LOW-ORDER CONTROLLABILITY AND KINEMATIC REDUCTIONS FOR AFFINE CONNECTION CONTROL SYSTEMS*

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Abstract. Controllability and kinematic modelling notions are investigated for a class of mechanical control systems. First, low-order controllability results are given for the class of mechanical control systems. Second, a precise connection is made between those mechanical systems which are dynamic (i.e., have forces as inputs) and those which are kinematic (i.e., have velocities as inputs). Interestingly and surprisingly, these two subjects are characterised and linked by a certain intrinsic vector-valued quadratic form that can be associated to an affine connection control system.

Key words. affine connection control systems, controllability, mechanics, driftless systems

AMS subject classifications. 70Q05, 93B03, 93B05, 93B29

1. Introduction.

The determination of useful necessary and sufficient conditions for local controllability of nonlinear systems remains an open problem, although significant progress has been made [2, 4, 19, 20, 34, 36]. In this paper, we investigate local controllability for a class of nonlinear systems with a rich geometric structure, namely affine connection control systems. For these systems, we provide first-order (in the sense that the conditions involve first derivatives of the system data) local controllability conditions. The results use a certain intrinsic vector-valued quadratic form. The use of vector-valued quadratic forms in control theory has been noticed in the context of optimal control (which has, of course, a relationship with controllability) by Agrachev [3], and they have been utilised explicitly for providing conditions for local controllability by Basto-Gonçalves [6] and Hirschorn and Lewis [21]. Other uses of vector-valued quadratic forms in control are outlined in the paper [10]. The controllability conditions we provide in Section 4 bear strong resemblance to the more general conditions of Hirschorn and Lewis [21], but we are able to provide more detail in this case because of the additional structure of the class of systems under consideration.

Affine connection control systems are a slight generalisation of a class of mechanical control systems, namely those which are Lagrangian with kinetic energy Lagrangian, and possibly with nonholonomic constraints. An initial systematic investigation of the local controllability properties of this class of systems was undertaken by Lewis and Murray [27]. The conditions for local accessibility in this work are characterised geometrically by the same authors [28] by utilising the characterisation of the so-called symmetric product provided by Lewis [24]. However, the sufficient conditions for local controllability provided by Lewis and Murray, following Sussmann [36], are not entirely satisfactory. One of the reasons for this is that these conditions are not feedback-invariant. The consequences of the lack of feedback invariance can be seen even in very simple examples, where a system can fail the sufficient condition test, but still be controllable. This points out the need to better understand local controllability, and one way to do this is to obtain conditions which are not depen-

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dent on a choice of basis for the input distribution. It is this that we do in this paper, at least for systems whose controllability can be determined by brackets of low-order.

A second objective of this paper is to characterise affine connection control systems in terms of equivalent lower-dimensional kinematic (or driftless) systems. The interest in low-complexity representations of affine connection control systems can be related to numerous previous efforts, including work on hybrid models for motion control systems [9], motion description languages [30], consistent control abstractions [32], hierarchical steering algorithms [31], and maneuver automata [18]. The key advantage of a low-complexity or reduced-order representation is the subsequent simplification of various control problems, including planning, stabilisation, and optimal control.

In Section 5, we introduce and characterise the notion of kinematic reductions as a reduced-order modelling technique adapted to affine connection control systems. This novel concept extends and unifies previous results by Lewis [25] and Bullo and Lynch [13]; see also the motivating work [5, 29, 15]. A kinematic model for an affine connection control system is one such that every controlled trajectory for the kinematic model can be realised as a trajectory, with a possible reparameterisation, of the full affine connection control system with some appropriate control. We also introduce and characterise the notion of maximally reducible affine connection control systems. For such systems, every trajectory of the affine connection control system, starting from initial velocities in the input distribution, can be implemented as a controlled trajectory of a maximal kinematic reduction. Some open problems concerning inverse kinematics and sufficient conditions for controllability are presented by Cortés, Martínez, and Bullo [16].

As a third contribution of this paper, the existence of, and the controllability properties of, kinematic reductions are related to the low-order controllability properties of the corresponding affine connection control system. Interestingly, all these concepts are characterised in terms of the vector-valued quadratic form mentioned above. Insightful relationships are established and presented in Figure 5.4. We illustrate our results with some example systems. For instance, it appears that numerous (but not all) interesting mechanical devices satisfying the low-order sufficient controllability condition are also kinematically controllable. This is surprising because the concept of kinematic controllability is not *a priori* related to the conditions for low-order controllability. We refer to [12] for a catalog of examples.

One of the byproducts of the intrinsic formulation of the controllability and kinematic reduction results we give is that they give a fairly complete characterisation of what can be done. The incompleteness of the characterisations we give results from a possible degeneracy of the vector-valued quadratic forms. Here, one will generally have to go to higher-order conditions for controllability. Sometimes it is possible to give results using quadratic forms, even in degenerate cases, and this is being explored in a paper by Tyner and Lewis [39], currently in preparation.

Let us briefly describe the layout of the paper. We begin in Section 2 with a general discussion of affine connection control systems, giving clear statements of the results of Lewis and Murray [27]. Background on vector-valued quadratic forms is presented in Section 3, along with the construction of a vector-valued quadratic form that can be associated with an affine connection control system. Our controllability results are motivated, stated, and proved in Section 4. Similarly, our kinematic reductions are discussed in Section 5. In this section are also presented a couple of physical examples, and a discussion of the relationships between low-order controllability and kinematic reductions.

