A catalog of inverse-kinematics planners for underactuated systems on matrix Lie groups

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Abstract— **This paper presents motion planning algorithms for underactuated systems evolving on rigid rotation and displacement groups. Motion planning is transcribed into (low-dimensional) combinatorial selection and inverse-kinematic problems. We present a catalog of solutions for all underactuated systems on** $\mathrm{SE}(2)$, $\mathrm{SO}(3)$ **and** $\mathrm{SE}(2) \times \mathbb{R}$ **classified according to their controllability properties.**

I. Introduction

This paper presents motion planning algorithms for underactuated mechanical control systems. We consider kinematic models that can switch between specified sets of admissible vector fields. These models are motivated by recent progress in kinematic modeling and kinematic reductions for mechanical control systems; see [1], [2], [3], [4], [5], [6]. In particular we focus on families of left-invariant vector fields defined on rigid displacements subgroups.

An important advantage of using a kinematic model as opposed to a full dynamic model is the simplification of the resulting control problem. In this way, motion planning is transcribed into low-dimensional combinatorial selection and inverse-kinematic problems. Although closed-form solutions and general methodologies for the motion planning problem remain unfeasible, the transcription into kinematic models renders individual systems easier to tackle; e.g., [4] discusses 3R planar manipulators and [7], [8] discuss the snakeboard system.

The literature on inverse kinematics suggests numerous techniques that have never been applied in the context of motion planning. Solution methods include (i) the Paden-Kahan subproblems approach as described in [9], [10], (ii) a linear programming approach for linear translational generators [11], and (iii) the general polynomial programming approach in [12]. The latter and more general method is based on simultaneously solving systems of algebraic equations and on tools from algebraic geometry.

In this paper we provide a catalog of solutions for some interesting and relevant systems. We consider systems evolving on proper subgroups of the group of rigid displacements in three-dimensional Euclidean space. Our solutions are closely related to the system controllability properties and attempt to minimize the number of switches.

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Problem statement

We consider left-invariant control systems evolving on a matrix Lie subgroup $\mathrm{G} \subset \mathrm{SE}(3)$. Examples include systems on $\mathrm{SE}(2)$, $\mathrm{SO}(3)$ and $\mathrm{SE}(2) \times \mathbb{R}$. As common in matters of Lie group theory, we identify left-invariant vector fields with their value at the identity. Given a family of left-invariant vector fields $\{V_1, \dots, V_m\}$ on G we consider the associated driftless control system

$$\dot{g}(t) = \sum_{i=1}^{m} V_i(g(t)) w_i(t) \qquad (1)$$

where $t \mapsto g(t) \in \mathrm{G}$ and where $t \mapsto (w_1, \dots, w_m) \in \{(\pm 1, 0, \dots, 0), (0, \pm 1, 0, \dots, 0), \dots, (0, \dots, 0, \pm 1)\}$. For these systems controllability can be assessed by algebraic means: it suffices to check the lack of involutivity of the Lie algebra subspace $\operatorname{span}\{V_1, \dots, V_m\}$. Recall that for matrix Lie algebras, Lie brackets are simply matrix commutators $[A, B] = AB - BA$.

We compute feasible motion plans for the control system (1) by the concatenation of a finite number of flows along the input vector fields. We call a flow along any input vector field a *maneuver* and its duration a *coasting time*. Therefore, motion planning is reduced to the problem of selecting a finite-length combination of k maneuvers $\{(i_1, \dots, i_k) \mid i_j \in \{1, \dots, m\}\}$ and computing appropriate coasting times $\{t_1, \dots, t_k\}$ that steer the system from the identity in the group to any target configuration $g_{\mathrm{f}} \in \mathrm{G}$. In mathematical terms, we need to solve

$$g_{\mathrm{f}} = \exp(t_1 V_{i_1}) \cdots \exp(t_k V_{i_k}).$$

No general methodology is currently available to solve these problems in closed-form. In this paper, we shall present a catalog of solutions for underactuated example systems defined on $\mathrm{SE}(2)$, $\mathrm{SO}(3)$, or $\mathrm{SE}(2) \times \mathbb{R}$. Based on a controllability analysis, we classify families of underactuated systems that pose qualitatively different planning problems. For each case, we solve the planning problem by providing *a* combination of k maneuvers and corresponding closed-form expressions for the coasting times. In each case, we attempt to select $k = \dim(\mathrm{G})$: this is the minimum necessary (but sometimes not sufficient) number of maneuvers needed. If the motion planning algorithm entails exactly $\dim(\mathrm{G})$ maneuvers, i.e., minimizes the number of switches, we will refer to it as a *switch-optimal* algorithm.

Notation

Here we briefly collect the notation used throughout the paper. Let $\mathrm{id}_S \colon S \to S$ denote the identity map on the set S and let $\mathrm{ind}_S \colon \mathbb{R} \to \mathbb{R}$ denote the characteristic function of the set S, i.e., $\mathrm{ind}_S(x) = 1$ if $x \in S$ and $\mathrm{ind}_S(x) = 0$ if $x \notin S$. Let $\arctan2\,(x, y)$ denote the arctangent of y/x taking into account which quadrant the point (x, y) is in. We make the convention $\arctan2\,(0, 0) = 0$. Let $\mathrm{sign} \colon \mathbb{R} \to \mathbb{R}$ be the sign function, $\mathrm{sign}(x) = 1$ if $x > 0$, $\mathrm{sign}(x) = -1$ if $x < 0$ and $\mathrm{sign}(0) = 0$. Let A_{ij} be the (i, j) element of the matrix A. Given $v, w \in \mathbb{R}^n$, let $\arg(v, w) \in [0, \pi[$ denote the angle between them. Finally, let $\|\cdot\|$ denote the Euclidean norm.

II. CATALOG FOR SE(2)

Let $\{e_\theta, e_x, e_y\}$ be the basis of $\mathfrak{se}(2)$:

$$e_\theta = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ e_x = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ e_y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then, $[e_\theta, e_x] = e_y$, $[e_y, e_\theta] = e_x$ and $[e_x, e_y] = 0$. For ease of presentation, we write $V \in \mathfrak{se}(2)$ as $V = a e_\theta + b e_x + c e_y \equiv (a, b, c)$, and $g \in \mathrm{SE}(2)$ as

$$g = \begin{bmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{bmatrix} \equiv (\theta, x, y).$$

With this notation, $\exp \colon \mathfrak{se}(2) \to \mathrm{SE}(2)$ is

$$\exp(a, b, c) = \left(a\,,\ \frac{\sin a}{a} b - \frac{1 - \cos a}{a} c\,,\ \frac{1 - \cos a}{a} b + \frac{\sin a}{a} c\right)$$

for $a \neq 0$, and $\exp(0, b, c) = (0, b, c)$.

Lemma 2.1 (Controllability conditions): Consider two left-invariant vector fields $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ in $\mathfrak{se}(2)$. Their Lie closure is full rank if and only if $a_1 b_2 - b_1 a_2 \neq 0$ or $c_1 a_2 - a_1 c_2 \neq 0$.

