denote the elements of $\mathcal{Q}$ by $q_1, \ldots, q_N$ and refer to them as *quantization points*. The sets $W_i := \mathrm{cl}\{x \in \mathcal{D} : q(x) = q_i\}$, $i \in \{1, \ldots, N\}$ associated with fixed values of the quantizer form a partition of the domain $\mathcal{D}$ and are called *quantization regions* (cl denotes closure). We will sometimes identify a quantizer q with the corresponding pair $(\mathcal{Q}, \mathcal{W})$, where $\mathcal{W} := \{W_1, \ldots, W_N\}$.

In the literature it is usually assumed that quantization regions are fixed in advance and have specific shapes, most often rectilinear. Here we are interested in the situation where the number N of quantizer values is a given information constraint, but the control designer has flexibility in choosing a specific configuration of quantization regions (whose shapes can in principle be arbitrary) and quantization points. While there has been some research on systems with quantization regions of arbitrary shapes [14], [13] and on the relationship between the choice of quantization regions and the behavior of the closed-loop system [9], [12], the general problem of determining the "best" quantizer for a particular control task such as feedback stabilization remains largely open.

A feedback law which globally asymptotically stabilizes a given system in the absence of quantization will in general fail to provide global asymptotic stability of the closed-loop system that arises in the presence of state quantization. In Section II we explain how the destabilizing effect of a given quantizer can be measured. We introduce the concept of a *destabilization measure* which, in conjunction with an arbitrary stabilizing feedback law and a corresponding Lyapunov function, can be used to determine an ultimate bound on solutions. One example of such a destabilization measure is the *worst-case quantization error* $\max_{x \in \mathcal{D}} |q(x) - x|$. However, it turns out that there exist other destabilization measures which are actually more suitable in the present context. Although the parameters of the control system are used in the stability analysis, the destabilization measure itself is a function of the quantization regions and quantization points only. The quantizer design problem then naturally reduces to an optimization problem which consists in minimizing such a measure over all quantizers satisfying the information constraint. We describe this procedure for three different types of quantizers arising from uniform, radial and spherical, and radially weighted quantization.

After casting quantizer design as an optimization problem, we proceed to explain how techniques from *optimal facility location* (or *locational optimization*) yield new insights into this problem as well as efficient algorithms for solving it. Facility location problems concern the location of a fixed number of facilities that provide service demanded by users; the objective is to minimize the average or maximal distance from sets of demand points to facilities. We focus here on settings continuous in the location of both the facilities and the demand points (i.e., both facilities and demand points take values in a continuum of points, such as a polytope or an ellipsoid). Facility location problems are surveyed in [7]. Relevant background on computational geometric methods in locational optimization is provided in Section III.

We will find that the problem relevant for our purposes is the *multicenter problem*, discussed in [17], [16]. It consists in choosing a collection of N points $q_1, q_2, \ldots, q_N$ in a bounded region $\mathcal{D} \subset \mathbb{R}^n$ so as to minimize the quantity $\max_{x \in \mathcal{D}} \min_{i \in \{1,\ldots,N\}} |q_i - x|$; it can also be stated as the problem of covering a given region with overlapping balls of minimal radius. The connection between the quantized control problem and the multicenter problem, although very natural, apparently has not been pursued before. In Section III we present a novel general formulation of the multicenter problem with weighting factors. We then discuss solutions of specific versions of this problem corresponding to the three types of quantization considered in Section II, all in terms of suitable Voronoi quantizers. We show how existing algorithms can handle the first two approaches, and then develop a new algorithm for the last one which gives less conservative results.

Simulation results (as well as existing studies of the related multimedian problem, such as [10]) indicate that by solving the quantized feedback stabilization problem with the help of locational optimization techniques, one may obtain quite interesting quantization patterns. For the multicenter problem in the plane, for example, a typical Voronoi region is a hexagon. Consequently, hexagonal quantization regions are capable of achieving better performance for planar systems than more traditional rectangular ones. Simulations results, as well as complete proofs, are omitted here for brevity; they can be found in [3].