2. Affine connection control systems. The basic differential geometric notation we use is that of [1]. When it is convenient to do so, we shall use the summation convention where summation over repeated indices is implied. For a vector bundle $\pi\colon E \to Q$, 0_q will denote the zero vector in the fibre E_q. Objects will be assumed real analytic (which we simply call "analytic") unless otherwise stated. We denote by $\Gamma(E)$ the set of analytic sections of the vector bundle $\pi\colon E \to Q$. Thus, in particular, $\Gamma(TQ)$ is the set of analytic vector fields on a manifold Q. The set of analytic functions on a manifold Q we denote by $\mathcal{F}(Q)$. We will assume the reader to be familiar with affine differential geometry to the extent that it is used in [27]. An excellent reference is [22]. Affine connection control systems represent a class of mechanical control systems. We shall not devote any space to the physics involved in this representation, but refer to [27] for a few words along these lines. These issues are addressed also in the books [8, 11].

We begin with the essential definitions for affine connection control systems, and provide definitions for what Lewis and Murray call "configuration controllability." Then we give the results of those authors which provide a launching point for what we do in the present paper. We provide fairly strong statements of the results of Lewis and Murray; stronger in fact than the original statements. All that we say, however, is readily implicit in the calculations of their original work.

2.1. Basic definitions. In this paper, an *affine connection control system* is a 5-tuple $\Sigma = (Q, \nabla, \mathsf{D}, \mathscr{Y}, U)$ where

1. Q is a analytic, finite-dimensional, manifold,
2. ∇ is an analytic affine connection on Q,
3. D is a constant rank analytic distribution on Q having the property that ∇ restricts to D (i.e., $\nabla_X Y \in \Gamma(\mathsf{D})$ for all $Y \in \Gamma(\mathsf{D})$ and for all $X \in \Gamma(TQ)$),
4. $\mathscr{Y} = \{Y_1, \dots, Y_m\}$ is a collection of analytic vector fields on Q taking values in D, and
5. $U \subset \mathbb{R}^m$.

The distribution D will not concern us much here, and we allow it in order to correctly model systems with nonholonomic constraints [26]. The essential geometry of our results are captured by thinking of $\mathsf{D} = TQ$. We will frequently be interested only in 4-tuples $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ satisfying the above conditions. Let us therefore agree to call this an *affine connection pre-control system*. This notion will be useful in discussions of properties of affine connection control systems that are independent of the control set U.

Associated with an affine connection control system $\Sigma = (Q, \nabla, \mathsf{D}, \mathscr{Y}, U)$ is the set of second-order control equations

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^{m} u^a(t) Y_a(\gamma(t)) \tag{2.1}$$

on Q. Thus a *controlled trajectory* for Σ is taken to be a pair (γ, u) where

1. $\gamma\colon I \to Q$ and $u\colon I \to U$ are both defined on the same interval $I \subset \mathbb{R}$,
2. u is locally integrable,
3. $\gamma'(t) \in \mathsf{D}_{\gamma(t)}$ for a.e. $t \in I$, and
4. (γ, u) together satisfy (2.1).

We denote by $\operatorname{conv}(U)$ and $\operatorname{aff}(U)$ the convex hull and affine hull, respectively, of $U \subset \mathbb{R}^m$. Thus $\operatorname{conv}(U)$ is the smallest convex set in $\mathbb{R}^m$ containing U, and $\operatorname{aff}(U)$ is the smallest affine subspace (i.e., shifted subspace) containing U. The control set

U is *proper* (resp. *almost proper*) if $0 \in \operatorname{int}(\operatorname{conv}(U))$ (resp. if $\operatorname{aff}(U) = \mathbb{R}^m$ and $0 \in \operatorname{conv}(U)$). (One may verify that for a control-affine system the property of the control set being almost proper is exactly that which ensures that the Lie algebra rank condition is equivalent to the reachable set having nonempty interior.) We denote by Y the input distribution, so that

$$\mathsf{Y}_q = \operatorname{span}_{\mathbb{R}}\{Y_1(q), \dots, Y_m(q)\}.$$

More generally if $\mathscr{V} \subset \Gamma(TQ)$ then we denote by V the distribution generated by the vector fields $\mathscr{V}$: $\mathsf{V}_q = \operatorname{span}_{\mathbb{R}}\{X(q) \mid X \in \mathscr{V}\}$. We also denote by $\Gamma(\mathsf{V})$ the set of analytic vector fields taking values in V. We make no *a priori* assumptions on the constancy of the rank of any of the distributions we encounter, including the input distribution Y.

Remark 2.1. Our allowing a distribution to have variable rank has consequences for the choice of generators. Let us make some comments on this. Consider a family $\mathscr{Y}$ of analytic vector fields, letting Y be the distribution generated as above. Then $\Gamma(\mathsf{Y})$ is a submodule of $\Gamma(TQ)$. If Y has constant rank, then it is true that the vector fields $\mathscr{Y}$ generate this submodule. This is essentially due to a theorem of Swan [38]. However, if the rank of Y is *not* constant (more precisely, locally constant), then it can be the case that the vector fields $\mathscr{Y}$ are *not* generators for $\Gamma(\mathsf{Y})$. However, we shall require always ask that our families of vector fields have the property that they are generators for the submodule of sections of the induced distribution. Locally, and in the analytic setting, this can be done without loss of generality, due to the Noetherian property of the ring of analytic functions. •