Proof: Given the equality $[V_1, V_2] = (0, c_1 a_2 - a_1 c_2\,, a_1 b_2 - b_1 a_2)$, one can see that $\mathrm{span}\,\{V_1, V_2, [V_1, V_2]\} = \mathfrak{se}(2)$ if and only if

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & c_1 a_2 - c_2 a_1 & b_2 a_1 - b_1 a_2 \end{bmatrix} = (a_1 b_2 - b_1 a_2)^2 + (c_1 a_2 - a_1 c_2)^2 \neq 0\,.$$

■

Let $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ satisfy the controllability condition in Lemma 2.1. Accordingly, either a_1 or a_2 is different from zero. Without loss of generality, we will assume that $a_1 \neq 0$, and take $a_1 = 1$. As a consequence of Lemma 2.1, there are two qualitatively different cases to be considered:

- $\mathbf{\Sigma_1} = \{(V_1, V_2) \in \mathfrak{se}(2) \times \mathfrak{se}(2) \mid V_1 = (1, b_1, c_1), V_2 = (0, b_2, c_2)$ and $b_2^2 + c_2^2 = 1\}$.
- $\mathbf{\Sigma_2} = \{(V_1, V_2) \in \mathfrak{se}(2) \times \mathfrak{se}(2) \mid V_1 = (1, b_1, c_1), V_2 = (1, b_2, c_2)$ and either $b_1 \neq b_2$ or $c_1 \neq c_2\}$.

Since $\dim(\mathfrak{se}(2)) = 3$, we need at least three maneuvers along the flows of $\{V_1, V_2\}$ to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(1)} \colon \mathbb{R}^3 \to \mathrm{SE}(2)$ defined by

$$\mathcal{FK}^{(1)}(t_1, t_2, t_3) = \exp(t_1 V_1) \exp(t_2 V_2) \exp(t_3 V_1). \quad (2)$$

In the following propositions, we compute solutions for each case.

Proposition 2.2: (Inversion for $\mathbf{\Sigma_1}$-systems on SE(2)) Let $(V_1, V_2) \in \mathbf{\Sigma_1}$. Consider the map $\mathcal{IK}^{\mathbf{\Sigma_1}} \colon \mathrm{SE}(2) \to \mathbb{R}^3$,

$$\mathcal{IK}^{\mathbf{\Sigma_1}}(\theta, x, y) = (\arctan2\,(\alpha, \beta)\,, \rho, \theta - \arctan2\,(\alpha, \beta)),$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} \right).$$

Then, $\mathcal{IK}^{\mathbf{\Sigma_1}}$ is a global right inverse of $\mathcal{FK}^{(1)}$, that is, it satisfies $\mathcal{FK}^{(1)} \circ \mathcal{IK}^{\mathbf{\Sigma_1}} = \mathrm{id}_{\mathrm{SE}(2)} \colon \mathrm{SE}(2) \to \mathrm{SE}(2)$.
Note that the algorithm provided in the proposition is not only switch-optimal, but also works globally.

Proof: The proof follows from the expression for the forward kinematics map. If $\mathcal{FK}^{(1)}(t_1, t_2, t_3) = (\theta, x, y)$, then

$$\theta = t_1 + t_3\,,$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} + \begin{bmatrix} b_2 & -c_2 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos t_1 \\ \sin t_1 \end{bmatrix} t_2\,.$$

The equation in $[x, y]^T$ can be rewritten as $[\alpha, \beta]^T = [\cos t_1, \sin t_1]^T t_2$. The selection $t_1 = \arctan2\,(\alpha, \beta)$, $t_2 = \rho$ solves this equation. ■

Proposition 2.3: (Inversion for $\mathbf{\Sigma_2}$-systems on SE(2)) Let $(V_1, V_2) \in \mathbf{\Sigma_2}$. Define the neighborhood of the identity in SE(2)

$$U = \{(\theta, x, y) \in \mathrm{SE}(2) \mid \ \|(c_1 - c_2, b_1 - b_2)\|^2 \geq \max\{\|(x, y)\|^2\,, 2(1 - \cos\theta)\|(b_1, c_1)\|^2\}.$$

Consider the map $\mathcal{IK}^{\mathbf{\Sigma_2}} \colon U \subset \mathrm{SE}(2) \to \mathbb{R}^3$ whose components are

$$\mathcal{IK}_1^{\mathbf{\Sigma_2}}(\theta, x, y) = \arctan2\left(\rho, \sqrt{4 - \rho^2}\right) + \arctan2\,(\alpha, \beta)\,,$$
$$\mathcal{IK}_2^{\mathbf{\Sigma_2}}(\theta, x, y) = \arctan2\left(2 - \rho^2, \rho\sqrt{4 - \rho^2}\right),$$
$$\mathcal{IK}_3^{\mathbf{\Sigma_2}}(\theta, x, y) = \theta - \mathcal{IK}_1^{\mathbf{\Sigma_2}}(\theta, x, y) - \mathcal{IK}_2^{\mathbf{\Sigma_2}}(\theta, x, y)\,,$$

and $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{\|(c_1 - c_2, b_1 - b_2)\|^2} \begin{bmatrix} c_1 - c_2 & b_2 - b_1 \\ b_1 - b_2 & c_1 - c_2 \end{bmatrix} \cdot \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} \right).$$

Then, $\mathcal{IK}^{\mathbf{\Sigma_2}}$ is a local right inverse of $\mathcal{FK}^{(1)}$, that is, it satisfies $\mathcal{FK}^{(1)} \circ \mathcal{IK}^{\mathbf{\Sigma_2}} = \mathrm{id}_U \colon U \to U$.

Proof: If $(\theta, x, y) \in U$, then

$$\rho = \|(\alpha, \beta)\| \leq \frac{1}{\|(c_1 - c_2, b_1 - b_2)\|} \cdot \left(\|(x,y)\| + \left\| \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1-\cos\theta \\ \sin\theta \end{bmatrix} \right\| \right) \leq 2\,,$$

and therefore $\mathcal{IK}^{\mathbf{\Sigma_2}}$ is well-defined on U. Let $\mathcal{IK}^{\mathbf{\Sigma_2}}(\theta, x, y) = (t_1, t_2, t_3)$. The components of $\mathcal{FK}^{(1)}(t_1, t_2, t_3)$ are

$$\mathcal{FK}^{(1)}_1(t_1,t_2,t_3) = t_1 + t_2 + t_3\,,$$
$$\begin{bmatrix} \mathcal{FK}^{(1)}_2(t_1,t_2,t_3) \\ \mathcal{FK}^{(1)}_3(t_1,t_2,t_3) \end{bmatrix} = \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1-\cos\theta \\ \sin\theta \end{bmatrix} + \begin{bmatrix} c_1 - c_2 & b_1 - b_2 \\ b_2 - b_1 & c_1 - c_2 \end{bmatrix} \begin{bmatrix} \cos t_1 - \cos(t_1+t_2) \\ \sin t_1 - \sin(t_1 + t_2) \end{bmatrix}.$$

After some computations, one can verify $\mathcal{FK}^{(1)}(t_1,t_2,t_3) = (\theta, x, y)$. ∎

Remark 2.4: The map $\mathcal{IK}^{\mathbf{\Sigma_2}}$ in Proposition 2.3 is a local right inverse to $\mathcal{FK}^{(1)}$ on a domain that strictly contains U. In other words, our estimate of the domain of $\mathcal{IK}^{\mathbf{\Sigma_2}}$ is conservative. For instance, for points of the form $(0, x, y) \in \mathrm{SE}(2)$, it suffices to ask for

$$\|(x,y)\| \leq 2\|(c_1 - c_2, b_1 - b_2)\|\,.$$

For a point $(\theta, 0, 0) \in \mathrm{SE}(2)$, it suffices to ask for

$$(1-\cos\theta)\|(b_1,c_1)\|^2 \leq 2\|(c_1 - c_2, b_1 - b_2)\|^2\,.$$

Additionally, without loss of generality, it is convenient to assume that the vector fields V_1, V_2 satisfy $b_1^2 + c_1^2 \leq b_2^2 + c_2^2$, so as to maximize the domain U.