II. Quantization and stability

A. Worst-case quantization error

Consider the linear system

$$\dot{x} = Ax + Bu, \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{1}$$

Suppose that it is stabilizable, so that for some matrix K the eigenvalues of $A + BK$ have negative real parts. Then there exists a unique positive definite symmetric matrix P such that

$$(A + BK)^T P + P(A + BK) = -I.$$

We let $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ denote the smallest and the largest eigenvalue of P, respectively. We denote the Euclidean norm by $|\cdot|$ and the corresponding induced matrix norm by $\|\cdot\|$.

The quantized state feedback control law

$$u = Kq(x)$$

yields the closed-loop system

$$\dot{x} = Ax + BKq(x) = (A + BK)x + BKe \tag{2}$$

where $e := q(x) - x$ represents the quantization error. The derivative of the function $V(x) := x^T Px$ along solutions of the system (2) satisfies

$$\dot{V} = -x^T x + 2x^T PBKe \le -|x|^2 + 2|x||PBKe|. \tag{3}$$

For an arbitrary small $\varepsilon > 0$, we have

$$|x| \ge 2(1 + \varepsilon)\|PBK\||e| \quad \Rightarrow \quad \dot{V} \le -\frac{\varepsilon}{1 + \varepsilon}|x|^2. \tag{4}$$

Pick a positive number M and consider the ball $\mathcal{B}_M := \{x \in \mathbb{R}^n : |x| \le M\}$. Define the worst-case quantization error

$$\Delta := \max_{x \in \mathcal{B}_M} |e|. \tag{5}$$

Consider the ellipsoids

$$\mathcal{R}_1 := \{x \in \mathbb{R}^n : x^T Px \le \lambda_{\min}(P)M^2\} \tag{6}$$

and

$$\mathcal{R}_2 := \{x \in \mathbb{R}^n : x^T Px \le \lambda_{\max}(P)4(1+\varepsilon)^2\|PBK\|^2\Delta^2\}. \tag{7}$$

The following is then straightforward to prove (see [13]).

Lemma 1 *Assume that*

$$\lambda_{\min}(P)M^2 > \lambda_{\max}(P)4(1 + \varepsilon)^2\|PBK\|^2\Delta^2. \tag{8}$$

Then the ellipsoids $\mathcal{R}_1$ and $\mathcal{R}_2$ defined by (6) *and* (7) *are invariant regions for the system* (2)*. Moreover, all solutions of* (2) *that start in the ellipsoid $\mathcal{R}_1$ enter the smaller ellipsoid $\mathcal{R}_2$ in finite time. An upper bound on this time is*

$$T = \frac{\lambda_{\min}(P)M^2 - \lambda_{\max}(P)4(1 + \varepsilon)^2\|PBK\|^2\Delta^2}{4\|PBK\|^2\Delta^2(1 + \varepsilon)\varepsilon}. \tag{9}$$

This lemma implies, in particular, that all solutions starting in $\mathcal{R}_1$ at time $t = t_0$ satisfy the ultimate bound

$$|x(t)| \le \sqrt{\frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}}2(1 + \varepsilon)\|PBK\|\Delta \qquad \forall t \ge t_0 + T$$

with T given by the formula (9). Decreasing ε to 0, we see that solutions (asymptotically) approach the ellipsoid

$$\{x \in \mathbb{R}^n : x^T Px \le \lambda_{\max}(P)4\|PBK\|^2\Delta^2\}.$$

We regard the quantity Δ defined by (5) as a *destabilization measure* of the quantizer q. It is not hard to see that if the number N of quantization regions is sufficiently large, then Δ can be made small enough for the inequality (8) to hold. Minimizing Δ—and consequently the size of the attracting invariant region $\mathcal{R}_2$—over all possible choices of the quantizer q corresponds to the following optimization problem:

$$\min_{\mathcal{Q},\,\mathcal{W}} \quad \max_{i \in \{1,\dots,N\}} \quad \max_{x \in W_i} \quad |q_i - x| \tag{10}$$

where $\mathcal{Q} = \{q_1, \dots, q_N\}$ is a set of quantization points and $\mathcal{W} = \{W_1, \dots, W_N\}$ is a partition of $\mathcal{B}_M$ into quantization regions. (We could work with partitions of $\mathcal{R}_1$ rather than $\mathcal{B}_M$, but this requires the knowledge of V.) The optimization problem (10) is known as the *multicenter problem* in computational geometry; we defer its discussion until Section III-A.