Let us clearly state our controllability definitions. First we provide notation for the reachable sets. For $T > 0$ and $q_0 \in Q$, let

$$\mathcal{R}^{\Sigma}_{TQ}(q_0, T) = \{\gamma'(T) \mid (\gamma, u) \text{ is a controlled trajectory on } [0,T] \text{ with } \gamma'(0) = 0_{q_0}\}$$

and let $\mathcal{R}^{\Sigma}_{TQ}(q_0, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}^{\Sigma}_{TQ}(q_0, t)$. These are therefore reachable *states* in TQ starting from zero initial velocity at the configuration q_0. We also consider the reachable configurations which we denote by

$$\mathcal{R}^{\Sigma}_{Q}(q_0, T) = \tau_Q(\mathcal{R}^{\Sigma}_{TQ}(q_0, T)), \qquad \mathcal{R}^{\Sigma}_{Q}(q_0, \leq T) = \tau_Q(\mathcal{R}^{\Sigma}_{TQ}(q_0, \leq T)),$$

where $\tau_Q \colon TQ \to Q$ is the tangent bundle projection. Note that, since D is invariant under ∇ and since the input vector fields are D-valued, solutions of (2.1) with initial conditions in D remain in D. In the following definition, $\operatorname{int}_{\mathsf{D}}(\cdot)$ means the interior in the relative topology on $\mathsf{D} \subset TQ$.

Definition 2.2. Let $\Sigma = (Q, \nabla, \mathsf{D}, \mathscr{Y}, U)$ be an affine connection control system and let $q_0 \in Q$.

(i) $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ is *accessible* from q_0 if, for every almost proper control set, there exists $T > 0$ such that $\operatorname{int}_{\mathsf{D}}(\mathcal{R}^{\Sigma}_{TQ}(q_0, \leq t)) \neq \emptyset$ for $t \in (0, T]$.

(ii) $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ is *configuration accessible* from q_0 if, for every almost proper control set, there exists $T > 0$ such that $\operatorname{int}(\mathcal{R}^{\Sigma}_{Q}(q_0, \leq t)) \neq \emptyset$ for $t \in (0, T]$.

(iii) Σ is *small-time locally controllable* (*STLC*) from q_0 if there exists $T > 0$ such that $0_{q_0} \in \operatorname{int}_{\mathsf{D}}(\mathcal{R}^{\Sigma}_{TQ}(q_0, \leq t)) \neq \emptyset$ for $t \in (0, T]$.

(a) $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ is *properly small-time locally controllable* (*properly STLC*) from q_0 if Σ is STLC from q_0 for every proper control set U.

(b) $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ is *small-time locally uncontrollable* (*STLUC*) from q_0 if Σ is not STLC from q_0 for any compact control set U.

(iv) Σ is *small-time locally configuration controllable* (*STLCC*) from q_0 if there exists $T > 0$ such that $0_{q_0} \in \operatorname{int}(\mathcal{R}_Q^\Sigma(q_0, \leq t)) \neq \emptyset$ for $t \in (0, T]$.

(a) $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ is *properly small-time locally configuration controllable* (*properly STLCC*) from q_0 if Σ is STLCC from q_0 for every proper control set U.

(b) $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ is *small-time locally configuration uncontrollable* (*STLCUC*) from q_0 if Σ is not STLCC from q_0 for any compact control set U. •

Remarks 2.3.

1. Note that we are careful in these definitions to distinguish those notions of controllability that depend only on the geometry of the affine connection pre-control system $(Q, \nabla, \mathsf{D}, \mathscr{Y})$, and those that also depend on the character of the control set U. Hirschorn and Lewis [21] illustrate various situations where the exact nature of the control set must be accounted for in the controllability analysis. For this reason we try to be careful about the exact manner in which the control set is considered.

2. A consequence of the classical theory of accessibility [37] is that for an affine connection pre-control system $(Q, \nabla, \mathsf{D}, \mathscr{Y})$, the reachable sets for $(Q, \nabla, \mathsf{D}, \mathscr{Y}, U)$ have nonempty interior for *all* almost proper control sets if and only if the reachable sets have nonempty interior for *some* almost proper control set.

3. It is clear that STLC implies STLCC and that STLCUC implies STLUC. The converse implications are generally false. What's more, even the relationships between STLCC and STLC *on the reachable set* are not completely understood at this time. •

2.2. Review of existing results. Let us briefly review the results of [27]. These results rely for their statement on the *symmetric product* defined by the affine connection ∇ by $\langle X : Y \rangle = \nabla_X Y + \nabla_X Y$. First let us provide a description of the set of points accessible from the zero vector 0_q in the tangent space $T_q Q$. We let $\Sigma = (Q, \nabla, \mathsf{D}, \mathscr{Y}, U)$ be an affine connection control system. As above, we denote by Y the distribution generated by the vector fields $\mathscr{Y}$, and we now define a sequence $\mathsf{Sym}^{(k)}(\mathsf{Y})$, $k \in \mathbb{N}$, of distributions by

$$\begin{aligned}
&\mathsf{Sym}^{(1)}(\mathsf{Y})_q = \mathsf{Y}_q + \operatorname{span}_{\mathbb{R}}\{\langle Y_a : Y_b \rangle |\ a, b \in \{1, \dots, m\}\},\\
&\mathsf{Sym}^{(k)}(\mathsf{Y})_q = \mathsf{Sym}^{(k-1)}(\mathsf{Y})_q\\
&\quad + \operatorname{span}_{\mathbb{R}}\{\langle Y_a : Y_b \rangle |\ Y_a \in \Gamma(\mathsf{Sym}^{(k_1)}(\mathsf{Y})),\ Y_b \in \Gamma(\mathsf{Sym}^{(k_2)}(\mathsf{Y})),\ k_1 + k_2 = k - 1\}.
\end{aligned}$$