We illustrate the performance of the algorithms in Figure 1.

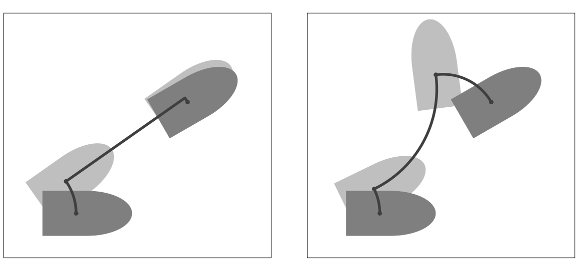

Fig. 1. We illustrate the inverse-kinematics planners for $\mathbf{\Sigma_1}$ and $\mathbf{\Sigma_2}$. The parameters of both systems are $(b_1, c_1) = (0, .5)$, $(b_2, c_2) = (1, 0)$. The target final location is $(\pi/6, 1, 1)$.

III. Catalog for SO(3)

Let $\{\widehat{e}_x, \widehat{e}_y, \widehat{e}_z\}$ be the basis of $\mathfrak{so}(3)$:

$$\widehat{e}_x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \widehat{e}_y = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \widehat{e}_z = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here we make use of the notation $\widehat{V} = a\widehat{e}_x + b\widehat{e}_y + c\widehat{e}_z \equiv \widehat{(a,b,c)}$ based on the Lie algebra isomorphism $\widehat{\cdot}: (\mathbb{R}^3, \times) \to (\mathfrak{so}(3), [\cdot,\cdot])$. An expression of the exponential $\exp : \mathfrak{so}(3) \to \mathrm{SO}(3)$ is given in terms of Rodrigues formula [10]:

$$\exp(\widehat{\eta}) = I_3 + \frac{\sin\|\eta\|}{\|\eta\|}\widehat{\eta} + \frac{1-\cos\|\eta\|}{\|\eta\|^2}\widehat{\eta}^2\,.$$

The commutator relations are $[\widehat{e}_x, \widehat{e}_z] = -\widehat{e}_y$, $[\widehat{e}_y, \widehat{e}_z] = \widehat{e}_x$ and $[\widehat{e}_x, \widehat{e}_y] = \widehat{e}_z$.

Lemma 3.1 (Controllability conditions): Consider two left-invariant vector fields $V_1 = (a_1, b_1, c_1)$ and $V_2 = (a_2, b_2, c_2)$ in $\mathfrak{so}(3)$. Their Lie closure is full rank if and only if $c_1a_2 - a_1c_2 \neq 0$ or $b_1c_2 - c_1b_2 \neq 0$ or $b_1a_2 - a_1b_2 \neq 0$.

Proof: Given the equality $[\widehat{V}_1, \widehat{V}_2] = \widehat{V_1 \times V_2}$, with $V_1 \times V_2 = (b_1c_2 - b_2c_1, c_1a_2 - c_2a_1, a_1b_2 - a_2b_1)$, one can see that $\mathrm{span}\{V_1, V_2, [V_1, V_2]\} = \mathfrak{so}(3)$ if and only if

$$\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ b_1c_2 - b_2c_1 & c_1a_2 - c_2a_1 & a_1b_2 - a_2b_1 \end{bmatrix} = (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2 \neq 0\,.$$

∎

Let V_1, V_2 satisfy the controllability condition in Lemma 3.1. Without loss of generality, we can assume $V_1 = e_z$ (otherwise we perform a suitable change of coordinates), and $\|V_2\|^2 = 1$. In what follows, we let $V_2 = (a, b, c)$. Since e_z and V_2 are linearly independent, necessarily $a^2 + b^2 \neq 0$ and $c \neq \pm 1$.

Since $\dim(\mathfrak{so}(3)) = 3$, we need at least three maneuvers to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(2)} \colon \mathbb{R}^3 \to \mathrm{SO}(3)$ defined by

$$\mathcal{FK}^{(2)}(t_1,t_2,t_3) = \exp(t_1\widehat{e}_z)\exp(t_2\widehat{V}_2)\exp(t_3\widehat{e}_z)\,. \quad (3)$$

Observe that equation (3) is similar to the formula for certain sets of Euler angles; see [10].

Proposition 3.2 (Inversion for systems on SO(3)*):* Let $V_1 = (0,0,1)$ and $V_2 = (a,b,c)$, with $a^2 + b^2 \neq 0$ and $c \neq \pm 1$. Define the neighborhood of the identity in SO(3)

$$U = \{R \in \mathrm{SO}(3) \mid R_{33} \in [2c^2 - 1, 1]\}.$$

Consider the map $\mathcal{IK} \colon U \subset \mathrm{SO}(3) \to \mathbb{R}^3$ whose components are

$$\mathcal{IK}_1(R) = \mathrm{arctan2}\left(w_1R_{13} + w_2R_{23}, -w_2R_{13} + w_1R_{23}\right)\,,$$
$$\mathcal{IK}_2(R) = \arccos\left(\frac{R_{33} - c^2}{1 - c^2}\right)\,,$$
$$\mathcal{IK}_3(R) = \mathrm{arctan2}\left(v_1R_{31} + v_2R_{32}, v_2R_{31} - v_1R_{32}\right)\,,$$

where, for $z = (1 - \cos(\mathcal{IK}_2(R)), \sin(\mathcal{IK}_2(R)))^T$,

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} ac & b \\ cb & -a \end{bmatrix} z\,, \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} ac & -b \\ cb & a \end{bmatrix} z.$$

Then, $\mathcal{IK}$ is a local right inverse of $\mathcal{FK}^{(2)}$, that is, it satisfies $\mathcal{FK}^{(2)} \circ \mathcal{IK} = \mathrm{id}_U : U \to U$.

Proof: Let $R \in U$. Then, $-1 \leq \frac{R_{33} - c^2}{1 - c^2} \leq 1$, and therefore $\mathcal{IK}(R)$ is well-defined. Denote $t_i = \mathcal{IK}_i(R)$

and let us show $R = \mathcal{FK}^{(2)}(t_1, t_2, t_3)$. Recall that the rows (resp. the columns) of a rotation matrix consist of orthonormal vectors in $\mathbb{R}^3$. Therefore, the matrix $\mathcal{FK}^{(2)}(t_1, t_2, t_3) \in \mathrm{SO}(3)$ is completely determined by its third column $\mathcal{FK}^{(2)}(t_1, t_2, t_3)e_z$ and its third row $e_z^T \mathcal{FK}^{(2)}(t_1, t_2, t_3)$.