B. Radial and spherical quantization

In the above developments, the required bounds on the quantization error do not depend on the size of the state. This leads to uniform quantization, in the sense that quantization points are distributed uniformly over the region of interest. However, it is well known that more efficient quantization schemes are those which provide lower precision far away from the origin and higher precision close to the origin. Quantizers with a logarithmic scale are particularly useful; see [9]. Loosely speaking, with logarithmic quantization one has the same number of quantization points in the vicinity of every sphere centered at the origin in the state space, whereas with uniform quantization this number grows with the radius. This observation suggests introducing a “direct product” of one quantizer on a unit sphere and another along the radial direction, which is what we do next.

Let us write $x = |x|\text{vers}(x)$ where $\text{vers}(x) := x/|x|$. We represent the quantizer accordingly as

$$q(x) = q^r(|x|)q^s(\text{vers}(x)) \tag{11}$$

where q^r takes N_1 positive real values, q^s takes N_2 values on or inside the unit sphere, and N_1 and N_2 are some positive integers such that $N_1N_2 \leq N$. This means that we introduce two separate quantizers, one for $|x|$ and the other for $\text{vers}(x)$. The set of quantization points for the resulting overall quantizer q is formed by the N pairwise products of values of q^r and q^s.

From the triangle inequality and the fact that $|q^s(\text{vers}(x))| \leq 1$ for all x by construction, we obtain

$$|q(x)-x| \leq |x| \left(\left| \frac{q^r(|x|)}{|x|} - 1 \right| + \left| q^s(\text{vers}(x)) - \text{vers}(x) \right| \right).$$

Use (3) and the definition of e to write

$$\dot{V} \leq -|x|^2 \left(1 - 2\|PBK\| \frac{|q(x)-x|}{|x|} \right). \tag{12}$$

Take some $\varepsilon > 0$. Then we have $\dot{V} \leq -\varepsilon|x|^2$ whenever

$$|q^s(\text{vers}(x)) - \text{vers}(x)| < \frac{1-\varepsilon}{2\|PBK\|} \tag{13}$$

and

$$\left| \frac{q^r(|x|)}{|x|} - 1 \right| \leq \frac{1-\varepsilon}{2\|PBK\|} - |q^s(\text{vers}(x)) - \text{vers}(x)|. \tag{14}$$

In view of (13), we introduce the worst-case quantization error on the unit sphere corresponding to q^s:

$$\Delta_s := \max_{|x|=1} |q^s(x) - x|. \tag{15}$$

Pick a positive number M. To handle (14), we take q^r to be a logarithmic quantizer. Define

$$a := 1 - \frac{1-\varepsilon}{2\|PBK\|} + \Delta_s, \quad b := 1 + \frac{1-\varepsilon}{2\|PBK\|} - \Delta_s.$$

If (13) holds for all x on the unit sphere, then it can be shown that $0 < a < 1 < b$. Let

$$q^r(s) := \frac{a^i}{b^{i-1}}M \ \text{ for } \ s \in \left(\frac{a^i}{b^i}M, \frac{a^{i-1}}{b^{i-1}}M \right),\ i = 1, \dots, N_1 \tag{16}$$

and define q^r at the endpoints of the above intervals to make it continuous from the right or from the left. Then it is easy to check that (14) holds for all x such that $(a/b)^{N_1}M \leq |x| \leq M$. Consider the ellipsoid

$$\mathcal{R}_2 := \{x \in \mathbb{R}^n : x^TPx \leq \lambda_{\max}(P)(a/b)^{2N_1}M^2\}. \tag{17}$$

The earlier Lyapunov analysis leads to the following result.