The smallest distribution containing these distributions we denote by $\mathsf{Sym}^{(\infty)}(\mathsf{Y})$, and we note that $\langle X : Y \rangle \in \Gamma(\mathsf{Sym}^{(\infty)}(\mathsf{Y}))$ for each $X, Y \in \Gamma(\mathsf{Sym}^{(\infty)}(\mathsf{Y}))$. The integrable distribution generated by $\mathsf{Sym}^{(\infty)}(\mathsf{Y})$ we denote by $\mathsf{Lie}^{(\infty)}(\mathsf{Sym}^{(\infty)}(\mathsf{Y}))$. Since this distribution is integrable, through each point $q_0 \in Q$ there is an immersed maximal integral manifold Λ_{q_0} with the property that $T_q \Lambda_{q_0} = \mathsf{Lie}^{(\infty)}(\mathsf{Sym}^{(\infty)}(\mathsf{Y}))_q$ for each $q \in \Lambda_{q_0}$. Note that since we are only thinking of local controllability, we may shrink Q so that Λ_{q_0} is an embedded submanifold, and thus $T_q \Lambda_{q_0}$ has its usual definition.

With this notation, we have the following theorem which describes the reachable set from $0_{q_0} \in TQ$. Note that the description we provide here is a little more complete than that originally given by Lewis and Murray, but what we state here is certainly implicit in the original paper.

Theorem 2.4. *Let $\Sigma = (Q, \nabla, \mathsf{D}, \mathscr{Y}, U)$ be an affine connection control system with U almost proper. Let Λ_{q_0} be the maximal integral manifold of $\mathsf{Lie}^{(\infty)}(\mathsf{Sym}^{(\infty)}(\mathsf{Y}))$ through $q_0 \in Q$, which we assume without loss of generality to be an embedded sub-*

manifold of Q. Let $\mathsf{S}(\mathsf{Y}, q_0)$ be the vector bundle over Λ_{q_0} whose fibre at $q \in \Lambda_{q_0}$ is $\mathtt{Sym}^{(\infty)}(\mathsf{Y})_q$. We have the following statements.

(i) *There exists $T > 0$ such that for each $t \in (0, T]$, $\mathcal{R}^{\Sigma}_{TQ}(q_0, \leq t)$ is contained in $\mathsf{S}(\mathsf{Y}, q_0)$, and contains a nonempty open subset of $\mathsf{S}(\mathsf{Y}, q_0)$.*

(ii) *In particular, there exists $T > 0$ such that for each $t \in (0, T]$, $\mathcal{R}^{\Sigma}_{Q}(q_0, \leq t)$ is contained in Λ_{q_0} and contains a nonempty open subset of Λ_{q_0}.*

Theorem 2.4 obviously leads to the following corollary.

Corollary 2.5. *An affine connection pre-control system $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ is*

(i) *accessible from q_0 if and only if $\mathtt{Sym}^{(\infty)}(\mathsf{Y})_{q_0} = \mathsf{D}_{q_0}$, and is*

(ii) *configuration accessible from q_0 if and only if $\mathtt{Lie}^{(\infty)}(\mathtt{Sym}^{(\infty)}(\mathsf{Y}))_{q_0} = T_{q_0}Q$.*

Now we turn to local configuration controllability. Let $P(\mathscr{Y})$ denote the set of iterated symmetric products of vector fields in $\mathscr{Y}$. A product $P_0 \in P(\mathscr{Y})$ is *bad* when it is comprised of an even number of each of the vector fields from $\mathscr{Y}$, and is otherwise *good*. The *degree* of $P_0 \in P(\mathscr{Y})$ is the total number of vector fields from $\mathscr{Y}$ which participate in P_0, counting multiplicities. Thus, for example, $\langle Y_a : \langle Y_b : Y_b\rangle\rangle$ is good and of degree 3, and $\langle\langle Y_a : Y_b\rangle : \langle Y_a : Y_b\rangle\rangle$ is bad and of degree 4. Let S_m be the symmetric group on m symbols. For $P_0 \in P(\mathscr{Y})$ and $\sigma \in S_m$, let $\sigma(P_0) \in P(\mathscr{Y})$ be obtained by permuting the occurrences of the vector fields from $\mathscr{Y}$ by σ. For example, if $P_0 = \langle Y_a : \langle Y_b : Y_c\rangle\rangle$ and if $\sigma = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right)$ then $\sigma(P_0) = \langle Y_b : \langle Y_c : Y_a\rangle\rangle$. With this notation, we have the following definition.

Definition 2.6. An affine connection pre-control system $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ *satisfies the good/bad hypothesis at* q_0 if, for each bad symmetric product $P_0 \in P(\mathscr{Y})$, there exist good symmetric products $P_1, \dots, P_k \in P(\mathscr{Y})$ of degree strictly less than P_0 and such that

$$\sum_{\sigma \in S_m} \sigma(P_0)(q_0) = \sum_{j=1}^{k} c_j P_j(q_0),$$

for some $c_1, \dots, c_k \in \mathbb{R}$. •

The following result of Lewis and Murray [27] is derived from a result of Sussmann [36]. Again, we provide a somewhat more thorough statement of the result than is given in [27].