The factors in (3) admit the following closed-form expressions. For $c_t = \cos t$ and $s_t = \sin t$, we compute

$$\exp(t\widehat{e_z}) = \begin{bmatrix} c_t & -s_t & 0 \\ s_t & c_t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $\exp(t\widehat{V}_2)$ equals

$$\begin{bmatrix} a^2 + (1-a^2)c_t & ba(1-c_t) - cs_t & ca(1-c_t) + bs_t \\ ab(1-c_t) + cs_t & b^2 + (1-b^2)c_t & cb(1-c_t) - as_t \\ ac(1-c_t) - bs_t & bc(1-c_t) + as_t & c^2 + (1-c^2)c_t \end{bmatrix}.$$

Now, using the fact that $\exp(t\widehat{e_z})e_z = e_z$, we compute

$$\begin{aligned} \mathcal{FK}^{(2)}(t_1, t_2, t_3)e_z &= \exp(t_1\widehat{e_z}) \exp(t_2\widehat{V}_2) \exp(t_3\widehat{e_z})e_z \\ &= \exp(t_1\widehat{e_z}) \exp(t_2\widehat{V}_2)e_z = \exp(t_1\widehat{e_z}) \begin{bmatrix} w_1 \\ w_2 \\ R_{33} \end{bmatrix} = Re_z\,. \end{aligned}$$

A similar computations shows that $e_z^T \mathcal{FK}^{(2)}(t_1, t_2, t_3) = e_z^T R$, which concludes the proof. ∎

Remark 3.3: If $\widehat{e_z}$ and V_2 are perpendicular, then $U = SO(3)$ and the map $\mathcal{IK}$ is a global right inverse of $\mathcal{FK}^{(2)}$. Otherwise, let us provide an equivalent formulation of the constraint $R_{33} \in [2c^2 - 1, 1]$ in terms of the axis/angle representation of the rotation matrix R. Recall that there always exist, possibly non-unique, a rotation angle $\theta \in [0, \pi]$ and an unit-length axis of rotation $\omega \in \mathbb{S}^2$ such that $R = \exp(\widehat{\omega}\theta)$. Because $\widehat{\omega}^2 = \omega^T\omega - I_3$, an equivalent statement of Rodrigues formula is

$$R = I_3 + \widehat{\omega}\sin\theta + (1-\cos\theta)(\omega^T\omega - I_3).$$

From $e_z^T\omega = \cos(\arg(e_z, \omega))$ we compute

$$\begin{aligned} e_z^T R e_z &= e_z^T e_z + (1-\cos\theta)((e_z^T\omega)^2 - e_z^T e_z) \\ &= 1 + (1-\cos\theta)((e_z^T\omega)^2 - 1) \\ &= 1 - \sin^2(\arg(e_z, \omega))(1-\cos\theta) \qquad (4) \end{aligned}$$

Therefore, $R_{33} \in [2c^2 - 1, 1]$ if and only if

$$\begin{aligned} &1 - \sin^2(\arg(e_z, \omega))(1-\cos\theta) \geq 2c^2 - 1 \\ &\iff \sin^2(\arg(e_z, \omega))(1-\cos\theta) \leq 2(1-c^2)\,. \end{aligned}$$

Two sufficient conditions are also meaningful. In terms of the rotation angle, if $|\theta| \leq \arccos(2c^2 - 1)$ then $1 - \cos\theta \leq 2(1-c^2)$, and in turn equation (4) is satisfied. In terms of the axis of rotation, a sufficient condition for equation (4) is $\sin^2(\arg(e_z, \omega)) \leq \sin^2(\arg(e_z, V_2)) = 1 - c^2$.

We illustrate the performance of the algorithms in Figure 2.

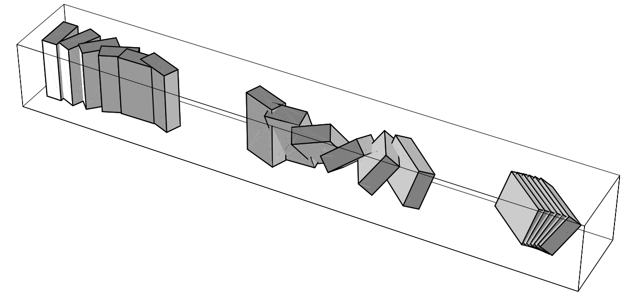

Fig. 2. We illustrate the inverse-kinematic planner on SO(3). The system parameters are $(a, b, c) = (0, 1/\sqrt{2}, 1/\sqrt{2})$. The target final rotation is $\exp(\pi/3, \pi/3, 0)$. To render the sequence of three rotations visible, the body is translated along the inertial x-axis.

IV. Catalog for $\mathrm{SE}(2) \times \mathbb{R}$

Let $\{(e_x, 0), (e_y, 0), (e_\theta, 0), (0, 0, 0, 1)\}$ be a basis of $\mathfrak{se}(2) \times \mathbb{R}$, where $\{e_x, e_y, e_\theta\}$ stands for the basis of $\mathfrak{se}(2)$ introduced in Section II. With a slight abuse of notation, we will denote by e_x the element $(e_x, 0)$, and so on. Also, we will use the shorthand notation $e_z = (0, 0, 0, 1)$. The Lie algebra commutators are given by

$$\begin{aligned} &[e_x, e_y] = [e_x, e_z] = [e_y, e_z] = [e_z, e_\theta] = 0\,, \\ &[e_x, e_\theta] = -e_y, \quad [e_y, e_\theta] = e_x\,. \end{aligned}$$

A left-invariant vector field V in $\mathfrak{se}(2) \times \mathbb{R}$ is written as $V = ae_\theta + be_x + ce_y + de_z \equiv (a, b, c, d)$, and $g \in \mathrm{SE}(2) \times \mathbb{R}$ as $g = (\theta, x, y, z)$. The exponential map, $\exp : \mathfrak{se}(2) \times \mathbb{R} \longrightarrow \mathrm{SE}(2) \times \mathbb{R}$, is given component-wise by the exponential on $\mathfrak{se}(2)$ and $\mathbb{R}$, respectively. That is, $\exp(V)$ is equal to

$$\left(a\,,\ \frac{\sin a}{a}b - \frac{1-\cos a}{a}c\,,\ \frac{1-\cos a}{a}b + \frac{\sin a}{a}c\,,\ d\right)$$

if $a \neq 0$, and $\exp(V) = (0, b, c, d)$ if $a = 0$.

Lemma 4.1: (Controllability conditions for $\mathrm{SE}(2) \times \mathbb{R}$ systems with 2 inputs) Consider two left-invariant vector fields $V_1 = (a_1, b_1, c_1, d_1)$ and $V_2 = (a_2, b_2, c_2, d_2)$ in $\mathfrak{se}(2) \times \mathbb{R}$. Their Lie closure is full rank if and only if $a_2 d_1 - d_2 a_1 \neq 0$, and either $c_1 a_2 - a_1 c_2 \neq 0$ or $a_1 b_2 - b_1 a_2 \neq 0$.