Lemma 2 *Assume that*

$$\lambda_{\min}(P)M^2 > \lambda_{\max}(P)(a/b)^{2N_1}M^2 \tag{18}$$

and

$$\Delta_s < \frac{1-\varepsilon}{2\|PBK\|}. \tag{19}$$

Then the ellipsoids $\mathcal{R}_1$ and $\mathcal{R}_2$ defined by (6) *and* (17) *are invariant regions for the system* (2) *with the quantizer* (11)*. Moreover, all solutions of* (2) *that start in the ellipsoid $\mathcal{R}_1$ enter the smaller ellipsoid $\mathcal{R}_2$ in finite time. An upper bound on this time is*

$$T = \frac{\lambda_{\min}(P)M^2 - \lambda_{\max}(P)(a/b)^{2N_1}M^2}{(a/b)^{2N_1}M^2\varepsilon}. \tag{20}$$

For fixed N_1 and N_2, the quantity Δ_s defined by (15) provides a destabilization measure (for q^s). When K is given and Δ_s satisfies the inequality (19) for some $\varepsilon > 0$, we can construct q^r via (16) and compute an ultimate bound on solutions using Lemma 2. Minimizing Δ_s corresponds to the following optimization problem:

$$\min_{\mathcal{Q}^s,\, \mathcal{W}^s} \quad \max_{i \in \{1,\dots,N_2\}} \quad \max_{x \in W_i^s} \quad |q_i^s - x| \tag{21}$$

where $\mathcal{Q}^s = \{q_1^s, \dots, q_{N_2}^s\}$ is a set of points on or inside the unit sphere and $\mathcal{W}^s = \{W_1^s, \dots, W_{N_2}^s\}$ is a partition of the unit sphere. An algorithm for solving this problem will be described in Section III-A. The quantity (21) will not exceed the right-hand side of (19) if N_2 is sufficiently large.

C. Radially weighted quantization

The need for logarithmic quantization patterns is evidenced by the fact that it is the ratio $|e|/|x|$, and not the absolute value of the quantization error $|e|$ itself, that needs to be small. This is clear from the formulas (4) and (12). The approach of Section II-B leads to an “aligned” logarithmic quantization pattern, in the sense that quantization points on spheres of different radii are aligned along the same radial directions. However, it is not hard to see that non-aligned quantization patterns may achieve better coverage. This suggests proceeding from (12) in a more direct fashion.

To this end, pick two numbers $M > m > 0$ and consider the ellipsoids $\mathcal{R}_1$ given by (6) and

$$\mathcal{R}_2 := \{x \in \mathbb{R}^n : x^TPx \leq \lambda_{\max}(P)m^2\}. \tag{22}$$

Define

$$\Delta_{rw} := \max_{x \in \mathcal{R}_1 \setminus \mathcal{R}_2} \frac{|q(x)-x|}{|x|}. \tag{23}$$

Take some $\varepsilon > 0$. The next result easily follows from (12).

Lemma 3 *Assume that*

$$\lambda_{\min}(P)M^2 > \lambda_{\max}(P)m^2 \tag{24}$$

and

$$\Delta_{rw} \leq \frac{1-\varepsilon}{2\|PBK\|}. \tag{25}$$

Then the ellipsoids $\mathcal{R}_1$ and $\mathcal{R}_2$ defined by (6) *and* (22) *are invariant regions for the system* (2)*. Moreover, all solutions of* (2) *that start in the ellipsoid $\mathcal{R}_1$ enter the smaller ellipsoid $\mathcal{R}_2$ in finite time. An upper bound on this time is*

$$T = \frac{\lambda_{\min}(P)M^2 - \lambda_{\max}(P)m^2}{m^2\varepsilon}. \qquad (26)$$

The quantity Δ_{rw} defined by (23) provides another destabilization measure for q, in relation to a pair of numbers $M > m > 0$. Given a stabilizing feedback gain K, we can check the inequalities (24) and (25) and, if they are satisfied, obtain an ultimate bound on solutions from Lemma 3. (It is also clear from (12) that Δ_{rw} provides a lower bound on the rate of decay of solutions in $\mathcal{R}_1 \setminus \mathcal{R}_2$.) This leads us to the following optimization problem:

$$\min_{\mathcal{Q},\,\mathcal{W}} \quad \max_{i\in\{1,\ldots,N\}} \quad \max_{x\in W_i} \quad \frac{|q_i - x|}{|x|} \qquad (27)$$

where $\mathcal{Q} = \{q_1, \ldots, q_N\}$ is a set of quantization points and $\mathcal{W} = \{W_1, \ldots, W_N\}$ is a partition of the annulus $\{x \in \mathbb{R}^n : m < |x| < M\}$ into quantization regions. The inequality (25) will hold for a given K if N is sufficiently large.