Theorem 2.7. *Let $\Sigma = (Q, \nabla, \mathsf{D}, \mathscr{Y}, U)$ be an affine connection control system with U proper, and let $q_0 \in Q$. If $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ satisfies the good/bad hypothesis at $q_0 \in Q$ then there exists $T > 0$ such that for each $t \in (0, T]$ the set $\mathcal{R}^{\Sigma}_{TQ}(q_0, \leq t)$ contains a neighbourhood of 0_{q_0} in the vector bundle $\mathsf{S}(\mathscr{Y}, q_0)$ over Λ_{q_0}.*

The result essentially says that when the good/bad hypothesis is satisfied, the system is locally controllable when restricted to its reachable set. In particular, we have the following corollary.

Corollary 2.8. *Let $\Sigma = (Q, \nabla, \mathsf{D}, \mathscr{Y}, U)$ be an affine connection control system with U proper and such that the pre-control system $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ satisfies the good/bad hypotheses at $q_0 \in Q$. Then*

(i) *Σ is locally controllable at q_0 if it is locally accessible at q_0, and*

(ii) *Σ is locally configuration controllable at q_0 if it is locally configuration accessible at q_0.*

The above results all follow from a detailed analysis of the Lie algebra of vector fields associated with the control system (2.1) when it is thought of as a control-affine system with state manifold TQ. The results reflect the fact that, when evaluated at zero velocity points, this Lie algebra structure simplifies enormously. We shall exploit this further when we prove our main results in Section 4. We remark that the structure of the Lie algebra at points of nonzero velocity is not currently well understood.

3. Vector-valued quadratic forms. In our controllability analysis we are led to investigate symmetric bilinear maps $B\colon V \times V \to W$ from a finite-dimensional $\mathbb{R}$-vector space V into a finite-dimensional $\mathbb{R}$-vector space W. In this section we first look at such objects in general, and then we construct a specific such object associated to an affine connection control system. Some other control theoretic problems where vector-valued quadratic forms arise are given by Bullo, Cortés, Lewis, and Martínez [10].

3.1. Basic definitions and properties. Let V and W be finite-dimensional $\mathbb{R}$-vector spaces and let $\Sigma_2(V;W)$ denote the set of symmetric $\mathbb{R}$-bilinear maps from $V \times V$ to W. For $B \in \Sigma_2(V;W)$, we define $Q_B\colon V \to W$ by $Q_B(v) = B(v,v)$. For $\lambda \in W^*$, we define $\lambda B\colon V \times V \to \mathbb{R}$ by $\lambda B(v_1, v_2) = \langle \lambda; B(v_1, v_2)\rangle$.

Definition 3.1. Let $B \in \Sigma_2(V;W)$.

(i) B is *definite* if there exists $\lambda \in W^*$ such that λB is positive-definite.
(ii) B is *essentially indefinite* if, for each $\lambda \in W^*$, λB is either
  (a) zero or
  (b) neither positive nor negative-semidefinite.

The following properties of symmetric bilinear maps will be important for us. The proof follows more or less directly from the definitions.

Lemma 3.2. *Let V and W be finite-dimensional $\mathbb{R}$-vector spaces with $B \in \Sigma_2(V;W)$. Suppose that $V \neq \{0\}$. The following statements hold:*

(i) *if $W = \{0\}$, then B is essentially indefinite;*
(ii) *if $W \neq \{0\}$, then B is essentially indefinite if and only if*

$$0 \in \mathrm{int}_{\mathrm{aff}(\mathrm{image}(Q_B))}(\mathrm{conv}(\mathrm{image}(Q_B)));$$

(iii) *if $W \neq \{0\}$, then B is definite if and only if there exists a hyperplane P through $0 \in W$ such that*
  (a) $\mathrm{image}(Q_B)$ *lies on one side of P and*
  (b) $\mathrm{image}(Q_B) \cap P = \{0\}$.

The matter of deciding whether a vector-valued quadratic form is essentially indefinite is known to be NP-complete, at least in the case when $\dim(W) > 1$.[1]

The following result gives some properties of $\mathbb{R}$-valued quadratic forms that will be useful in our discussion. We refer to Hirschorn and Lewis [21] for a proof.

Lemma 3.3. *Let V be a finite-dimensional $\mathbb{R}$-vector space and let $B \in \Sigma_2(V;\mathbb{R})$. For a basis $\mathscr{V} = \{v_1, \dots, v_n\}$ for V, let $[B]_{\mathscr{V}}$ be the $n \times n$ matrix representation of B. The following statements are equivalent:*

(i) *there exists a basis $\mathscr{V}$ for V for which the sum of the diagonal entries in the matrix $[B]_{\mathscr{V}}$ is zero;*
(ii) *there exists a basis $\mathscr{V}$ for V for which the diagonal entries in the matrix $[B]_{\mathscr{V}}$ are all zero;*
(iii) *B is essentially indefinite.*

3.2. Vector-valued quadratic forms and affine connection control systems. Let $\Sigma = (Q, \nabla, \mathrm{D}, \mathscr{Y}, U)$ be an affine connection control system and let $q \in Q$.

[1] This was pointed out to the authors by a reviewer for [10].