Proof: Since $[V_1, V_2] = (0\,, c_1 a_2 - a_1 c_2\,, a_1 b_2 - b_1 a_2\,, 0) \neq 0$, we deduce that either $c_1 a_2 - a_1 c_2 \neq 0$ or $a_1 b_2 - b_1 a_2 \neq 0$. In particular, this implies that necessarily $a_1 \neq 0$ or $a_2 \neq 0$. Assume $a_1 \neq 0$. Now,

$$[V_1, [V_1, V_2]] = (0, a_1(-b_2 a_1 + b_1 a_2), a_1(c_1 a_2 - c_2 a_1), 0)\,,$$

and note that $[V_2, [V_1, V_2]] = (a_2/a_1)[V_1, [V_1, V_2]]$. Finally, $\overline{\mathrm{Lie}}(\{V_1, V_2\}) = \mathfrak{se}(2) \times \mathbb{R}$ if and only if

$$\det \begin{bmatrix} b_1 & c_1 & d_1 & a_1 \\ b_2 & c_2 & d_2 & a_2 \\ c_1 a_2 - c_2 a_1 & b_2 a_1 - b_1 a_2 & 0 & 0 \\ a_1(-b_2 a_1 + b_1 a_2) & a_1(c_1 a_2 - c_2 a_1) & 0 & 0 \end{bmatrix} =$$

$$a_1(a_2 d_1 - d_2 a_1)\left[(c_1 a_2 - c_2 a_1)^2 + (-b_2 a_1 + b_1 a_2)^2\right] \neq 0.$$

Since $[V_1, V_2] \neq 0$, this condition reduces to $a_2 d_1 - d_2 a_1 \neq 0$. ∎

Let V_1, V_2 satisfy the controllability condition in Lemma 4.1. Without loss of generality, we can assume $a_1 = 1$. As in the case of SE(2), there are two qualitatively different situations to be considered:

$\boldsymbol{\Lambda_1} = \{(V_1, V_2) \in (\mathfrak{se}(2) \times \mathbb{R})^2 \,|\, V_1 = (1, b_1, c_1, d_1), V_2 = (0, b_2, c_2, 1) \text{ and } b_2^2 + c_2^2 \neq 0\}$.

$\boldsymbol{\Lambda_2} = \{(V_1, V_2) \in (\mathfrak{se}(2) \times \mathbb{R})^2 \,|\, V_1 = (1, b_1, c_1, d_1), V_2 = (1, b_2, c_2, d_2), d_1 \neq d_2 \text{ and either } b_1 \neq b_2 \text{ or } c_1 \neq c_2\}$.

Lemma 4.2: (Controllability conditions for $\mathrm{SE}(2) \times \mathbb{R}$ systems with 3 inputs) Consider three left-invariant vector fields $V_i = (a_i, b_i, c_i, d_i)$, $i = 1, 2, 3$ in $\mathfrak{se}(2) \times \mathbb{R}$. Assume $\overline{\mathrm{Lie}}(\{V_{i_1}, V_{i_2}\}) \subsetneq \mathfrak{se}(2) \times \mathbb{R}$, for $i_j \in \{1, 2, 3\}$ and $\overline{\mathrm{Lie}}(\{V_1, V_2, V_3\}) = \mathfrak{se}(2) \times \mathbb{R}$. Then, possibly after a reordering of the vector fields, they must fall in one of the following cases:

$\boldsymbol{\Lambda_3} = \{(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \,|\, V_1 = (1, b_1, c_1, d_1), V_2 = (0, b_2, c_2, 0), V_3 = (1, b_1, c_1, d_3), d_1 \neq d_3 \text{ and } b_2^2 + c_2^2 \neq 0\}$.

$\boldsymbol{\Lambda_4} = \{(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \,|\, V_1 = (1, b_1, c_1, d_1), V_2 = (0, b_2, c_2, 0), V_3 = (0, 0, 0, d_3), 0 \neq d_3 \neq d_1 \text{ and } b_2^2 + c_2^2 \neq 0\}$.

$\boldsymbol{\Lambda_5} = \{(V_1, V_2, V_3) \in (\mathfrak{se}(2) \times \mathbb{R})^3 \,|\, V_1 = (1, b_1, c_1, d_1), V_2 = (1, b_2, c_2, d_1), V_3 = (0, 0, 0, d_3), d_3 \neq 0 \text{ and either } b_2 \neq b_1 \text{ or } c_1 \neq c_2\}$.

Proof: Without loss of generality, we can assume that $[V_1, V_2] \neq 0$ and $a_1 = 1$. Since $\overline{\mathrm{Lie}}(\{V_1, V_2\}) \neq \mathfrak{se}(2) \times \mathbb{R}$, then $a_2 d_1 = d_2$. Given that the Lie closure of $\{V_1, V_2, V_3\}$ is full-rank, and $\dim(\mathrm{span}\,\{V_1, V_2, [V_1, V_2]\}) = 3$, we have that $d_3 \neq a_3 d_1$. This latter fact, together with $\overline{\mathrm{Lie}}(\{V_1, V_3\}) \subsetneq \mathfrak{se}(2) \times \mathbb{R}$, implies that $[V_1, V_3] = 0$, and therefore $b_3 = a_3 b_1$, $c_1 a_3 = c_3$.

We distinguish now two situations depending on $[V_2, V_3]$ being zero or not.

(a) $[V_2, V_3] \neq 0$. Necessarily, $a_3 \neq 0$. Therefore, we can assume $a_3 = 1$. Since $\overline{\mathrm{Lie}}(\{V_2, V_3\})$ is not full-rank, then $a_2 = 0$. We then have a $\boldsymbol{\Lambda_3}$-system.

(b) $[V_2, V_3] = 0$. Necessarily, $b_3 a_2 = b_2 a_3$ and $c_2 a_3 = c_3 a_2$. Depending on the values of a_2 and a_3, there are four subcases:

(i) If $a_2 = a_3 = 0$, then $d_2 = 0$, $d_3 \neq 0$, $b_3 = c_3 = 0$. Then, this is a $\boldsymbol{\Lambda_4}$-system.

(ii) If $a_2 = 0$, and $a_3 = 1$, then $b_2 = b_3 a_2 = 0$, $c_2 = c_3 a_2 = 0$ and also $d_2 = d_1 a_2 = 0$. This is not possible as it would make $V_2 = 0$.

(iii) If $a_2 = 1$ and $a_3 = 0$, then $b_3 = c_3 = 0$, and $d_2 = d_1$. Therefore, this is a $\boldsymbol{\Lambda_5}$-system.

(iv) Finally, if $a_2 = 1$ and $a_3 = 1$, then $b_1 = b_2$, $c_1 = c_2$, and $d_1 = d_2$, which makes V_1 and V_2 linearly dependent.

■

A. Two-dimensional input distribution

Let V_1, V_2 satisfy the controllability condition in Lemma 4.1. Since $\dim(\mathfrak{se}(2) \times \mathbb{R}) = 4$, we need at least four maneuvers to plan any motion between two desired configurations. Consider the map $\mathcal{FK}^{(3)} \colon \mathbb{R}^4 \to \mathrm{SE}(2) \times \mathbb{R}$,

$$\begin{aligned}\mathcal{FK}^{(3)}(t_1, t_2, t_3, t_4) = \exp(t_1 V_2) \exp(t_2 V_1)\\ \cdot \exp(t_3 V_2) \exp(t_4 V_1)\,. \quad (5)\end{aligned}$$

Proposition 4.3: (Lack of switch-optimal inversion for $\boldsymbol{\Lambda_1}$-systems on $\mathrm{SE}(2) \times \mathbb{R}$) Let $(V_1, V_2) \in \boldsymbol{\Lambda_1}$. Then, the map $\mathcal{FK}^{(3)}$ in not invertible at any neighborhood of the origin.

Proof: Let $\mathcal{FK}^{(3)}(t_1, t_2, t_3, t_4) = (\theta, x, y, z)$. Then,

$$\begin{aligned}
&\theta = t_2 + t_4\,,\\
&z = t_1 + t_3 + d_1(t_2 + t_4) = t_1 + t_3 + d_1\theta\,,\\
&\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -c_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} c_1 & b_1 \\ -b_1 & c_1 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}\\
&\qquad + \begin{bmatrix} b_2 \\ c_2 \end{bmatrix} t_1 + \begin{bmatrix} b_2 & -c_2 \\ c_2 & b_2 \end{bmatrix} \begin{bmatrix} \cos t_2 \\ \sin t_2 \end{bmatrix} t_3\,.
\end{aligned}$$

Consider a configuration with $\theta = z = 0$. Then, the equation in (x, y) is invertible if and only if the map $f \colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\begin{bmatrix} t_2 \\ t_3 \end{bmatrix} \longmapsto \begin{bmatrix} \cos t_2 - 1 \\ \sin t_2 \end{bmatrix} t_3$$

is invertible. But f can not be inverted in $(0, \beta)$, $\beta \neq 0$. ■

Remark 4.4: A similar situation occurs if we start taking maneuvers along the flow of V_1 instead of V_2.