The optimization problem (27) is different in structure from the ones we encountered earlier, and apparently has not been studied in the locational optimization literature. We henceforth call it the *radially weighted multicenter problem*. It turns out that while this problem is more challenging than the others, it is still computationally tractable. We will develop an algorithm for solving it in Section III-B.

III. Multicenter problems in facility location

In this section we present a class of optimization problems related to the field of facility location, which contains as special cases the optimization problems studied in Section II, and in particular the problems (10) from Section II-A, (21) from Section II-B, and (27) from Section II-C.

Let us review some preliminary concepts. Given a compact region $\mathcal{D} \subset \mathbb{R}^n$ and a set of N points $\mathcal{Q} = \{q_1, \ldots, q_N\}$ in $\mathbb{R}^n$, the *Voronoi partition* $\mathcal{V} = \{V_1, \ldots, V_N\}$ of $\mathcal{D}$ generated by $\mathcal{Q}$ is defined according to

$$V_i := \{x \in \mathcal{D} : \ |x - q_i| \le |x - q_j| \ \ \forall j \ne i\}. \qquad (28)$$

When it is useful to emphasize the dependency on $\mathcal{Q}$, we shall write $\mathcal{V}(\mathcal{Q})$ or $V_i(\mathcal{Q})$. When $\mathcal{D}$ is a polytope in $\mathbb{R}^n$, each Voronoi region V_i is a polytope, otherwise V_i is the intersection between a polytope and $\mathcal{D}$. The faces of the polytope which defines V_i are given by hyperplanes of points in $\mathbb{R}^n$ that are equidistant from q_i and q_j, $j \ne i$; among the latter, only "neighboring" points play a role. Note that this (standard) construction remains valid when $\mathcal{D}$ is a lower-dimensional subset of $\mathbb{R}^n$, such as a sphere. We refer to [6], [15] for comprehensive treatments of Voronoi partitions.

Let $\mathcal{Q} = \{q_1, \ldots, q_N\}$ be a collection of points in $\mathbb{R}^n$ and let $\mathcal{W} = \{W_1, \ldots, W_N\}$ be a partition of $\mathcal{D}$. In what follows, we shall concern ourselves with the function

$$\mathcal{H}(\mathcal{Q}, \mathcal{W}) := \max_{i\in\{1,\ldots,N\}} \quad \max_{x\in W_i} \quad \phi(x) f(|x - q_i|) \qquad (29)$$

where $\phi : \mathcal{D} \to [0, \infty)$ is continuous non-negative and $f : [0, \infty) \to [0, \infty)$ is continuous, non-decreasing and unbounded. We also assume that ϕ does not identically vanish on $\mathcal{D}$. We investigate the optimization problem

$$\min_{\mathcal{Q},\mathcal{W}} \quad \mathcal{H}(\mathcal{Q}, \mathcal{W}) \qquad (30)$$

and refer to it as the *weighted multicenter problem*. In general, $\mathcal{H}$ is a nonlinear non-convex function of the locations $\mathcal{Q}$ and of the partition $\mathcal{W}$. Accordingly, its global minima can be obtained only numerically via nonlinear programming algorithms. However, this and related facility location problems [17], [16], [8] have some peculiar structure that helps us characterize optimal solutions and design useful iterative algorithms. Let us start by considering the *weighted 1-center problem* over $\mathcal{D}$, i.e., take $N = 1$.