If $S_q \subset T_qQ$ is a subspace, then we define $B_{\mathsf{Y}_q}(S_q)\colon \mathsf{Y}_q \times \mathsf{Y}_q \to T_qQ/S_q$ as the T_qQ/S_q-valued symmetric, bilinear mapping on Y_q given by

$$B_{\mathsf{Y}_q}(S_q)(v_1, v_2) = \pi_{S_q}(\langle V_1 : V_2\rangle(q)), \tag{3.1}$$

where V_1 and V_2 are vector fields extending $v_1, v_2 \in \mathsf{Y}_q$, and where $\pi_{S_q}\colon T_qQ \to T_qQ/S_q$ is the canonical projection. Note that $B_{\mathsf{Y}_q}(S_q)$ is not necessarily well-defined.

Lemma 3.4. *If $\mathsf{Y}_q \subset S_q$ then $B_{\mathsf{Y}_q}(S_q)$ is well-defined.*

Proof. We need to show that the definition in (3.1) does not depend on the extensions V_1 and V_2 of v_1 and v_2. This will follow if $\pi_{S_q}(\langle V_1 : V_2\rangle(q))$ depends only on the values of V_1 and V_2 at q, and not on their derivatives. Let $\phi_1, \phi_2 \in \mathcal{F}(Q)$ and compute

$$\langle \phi_1 V_1 : \phi_2 V_2\rangle = \phi_1\phi_2\langle V_1 : V_2\rangle + \phi_1(\mathscr{L}_{V_1}\phi_2)V_2 + \phi_2(\mathscr{L}_{V_2}\phi_1)V_1.$$

Thus $\pi_{S_q}(\langle \phi_1 V_1 : \phi_2 V_2\rangle(q)) = \phi_1(q)\phi_2(q)\pi_{S_q}(\langle V_1 : V_2\rangle(q))$, showing that $\pi_{S_q}(\langle V_1 : V_2\rangle(q))$ does not depend on the derivatives of V_1 and V_2 at q, and so the result follows. □

Remark 3.5. Note that $(T_qQ/S_q)^* \simeq \operatorname{ann}(S_q)$. Therefore, the definition of $\lambda B_{\mathsf{Y}_q}(S_q)$, $\lambda \in (T_qQ/S_q)^*$ is concrete, in that one need to worry about objects in the quotient. •

If Y has constant rank, then one can define a TQ/Y-valued quadratic form B_{Y} globally by

$$B_{\mathsf{Y}}(V_1, V_2) = \pi_{\mathsf{Y}}(\langle V_1 : V_2\rangle)$$

for $V_1, V_2 \in \Gamma(\mathsf{Y})$ where $\pi_{\mathsf{Y}}\colon TQ \to TQ/\mathsf{Y}$ is the projection.

4. Controllability results. In this section we undertake the formulation and discussion of novel controllability results. Our objective is to obtain controllability conditions that are independent of the basis for the input distribution Y. We achieve this by means of controllability tests that do not entail good/bad conditions, but rather are expressed in terms of properties of a vector-valued quadratic form. Before we state the results we need some preliminary constructions.

4.1. Constructions concerning vanishing input vector fields. We let $\Sigma = (Q, \nabla, \mathsf{D}, \mathscr{Y}, U)$ be an analytic affine connection control system and we let $q_0 \in Q$. One of the generalisations we wish to allow is the case when q_0 may not a regular point for the distribution Y generated by $\mathscr{Y}$. In this case the vector fields $\mathscr{Y}$ cannot be linearly independent at q_0. It may also happen that, even if q_0 *is* a regular point for Y, the vector fields may still not be linearly independent. For example, if one wishes to globally define a control system for which the input distribution Y has constant rank, but is not trivial, then one will necessarily have to choose more input vector fields than $\operatorname{rank}(\mathsf{Y})$, implying that the input vector fields will never be linearly independent. It will be convenient to organise the vector fields in $\mathscr{Y}$ in a manner consistent with these possibilities. The following result gives a useful way of doing this.

Lemma 4.1. *Let $(Q, \nabla, \mathsf{D}, \mathscr{Y} = \{Y_1, \dots, Y_m\})$ be an analytic affine connection pre-control system with $q_0 \in Q$. There exists $\boldsymbol{T} \in GL(m; \mathbb{R})$ with the property that, if $\tilde{Y}_a = T^b_a Y_a$, $a \in \{1, \dots, m\}$, then*

(i) *$\{\tilde{Y}_1(q_0), \dots, \tilde{Y}_k(q_0)\}$ form a basis for Y_{q_0} and*
(ii) *the vector fields $\tilde{Y}_{k+1}, \dots, \tilde{Y}_m$ vanish at q_0.*

Proof. We let $k = \dim(\mathsf{Y}_{q_0})$. Since $\mathscr{Y}$ generates Y, we may find $\boldsymbol{R} \in GL(m;\mathbb{R})$ with the property that, if $X_a = R^b_a Y_b$, $a \in \{1,\dots,m\}$, then $\{X_1(q_0),\dots,X_k(q_0)\}$ form a basis for Y_{q_0}. Now let $L_{q_0} : \mathbb{R}^m \to \mathsf{Y}_{q_0}$ be defined by $L_{q_0}(\boldsymbol{u}) = \sum_{a=1}^m u^a X_a(q_0)$. Let $\boldsymbol{u}_{k+1},\dots,\boldsymbol{u}_m \in \mathbb{R}^m$ be a basis for $\ker(L_{q_0})$ and define $\boldsymbol{S} \in GL(m;\mathbb{R})$ by

$$\boldsymbol{S} = \left[\ \boldsymbol{e}_1 \,\middle|\, \cdots \,\middle|\, \boldsymbol{e}_k \,\middle|\, \boldsymbol{u}_{k+1} \,\middle|\, \cdots \,\middle|\, \boldsymbol{u}_m\ \right].$$