Consider the map $\mathcal{FK}^{(4)} \colon \mathbb{R}^5 \to \mathrm{SE}(2) \times \mathbb{R}$ defined by

$$\begin{aligned}\mathcal{FK}^{(4)}(t_1, t_2, t_3, t_4, t_5) = \exp(t_1 V_1) \exp(t_2 V_2)\\ \cdot \exp(t_3 V_1) \exp(t_4 V_2) \exp(t_5 V_1)\,. \quad (6)\end{aligned}$$

Proposition 4.5 (Inversion for $\boldsymbol{\Lambda_1}$-systems on $\mathrm{SE}(2) \times \mathbb{R}$): Let $(V_1, V_2) \in \boldsymbol{\Lambda_1}$. Consider the map $\mathcal{IK}^{\boldsymbol{\Lambda_1}} \colon \mathrm{SE}(2) \times \mathbb{R} \to \mathbb{R}^5$ whose components are

$$\begin{aligned}
\mathcal{IK}_1^{\boldsymbol{\Lambda_1}}(\theta, x, y, z) &= \pi \mathrm{ind}_{]-\infty,0[}(\gamma - \rho) + \mathrm{arctan2}\left((\rho + \gamma)/2, 0\right)\\
&\quad + \mathrm{arctan2}\left(\alpha, \beta\right),\\
\mathcal{IK}_2^{\boldsymbol{\Lambda_1}}(\theta, x, y, z) &= (\gamma - \rho)/2,\\
\mathcal{IK}_3^{\boldsymbol{\Lambda_1}}(\theta, x, y, z) &= \mathrm{arctan2}\left((\rho - \gamma)/2, 0\right) - \mathrm{arctan2}\left((\rho + \gamma)/2, 0\right)\\
&\quad + \pi(\mathrm{ind}_{]-\infty,0[}(\gamma + \rho) - \mathrm{ind}_{]-\infty,0[}(\gamma - \rho)),\\
\mathcal{IK}_4^{\boldsymbol{\Lambda_1}}(\theta, x, y, z) &= (\gamma + \rho)/2,\\
\mathcal{IK}_5^{\boldsymbol{\Lambda_1}}(\theta, x, y, z) &= \theta - \mathcal{IK}_1^{\boldsymbol{\Lambda_1}}(\theta, x, y, z) - \mathcal{IK}_3^{\boldsymbol{\Lambda_1}}(\theta, x, y, z),
\end{aligned}$$

where $\rho = \sqrt{\alpha^2 + \beta^2}$ and

$$\begin{aligned}
&\gamma = z - d_1\theta\,,\\
&\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \frac{1}{b_2^2 + c_2^2} \begin{bmatrix} b_2 & c_2 \\ -c_2 & b_2 \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} -c_1 & b_1 \\ b_1 & c_1 \end{bmatrix} \begin{bmatrix} 1 - \cos\theta \\ \sin\theta \end{bmatrix} \right).
\end{aligned}$$

Then, $\mathcal{IK}^{\boldsymbol{\Lambda_1}}$ is a global right inverse of $\mathcal{FK}^{(4)}$, that is, it satisfies $\mathcal{FK}^{(4)} \circ \mathcal{IK}^{\boldsymbol{\Lambda_1}} = \mathrm{id}_{\mathrm{SE}(2) \times \mathbb{R}} \colon \mathrm{SE}(2) \times \mathbb{R} \to \mathrm{SE}(2) \times \mathbb{R}$.

Proof: The proof follows from the expression for the forward kinematics map. If $\mathcal{FK}^{(4)}(t_1,t_2,t_3,t_4,t_5) = (\theta,x,y,z)$, then

$$\begin{aligned}
\theta &= t_1 + t_3 + t_5\,,\\
z &= t_2 + t_4 + d_1\theta\,,\\
\begin{bmatrix} x\\ y\end{bmatrix} &= \begin{bmatrix} -c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix} 1-\cos\theta\\ \sin\theta\end{bmatrix}\\
&\quad + \begin{bmatrix} b_2 & -c_2\\ c_2 & b_2\end{bmatrix}\left(\begin{bmatrix}\cos t_1\\ \sin t_1\end{bmatrix} t_2 + \begin{bmatrix}\cos(t_1+t_3)\\ \sin(t_1+t_3)\end{bmatrix} t_4\right).
\end{aligned}$$

The equation in $[x,y]^T$ can be rewritten as

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\cos t_1\\ \sin t_1\end{bmatrix} t_2 + \begin{bmatrix}\cos(t_1+t_3)\\ \sin(t_1+t_3)\end{bmatrix} t_4\,,$$

which is solved by the selection of coasting times given by the components of the map $\mathcal{IK}^{\mathbf{\Lambda_1}}$. ■

Proposition 4.6 (Inversion for $\mathbf{\Lambda_2}$-systems on SE(2) $\times\,\mathbb{R}$*):* Let $(V_1,V_2)\in\mathbf{\Lambda_2}$. Define the neighborhood of the identity in SE(2) $\times\,\mathbb{R}$

$$\begin{aligned}
U = \Big\{&(\theta,x,y,z)\in \mathrm{SE}(2)\times\mathbb{R} \mid\ 4\,\|(c_1-c_2,b_1-b_2)\|^2 \geq \\
&\max\{\|(x,y)\|^2, 2(1-\cos\theta)\|(b_1,c_1)\|^2\}\,,\\
&|z-d_1\theta| \leq 2|d_2-d_1|\arccos\left(-1+\frac{1}{\|(c_1-c_2,b_1-b_2)\|}\right.\\
&\left.\cdot\left(\|(x,y)\|+\|(b_1,c_1)\|\sqrt{2(1-\cos\theta)}\right)\right)\Big\}.
\end{aligned}$$