Lemma 4 *The function $\mathcal{H}_1 : \mathbb{R}^n \to [0, \infty)$ defined by*

$$\mathcal{H}_1(q) := \mathcal{H}(\{q\}, \{\mathcal{D}\}) = \max_{x\in\mathcal{D}} \quad \phi(x) f(|x - q|)$$

is continuous, radially unbounded, and quasiconvex.[1] *If f is convex and ϕ is constant, then $\mathcal{H}_1$ is convex.*

Lemma 5 *The set of global minimum points for $\mathcal{H}_1$ is compact, convex and has a non-empty intersection with* $\mathrm{co}(\mathcal{D})$*. If f is strictly increasing, then all global minimum points belong to* $\mathrm{co}(\mathcal{D})$*.*

We call $q^*(\mathcal{D})$ a *weighted center* of the region $\mathcal{D}$ if it is a (possibly non-unique) global minimum point:

$$q^*(\mathcal{D}) := \operatorname*{argmin}_{q\in\mathrm{co}(\mathcal{D})} \quad \max_{x\in\mathcal{D}} \quad \phi(x) f(|x - q|).$$

Now, it is useful to return to the general weighted multicenter problem (29), (30) and define $\mathcal{W} \mapsto \mathcal{Q}^*(\mathcal{W})$ as the map that associates to $\mathcal{W}$ a collection of N (possibly non-unique) global minimum points for the corresponding weighted 1-center problems; in other words, $\mathcal{Q}^*(\{W_1, \ldots, W_N\}) := \{q^*(W_1), \ldots, q^*(W_N)\}$. Note that these weighted centers are well defined in view of the above discussion since each W_i is compact. Finally, define the *Lloyd map* (or the Lloyd algorithm) $\mathcal{L} : (\mathcal{Q}, \mathcal{W}) \mapsto (\mathcal{Q}', \mathcal{W}')$ where $\mathcal{W}' := \mathcal{V}(\mathcal{Q})$ and $\mathcal{Q}' := \mathcal{Q}^*(\mathcal{W}')$. The following result is a relatively straightforward consequence of LaSalle Invariance Principle for discrete-time dynamical systems; further convergence properties are under current investigation in [5].

Lemma 6 *At a fixed $\mathcal{Q}$, the global minimum of $\mathcal{W} \mapsto \mathcal{H}(\mathcal{Q}, \mathcal{W})$ is achieved at $\mathcal{W} = \mathcal{V}(\mathcal{Q})$. At a fixed $\mathcal{W}$, a global*

[1] Recall that a *quasiconvex* function is a function defined on a convex domain and with convex sublevel sets.

minimum of $\mathcal{Q} \mapsto \mathcal{H}(\mathcal{Q}, \mathcal{W})$ is achieved at $\mathcal{Q} = \mathcal{Q}^*(\mathcal{W})$. The Lloyd map is a descent algorithm for the cost function $\mathcal{H}$, i.e., an application of the map is guaranteed not to increase $\mathcal{H}$. The cost is guaranteed to decrease in one iteration if no active[2] point $q_j \in \mathcal{Q}$ is a weighted center of its region W_j. Given an initial pair $(\mathcal{Q}_0, \mathcal{W}_0)$, the sequence $\{\mathcal{L}^k(\mathcal{Q}_0, \mathcal{W}_0),\ \ k \geq 0\}$ approaches the largest set invariant under $\mathcal{L}$ such that $\mathcal{H}(\mathcal{L}(\mathcal{Q}, \mathcal{W})) = \mathcal{H}(\mathcal{Q}, \mathcal{W})$.

Fixed points of the Lloyd map are *weighted central Voronoi quantizers*, i.e., pairs $(\mathcal{Q}, \mathcal{W})$ such that $\mathcal{W}$ is the Voronoi partition generated by $\mathcal{Q}$ and at the same time the points in $\mathcal{Q}$ are weighted centers for $\mathcal{W}$. It is an open conjecture that the iteration described in the lemma converges to local minima of $\mathcal{H}$. Nevertheless, the algorithm is of interest to us because it is guaranteed to improve a given quantizer design and provides a good indication as to whether or not N is large enough to achieve the control objective.