It is then clear that if we take $\tilde{Y}_a = S^b_a X_b$, $a \in \{1,\dots,m\}$, then $\{\tilde{Y}_1(q_0),\dots,\tilde{Y}_k(q_0)\}$ form a basis for Y_{q_0}, and that $\tilde{Y}_{k+1},\dots,\tilde{Y}_m$ vanish at q_0. Now we take $\boldsymbol{T} = \boldsymbol{RS}$. □

REMARKS 4.2.

1. If the vector fields $\mathscr{Y}$ are linearly independent at q_0 then one may take $\boldsymbol{T} = \boldsymbol{I}_m$ in the lemma.
2. Suppose that we have a control set U for $(Q,\nabla,\mathsf{D},\mathscr{Y})$. If we take $\boldsymbol{T} \in GL(m;\mathbb{R})$ and $\tilde{\mathscr{Y}} = \{\tilde{Y}_1,\dots,\tilde{Y}_m\}$ as in the lemma, and if we define $\tilde{U} = \{\boldsymbol{T}^{-1}\boldsymbol{u} \mid\ \boldsymbol{u} \in U\}$, this gives an affine connection control system $\tilde{\Sigma} = (Q,\nabla,\mathsf{D},\tilde{\mathscr{Y}},\tilde{U})$. Clearly the controlled trajectories for $\Sigma = (Q,\nabla,\mathsf{D},\mathscr{Y},U)$ and $\tilde{\Sigma}$ agree, so we can without loss of generality assume that the input vector fields for an affine connection control system satisfy conditions (i) and (ii) of the lemma. Input vector fields satisfying these conditions at q_0 will be said to be *adapted at* q_0. •

Let $X, Y \in \Gamma(Q)$. If $X(q_0) = 0_{q_0}$ then the expression $\langle X : Y\rangle(q_0)$ may be verified (in coordinates, for example) to depend only on the value of Y at q_0. That is to say, we may define a linear map $\mathrm{sym}_X : T_{q_0}Q \to T_{q_0}Q$ by $v \mapsto \langle X : V\rangle(q_0)$ where V is any extension of $v \in T_{q_0}Q$. If $\mathscr{Y}$ is adapted at q_0, then we denote by $Z_{q_0}(\mathscr{Y})$ the set of linear maps sym_{Y_a}, $a \in \{k+1,\dots,m\}$, where $k = \dim(\mathsf{Y}_{q_0})$. For a $\mathbb{R}$-vector space W, an arbitrary subset $\mathscr{L}$ of linear transformations of W, and a subspace $S \subset W$, we denote by $\langle \mathscr{L}, S\rangle$ the smallest subspace of W containing S and which is an invariant subspace for each of the linear maps from $\mathscr{L}$. One readily verifies that $\langle \mathscr{L}, S\rangle$ is generated by vectors of the form

$$L_1 \circ \cdots \circ L_{k-1}(v), \qquad L_1,\dots,L_{k-1} \in \mathscr{L},\ v \in S,\ k \in \mathbb{N}. \tag{4.1}$$

We will be interested in subspaces of the form $\langle Z_{q_0}(\mathscr{Y}), S_{q_0}\rangle$ where S_{q_0} is a subspace of $T_{q_0}Q$. In order for such constructions to make sense (in that they are independent of the choice of adapted family of vector fields) the subspace S_{q_0} should have some properties.

LEMMA 4.3. *Let* $\Sigma = (Q,\nabla,\mathsf{D},\mathscr{Y},U)$ *and* $\tilde{\Sigma} = (Q,\nabla,\mathsf{D},\tilde{\mathscr{Y}},\tilde{U})$ *be affine connection control systems satisfying*

(i) $\mathsf{Y} = \tilde{\mathsf{Y}}$ *and*

(ii) $\mathscr{Y}$ *and* $\tilde{\mathscr{Y}}$ *are adapted at* q_0.

Then $\langle Z_{q_0}(\tilde{\mathscr{Y}}), S_{q_0}\rangle = \langle Z_{q_0}(\mathscr{Y}), S_{q_0}\rangle$ *for any subspace* S_{q_0} *containing* Y_{q_0}.

Proof. We write $\mathscr{Y} = \{Y_1,\dots,Y_m\}$ and $\tilde{\mathscr{Y}} = \{\tilde{Y}_1,\dots,\tilde{Y}_{\tilde{m}}\}$. Since $\mathsf{Y} = \tilde{\mathsf{Y}}$, we must have

$$\tilde{Y}_\alpha = \sum_{a=1}^m \Lambda^a_\alpha Y_a, \qquad \alpha \in \{1,\dots,\tilde{m}\}$$

for functions $\Lambda^a_\alpha : Q \to \mathbb{R}$, $a \in \{1,\dots,m\}$, $\alpha \in \{1,\dots,\tilde{m}\}$. (Here we make use of the assumption stated in Remark 2.1.) Assume that $\dim(\mathsf{Y}_{q_0}) = k$ so that both $\{Y_1(q_0),\dots,Y_k(q_0)\}$ and $\tilde{Y}_1(q_0),\dots,\tilde{Y}_k(q_0)\}$ are bases for Y_{q_0} and so that $Y_{k+1},\dots,Y_m$ and $\tilde{Y}_{k+1},\dots,\tilde{Y}_{\tilde{m}}$ all vanish at q_0. Note that $\langle Z_{q_0}(\mathscr{Y}), S_{q_0}\rangle$ is generated by those tangent vectors at q_0 of the form

$$\mathrm{sym}_{Y_{a_{\ell-1}}} \circ \cdots \circ \mathrm{sym}_{Y_{a_1}}(v), \qquad a_1, \ldots, a_{\ell-1} \in \{k+1, \ldots, m\},\ \ell \in \mathbb{N},\ v \in S_{q_0}.$$