Consider the map $\mathcal{IK}^{\mathbf{\Lambda_2}}\colon \mathrm{SE}(2)\times\mathbb{R}\to\mathbb{R}^5$ whose components are

$$\begin{aligned}
\mathcal{IK}_1^{\mathbf{\Lambda_2}}(\theta,x,y,z) &= \arctan2\left(l,\sqrt{4-l^2}\right) + \arctan2\left(\alpha,\beta\right),\\
\mathcal{IK}_2^{\mathbf{\Lambda_2}}(\theta,x,y,z) &= 2\arctan2\left(\sqrt{4-l^2}, l\right),\\
\mathcal{IK}_3^{\mathbf{\Lambda_2}}(\theta,x,y,z) &= -\arctan2\left(\rho-l,\sqrt{4-(\rho-l)^2}\right)\\
&\quad - \mathcal{IK}_1^{\mathbf{\Lambda_2}}(\theta,x,y,z) - \mathcal{IK}_2^{\mathbf{\Lambda_2}}(\theta,x,y,z),\\
\mathcal{IK}_4^{\mathbf{\Lambda_2}}(\theta,x,y,z) &= \gamma - \mathcal{IK}_2^{\mathbf{\Lambda_2}}(\theta,x,y,z)\\
\mathcal{IK}_5^{\mathbf{\Lambda_2}}(\theta,x,y,z) &= \theta - \sum_{i=1}^{4}\mathcal{IK}_i^{\mathbf{\Lambda_2}}(\theta,x,y,z),
\end{aligned}$$

where $\rho=\sqrt{\alpha^2+\beta^2}$, $s=\sin(\gamma/2)$, $c=\cos(\gamma/2)$ and

$$\begin{aligned}
\gamma &= (z-d_1\theta)/(d_2-d_1)\,,\\
l &= \frac{\rho(1+c)+\operatorname{sign}(\gamma)\sqrt{\rho^2(1+c)^2-(1+c)(2\rho^2-8s^2)}}{2(1+c)},\\
\begin{bmatrix}\alpha\\ \beta\end{bmatrix} &= \frac{1}{\|(d_1-d_2,c_1-c_2)\|^2}\begin{bmatrix} d_1-d_2 & c_2-c_1\\ c_1-c_2 & d_1-d_2\end{bmatrix}\\
&\quad\cdot\left(\begin{bmatrix} x\\ y\end{bmatrix} - \begin{bmatrix} -d_1 & c_1\\ c_1 & d_1\end{bmatrix}\begin{bmatrix} 1-\cos\theta\\ \sin\theta\end{bmatrix}\right).
\end{aligned}$$

Then, $\mathcal{IK}^{\mathbf{\Lambda_2}}$ is a local right inverse of $\mathcal{FK}^{(4)}$, that is, it satisfies $\mathcal{FK}^{(4)}\circ\mathcal{IK}^{\mathbf{\Lambda_2}} = \mathrm{id}_U\colon U\to U$.

Proof: If $(\theta,x,y,z)\in U$, then $\rho\leq 4$ and $|\gamma|\leq 2\arccos(-1+\rho/2)$. This in turn implies that

$$c = \cos\left(\frac{\gamma}{2}\right) \geq -1+\frac{\rho}{2} \geq -1+\frac{\rho^2}{8}$$

over $\rho\leq 4$. The second inequality guarantees that l is well-defined. The first one implies $l\in[\rho-2,2]$, which makes $\mathcal{IK}^{\mathbf{\Lambda_2}}$ well-defined on U. Let $\mathcal{IK}^{\mathbf{\Lambda_2}}(\theta,x,y,z) = (t_1,t_2,t_3,t_4,t_5)$. The components of $\mathcal{FK}^{(4)}(t_1,t_2,t_3,t_4,t_5)$ are the following

$$\begin{aligned}
\theta &= t_1+t_2+t_3+t_4+t_5\,,\\
z &= d_1\theta + (d_2-d_1)(t_2+t_4)\,,\\
\begin{bmatrix} x\\ y\end{bmatrix} &= \begin{bmatrix} -c_1\\ b_1\end{bmatrix} + \begin{bmatrix} c_1 & b_1\\ -b_1 & c_1\end{bmatrix}\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} + \begin{bmatrix} c_1-c_2 & b_1-b_2\\ b_2-b_1 & c_1-c_2\end{bmatrix}\\
&\quad\begin{bmatrix}\cos t_1 - \cos(t_1+t_2) + \cos(t_1+t_2+t_3) - \cos(\sum_{i=1}^4 t_i)\\ \sin t_1 - \sin(t_1+t_2) + \sin(t_1+t_2+t_3) - \sin(\sum_{i=1}^4 t_i)\end{bmatrix}
\end{aligned}$$

After some rather involved computations, one can verify $\mathcal{FK}^{(4)}(t_1,t_2,t_3,t_4,t_5) = (\theta,x,y,z)$. ■

B. Three-dimensional input distribution

Let V_1, V_2, V_3 satisfy the controllability condition in Lemma 4.2. Consider the map $\mathcal{FK}^{(5)}\colon\mathbb{R}^4\to\mathrm{SE}(2)\times\mathbb{R}$ defined by

$$\begin{aligned}
\mathcal{FK}^{(5)}(t_1,t_2,t_3,t_4) &= \exp(t_1V_1)\exp(t_2V_3)\\
&\quad\cdot\exp(t_3V_2)\exp(t_4V_1)\,. \qquad (7)
\end{aligned}$$

Proposition 4.7 (Inversion for $\mathbf{\Lambda_3}$-systems on SE(2) $\times\,\mathbb{R}$*):* Let $(V_1,V_2,V_3)\in\mathbf{\Lambda_3}$. Consider the map $\mathcal{IK}^{\mathbf{\Lambda_3}}\colon\mathrm{SE}(2)\times\mathbb{R}\to\mathbb{R}^4$ whose components are

$$\begin{aligned}
\mathcal{IK}_1^{\mathbf{\Lambda_3}}(\theta,x,y,z) &= \arctan2\left(\alpha,\beta\right) - \mathcal{IK}_2^{\mathbf{\Lambda_3}}(\theta,x,y,z)\,,\\
\mathcal{IK}_2^{\mathbf{\Lambda_3}}(\theta,x,y,z) &= \frac{z-d_1\theta}{d_3-d_1},\\
\mathcal{IK}_3^{\mathbf{\Lambda_3}}(\theta,x,y,z) &= \rho\,,\\
\mathcal{IK}_4^{\mathbf{\Lambda_3}}(\theta,x,y,z) &= \theta - \arctan2\left(\alpha,\beta\right)\,,
\end{aligned}$$

where $\rho=\sqrt{\alpha^2+\beta^2}$ and

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{b_2^2+c_2^2}\begin{bmatrix} b_2 & c_2\\ -c_2 & b_2\end{bmatrix}\left(\begin{bmatrix} x\\ y\end{bmatrix} - \begin{bmatrix} -c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix} 1-\cos\theta\\ \sin\theta\end{bmatrix}\right).$$

Then, $\mathcal{IK}^{\mathbf{\Lambda_3}}$ is a global right inverse of $\mathcal{FK}^{(5)}$, that is, it satisfies $\mathcal{FK}^{(5)}\circ\mathcal{IK}^{\mathbf{\Lambda_3}} = \mathrm{id}_{\mathrm{SE}(2)\times\mathbb{R}}\colon\mathrm{SE}(2)\times\mathbb{R}\to\mathrm{SE}(2)\times\mathbb{R}$.