The classic Lloyd algorithm is tailored to the continuous multimedian problem as it appears, for example, in the problem of fixed-rate minimum-distorsion quantizer design; see [7], [11]. The classic Lloyd algorithm differs from the one considered here only in the fact that the points in $\mathcal{Q}$ are moved to the centroids—as opposed to the weighted centers—of the respective Voronoi regions. (Centroids are solutions of the 1-median problems.)

Next, we consider the specific settings that arise in the quantizer design problems discussed in the previous section. We study the multicenter problem (10), the spherical multicenter problem (21), and the radially weighted multicenter problem (27). To implement the Lloyd algorithm, two tasks must be carried out repeatedly. One consists in computing the Voronoi partition for a given set of points $\mathcal{Q}$, which is accomplished by the standard procedure described earlier. The other amounts to computing a weighted center for each set W_i in a given partition. Thus for each of the specific multicenter problems studied below, we only need to explain how to solve the corresponding 1-center problem.

A. Multicenter problem

Let us consider the problem (10) arising in Section II-A. The domain is a ball centered at the origin or, more generally, an ellipsoid, i.e., $\mathcal{D} = \{x \in \mathbb{R}^n : x^T P x \leq 1\}$ for some positive definite symmetric matrix P. The weighting function ϕ and the performance function f are the identity maps. Under these conditions, we refer to the optimization problem (30) simply as the multicenter problem [17], [16].

Let us analyze the 1-center problem. From Lemma 4 we know that this is a convex optimization problem. For each region V_i, the optimal solution $q^*(V_i)$ is the unique center of the minimal-radius enclosing sphere for V_i. When $V_i \subset \mathbb{R}^2$ is a polygon, this sphere is referred to as the smallest enclosing circle and algorithms are available to compute it; see [6, Chapter 4]. When $V_i \subset \mathbb{R}^n$ is a polytope, the smallest enclosing ellipsoid (in particular, sphere) can be computed via iterative convex optimization algorithms; see [1]. For a Voronoi region V_i near the boundary of $\mathcal{D}$, which is not a polytope, we can under-approximate it by a polytope generated by the vertices of V_i and suitable additional points on the intersection of V_i with the boundary of $\mathcal{D}$, and then compute the center of this polytope. For a sufficiently close under-approximation, this center will also be the center of V_i.

The spherical multicenter problem (21) from Section II-B corresponds to the setting where $\mathcal{D} = \{x \in \mathbb{R}^n : |x| = 1\}$ is the unit sphere in $\mathbb{R}^n$. Since the spherical multicenter problem is formulated in terms of the Euclidean distance in $\mathbb{R}^n$, Voronoi partitions of the sphere can be constructed as explained earlier for the general case. Voronoi regions will be intersections of polytopes with the unit sphere. The center of each Voronoi region V_i is the center of the minimal-radius enclosing sphere for V_i. We can consider a polytope in $\mathbb{R}^n$ generated by the vertices of V_i and perhaps some other points in V_i. If enough points are taken, then the center of this polytope will also be the center of V_i. As we explained earlier, computing the center of a polytope is a computationally tractable task.

B. Radially weighted multicenter problem

Here, we study the problem (27) formulated in Section II-C, where the domain is the spherical annulus $\mathcal{D} = \{x \in \mathbb{R}^n : m < |x| < M\}$. We consider the corresponding radially weighted 1-center problem over a set $V \subset \mathcal{D}$:

$$\min_{q \in \mathrm{co}(V)} \quad \max_{x \in V} \quad \frac{|q - x|}{|x|}. \tag{31}$$

The problem is well-posed because V is a subset of $\mathcal{D}$ and therefore does not contain the origin. In what follows, we take V to be a polytope; if it is not, we approximate it by a polytope as before.

Lemma 7 *The optimal cost in the problem* (31) *is smaller than* 1 *if and only if the set V is separated from the origin by a hyperplane.*

We shall henceforth assume that the set V is separated from the origin by a hyperplane. For N sufficiently large, the initial quantization points can be chosen in such a way that each of the resulting Voronoi regions indeed has this property. Since by Lemma 6 the Lloyd algorithm does not increase the cost, Lemma 7 implies that all Voronoi regions will then have this property at every step of the iteration.

We know from Lemmas 4 and 5 that the weighted 1-center problem is a quasiconvex optimization problem, i.e., it consists in minimizing the quasiconvex function $\mathcal{H}_1$ over the convex set $\mathrm{co}(\mathcal{D})$. Every quasiconvex optimization problem can be solved by iterative techniques (via a bisection algorithm solving a convex feasibility problem at each step;

[2] We call q_j *active* if $\mathcal{H}(\mathcal{Q}, \mathcal{W}) = \max_{x \in W_j} \phi(x) f(|x - q_j|)$, i.e., the maximum over i is achieved at the index j.

see Section 3.2 in [1]). However, the structure of the problem (31) can be used to obtain a solution more constructively.

Lemma 8 *Let V be a polytope separated from the origin by a hyperplane. Consider the problem of finding the sphere with center c and radius r which encloses V and minimizes $r/|c|$. Let (c^*, r^*) be the parameters of the optimal sphere. Then the optimal value for the problem* (31) *is $\gamma^* := r^*/|c^*|$ and the optimal point is $q^* := \big(1-(\gamma^*)^2\big)c^*$.*

The above result leads us to considering the problem

$$\min_{c,r} \gamma^2(c,r) := \frac{r^2}{|c|^2} \quad \text{where } |c-v_i|^2 \le r^2,\ i=1,\dots,p \tag{32}$$

where $v_1, \dots, v_p$ are the vertices of the polytope V. This is an optimization problem subject to inequality constraints, which can be solved with a finite number of computations. The idea is to enumerate active constraints, according to the following algorithm:

1: **for** all subsets S of the set of vertices of V **do**
2: compute the (c_S, r_S)-sphere minimizing γ^2 among all (c,r)-spheres touching all points in S
3: **end for**
4: discard (c_S, r_S)-spheres not containing all vertices of V
5: find global minima for (32) by comparing the values of $r_S^2/|c_S|^2$ among all remaining candidate spheres

Steps 4 and 5 are straightforward comparison checks. Regarding step 1, it turns out we can restrict our search to sets S containing at least two vertices of V.

Lemma 9 *The optimal sphere for the problem* (32) *touches at least two vertices of V, i.e., at least two constraints are active at the minimum.*

Regarding step 2, we need to minimize γ^2 over spheres passing through two or more vertices of V. Spheres passing through l generic points in $\mathbb{R}^n$ are parameterized by $n+1-l$ variables. A convenient parameterization is obtained by intersecting hyperplanes of points equidistant from pairs of points from a given set. Coordinates of the points on the intersection are given by affine functions of $n+1-l$ free parameters. Note that the radius r of the sphere is uniquely determined by its center c and the vertices of V which lie on the sphere. It is not hard to verify that the function γ^2 in (32) is a rational function whose numerator and denominator are quadratic inhomogeneous polynomials in these free parameters, and that critical points of γ^2 are solutions of $n+1-l$ quadratic equations in the same number of unknowns. According to Bezout's theorem, this generically gives 2^{n+1-l} candidate optimal spheres [4]. Step 2 is completed by choosing the one with the smallest radius.

As an example of step 2, let us work out the planar case. When $n=2$, the problem reduces to finding critical points of γ for circles passing through l vertices of V, where $l>1$ by Lemma 7. Since for $l>2$ there is at most one circle passing through the corresponding vertices, we only need to explain how to solve this problem for $l=2$. For convenience, let us consider an affine change of coordinates which places the two vertices at $(1,0)^T$ and $(-1,0)^T$ and the origin at some point $(x_0,y_0)^T$. Without loss of generality, assume that $y_0 \ge 0$. The center of the circle is denoted by $c=(\bar{x},\bar{y})^T$. We know that c must be equidistant from the two vertices, hence $\bar{x}=0$. In the special case when $y_0=0$, the solution is $\bar{y}=0$ (as is clear from symmetry). When $y_0 \neq 0$, the minimum is achieved at

$$\bar{y} = \frac{x_0^2+y_0^2-1-\sqrt{(x_0^2+y_0^2-1)^2+4y_0^2}}{2y_0} < 0.$$

IV. Acknowledgment

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V. References

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