Proof: The proof follows from the expression for the map $\mathcal{FK}^{(5)}$. If $\mathcal{FK}^{(5)}(t_1,t_2,t_3,t_4) = (\theta,x,y,z)$, then

$$\begin{aligned}
\theta &= t_1+t_2+t_4\,,\\
z &= d_1t_1 + d_3t_2 + d_1t_4 = d_1\theta + (d_3-d_1)t_2\,,\\
\begin{bmatrix} x\\ y\end{bmatrix} &= \begin{bmatrix} -c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix} 1-\cos\theta\\ \sin\theta\end{bmatrix} + \begin{bmatrix} b_2 & -c_2\\ c_2 & b_2\end{bmatrix}\begin{bmatrix}\cos(t_1+t_2)\\ \sin(t_1+t_2)\end{bmatrix} t_3\,.
\end{aligned}$$

The equation in $[x,y]^T$ can be rewritten as

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}\cos(t_1+t_2)\\ \sin(t_1+t_2)\end{bmatrix} t_3\,,$$

which is solved by the selection given by $(t_1,t_2,t_3,t_4) = \mathcal{IK}^{\mathbf{\Lambda_3}}(\theta,x,y,z)$. ■

Consider the map $\mathcal{FK}^{(6)}\colon \mathbb{R}^4 \to \mathrm{SE}(2)\times\mathbb{R}$ defined by

$$\mathcal{FK}^{(6)}(t_1,t_2,t_3,t_4) = \exp(t_1V_1)\exp(t_2V_2) \cdot \exp(t_3V_1)\exp(t_4V_3)\,. \quad (8)$$

Proposition 4.8 (Inversion for $\mathbf{\Lambda_4}$-systems on $\mathrm{SE}(2)\times\mathbb{R}$*):* Let $(V_1,V_2,V_3) \in \mathbf{\Lambda_4}$. Consider the map $\mathcal{IK}^{\mathbf{\Lambda_4}}\colon \mathrm{SE}(2)\times\mathbb{R} \to \mathbb{R}^4$ given by

$$\mathcal{IK}^{\mathbf{\Lambda_4}}(\theta,x,y,z) = \left(\operatorname{arctan2}(\alpha,\beta), \rho, \theta - \operatorname{arctan2}(\alpha,\beta), \frac{z-d_1\theta}{d_3}\right),$$

where $\rho = \sqrt{\alpha^2+\beta^2}$ and

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{b_2^2+c_2^2}\begin{bmatrix}b_2 & c_2\\ -c_2 & d_2\end{bmatrix}\left(\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix}\right).$$

Then, $\mathcal{IK}^{\mathbf{\Lambda_4}}$ is a global right inverse of $\mathcal{FK}^{(6)}$, that is, it satisfies $\mathcal{FK}^{(6)}\circ\mathcal{IK}^{\mathbf{\Lambda_4}} = \mathrm{id}_{\mathrm{SE}(2)\times\mathbb{R}}\colon \mathrm{SE}(2)\times\mathbb{R} \to \mathrm{SE}(2)\times\mathbb{R}$.

Proof: If $\mathcal{FK}^{(6)}(t_1,t_2,t_3,t_4) = (\theta,x,y,z)$, then

$$\theta = t_1+t_3\,,$$
$$\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix} + \begin{bmatrix}b_2 & -c_2\\ c_2 & b_2\end{bmatrix}\begin{bmatrix}\cos t_1\\ \sin t_1\end{bmatrix} t_2\,,$$
$$z = d_1(t_1+t_3)+d_3t_4\,.$$

The equation in $[x,y]^T$ can be rewritten as $[\alpha,\beta]^T = [\cos t_1, \sin t_1]^T t_2$. As in the proof of Proposition 2.2, the selection $t_1 = \operatorname{arctan2}(\alpha,\beta)$, $t_2=\rho$ solves it. ∎

Proposition 4.9 (Inversion for $\mathbf{\Lambda_5}$-systems on $\mathrm{SE}(2)\times\mathbb{R}$*):* Let $(V_1,V_2,V_3)\in\mathbf{\Lambda_5}$. Define the neighborhood of the identity in $\mathrm{SE}(2)\times\mathbb{R}$

$$U = \{(\theta,x,y)\in\mathrm{SE}(2)\times\mathbb{R} \mid \|(c_1-c_2,b_1-b_2)\|^2 \geq \max\{\|(x,y)\|^2, 2(1-\cos\theta)\|(b_1,c_1)\|^2\}.$$

Consider the map $\mathcal{IK}^{\mathbf{\Lambda_5}}\colon U\subset \mathrm{SE}(2)\times\mathbb{R}\to\mathbb{R}^4$ whose components are

$$\mathcal{IK}_1^{\mathbf{\Lambda_5}}(\theta,x,y,z) = \operatorname{arctan2}\left(\rho,\sqrt{4-\rho^2}\right) + \operatorname{arctan2}(\alpha,\beta)\,,$$
$$\mathcal{IK}_2^{\mathbf{\Lambda_5}}(\theta,x,y,z) = \operatorname{arctan2}\left(2-\rho^2, \rho\sqrt{4-\rho^2}\right),$$
$$\mathcal{IK}_3^{\mathbf{\Lambda_5}}(\theta,x,y,z) = \theta - \mathcal{IK}_1^{\mathbf{\Lambda_5}}(\theta,x,y) - \mathcal{IK}_2^{\mathbf{\Lambda_5}}(\theta,x,y)\,,$$
$$\mathcal{IK}_4^{\mathbf{\Lambda_5}}(\theta,x,y,z) = \frac{z-d_1\theta}{d_3}\,,$$

and $\rho=\sqrt{\alpha^2+\beta^2}$ and

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \frac{1}{\|(c_1-c_2,b_1-b_2)\|^2}\begin{bmatrix}c_1-c_2 & b_2-b_1\\ b_1-b_2 & c_1-c_2\end{bmatrix} \cdot\left(\begin{bmatrix}x\\ y\end{bmatrix} - \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix}\right).$$

Then, $\mathcal{IK}^{\mathbf{\Lambda_5}}$ is a local right inverse of $\mathcal{FK}^{(6)}$, that is, it satisfies $\mathcal{FK}^{(6)}\circ\mathcal{IK}^{\mathbf{\Lambda_5}} = \mathrm{id}_U\colon U\to U$.

Proof: If $(\theta,x,y,z)\in U$, then one can see that $\rho = \|(\alpha,\beta)\| \leq 2$, and therefore $\mathcal{IK}^{\mathbf{\Lambda_5}}$ is well-defined on U. Let $\mathcal{IK}^{\mathbf{\Lambda_5}}(\theta,x,y,z) = (t_1,t_2,t_3,t_4)$. The components of $\mathcal{FK}^{(6)}(t_1,t_2,t_3,t_4)$ are

$$\mathcal{FK}_1^{(6)}(t_1,t_2,t_3,t_4) = t_1+t_2+t_3\,,$$
$$\begin{bmatrix}\mathcal{FK}_2^{(6)}(t_1,t_2,t_3,t_4)\\ \mathcal{FK}_3^{(6)}(t_1,t_2,t_3,t_4)\end{bmatrix} = \begin{bmatrix}-c_1 & b_1\\ b_1 & c_1\end{bmatrix}\begin{bmatrix}1-\cos\theta\\ \sin\theta\end{bmatrix} + \begin{bmatrix}c_1-c_2 & b_1-b_2\\ b_2-b_1 & c_1-c_2\end{bmatrix}\begin{bmatrix}\cos t_1 - \cos(t_1+t_2)\\ \sin t_1 - \sin(t_1+t_2)\end{bmatrix},$$
$$\mathcal{FK}_4^{(6)}(t_1,t_2,t_3,t_4) = d_1(t_1+t_2+t_3)+d_3t_4\,.$$

After some computations, one can verify $\mathcal{FK}^{(6)}(t_1,t_2,t_3,t_4) = (\theta,x,y,z)$. ∎

V. Conclusions

We have presented a catalog of feasible motion planning algorithms for underactuated controllable systems on SE(2), SO(3) and $\mathrm{SE}(2)\times\mathbb{R}$. Future directions of research include (i) considering other relevant classes of underactuated systems on SE(3), (ii) computing catalogs of optimal sequences of maneuvers, and (iii) developing hybrid feedback schemes that rely on the proposed open-loop planners to achieve point stabilization and trajectory tracking.

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