We will show by induction on ℓ that each of these generators lies in $\langle Z_{q_0}(\tilde{\mathscr{Y}}), S_{q_0}\rangle$. This is clearly true for $\ell = 1$, so suppose it true for $\ell = j$ and let $a_j \in \{k+1, \ldots, m\}$. Then. for any $V \in \Gamma(TQ)$, we have

$$\langle Y_{a_j} : V\rangle = \langle \Lambda^\alpha_{a_j}(\tilde{Y}_\alpha) : V\rangle = \Lambda^\alpha_a \langle \tilde{Y}_\alpha : V\rangle + \sum_{\alpha=1}^{\tilde{m}} (\mathscr{L}_V \Lambda^\alpha_{a_j})\tilde{Y}_\alpha,$$

from which we ascertain that

$$\mathrm{sym}_{Y_{a_j}} = \sum_{\alpha=k+1}^{\tilde{m}} \Lambda^\alpha_{a_j}(q_0)\mathrm{sym}_{\tilde{Y}_\alpha} + \sum_{\alpha=1}^{k} \tilde{Y}_\alpha(q_0) \otimes \boldsymbol{d}^\alpha_{a_j}(q_0),$$

since $\Lambda^\alpha_a(q_0) = 0$ for $\alpha \in \{1, \ldots, k\}$ and $a \in \{k+1, \ldots, m\}$. Therefore, by the induction hypothesis, we conclude that

$$\mathrm{sym}_{Y_{a_j}} \circ \mathrm{sym}_{Y_{a_{j-1}}} \circ \cdots \circ \mathrm{sym}_{Y_{a_1}}(v) \in \langle Z_{q_0}(\tilde{\mathscr{Y}}), S_{q_0}\rangle.$$

This shows that $\langle Z_{q_0}(\mathscr{Y}), S_{q_0}\rangle \subset \langle Z_{q_0}(\tilde{\mathscr{Y}}), S_{q_0}\rangle$. The opposite inclusion follows as above, but swapping $\mathscr{Y}$ and $\tilde{\mathscr{Y}}$. □

The preceding result shows the invariance of the definition of a subspace on the choice of adapted generators for Y. The next result gives the same conclusion for a vector-valued quadratic form.

LEMMA 4.4. *Let $\Sigma = (Q, \nabla, \mathsf{D}, \mathscr{Y}, U)$ and $\tilde{\Sigma} = (Q, \nabla, \mathsf{D}, \tilde{\mathscr{Y}}, \tilde{U})$ be affine connection control systems satisfying*

(i) $\mathsf{Y} = \tilde{\mathsf{Y}}$ *and*
(ii) $\mathscr{Y}$ *and* $\tilde{\mathscr{Y}}$ *are adapted at* q_0.

If $S_{q_0} \subset T_{q_0}Q$ is a subspace containing Y_{q_0}, then $B_{\tilde{\mathsf{Y}}_{q_0}}(S_{q_0}) = B_{\mathsf{Y}_{q_0}}(S_{q_0})$.

Proof. As in the proof of Lemma 4.3 we have

$$\tilde{Y}_\alpha = \sum_{a=1}^{m} \Lambda^a_\alpha Y_a, \qquad \alpha \in \{1, \ldots, \tilde{m}\},$$

for functions $\Lambda^a_\alpha \colon Q \to \mathbb{R}$, $a \in \{1, \ldots, m\}$, $\alpha \in \{1, \ldots, \tilde{m}\}$. We then compute

$$\langle Y_a : Y_b\rangle = \Lambda^\alpha_a \Lambda^\beta_b \langle \tilde{Y}_\alpha : \tilde{Y}_\beta\rangle + \sum_{\alpha,\beta=1}^{\tilde{m}} \Lambda^\beta_b (\mathscr{L}_{\tilde{Y}_\beta} \Lambda^\alpha_a)\tilde{Y}_\alpha + \sum_{\alpha,\beta=1}^{\tilde{m}} \Lambda^\alpha_a (\mathscr{L}_{\tilde{Y}_\alpha} \Lambda^\beta_b)\tilde{Y}_\beta + \Lambda^\alpha_a \Lambda^\beta_b S^\delta(\tilde{Y}_\alpha, \tilde{Y}_\beta)\tilde{Y}_\delta.$$

The lemma follows directly from this formula since the terms in $\Gamma(\mathsf{Y})$ will go to zero when projected by $\pi_{S_{q_0}}$ since $\mathsf{Y}_{q_0} \subset S_{q_0}$. □

4.2. Main results. Our main results may now be stated. Let us first state a sufficient condition for controllability.

THEOREM 4.5. *Let $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ be an analytic affine connection pre-control system, and suppose that $\mathscr{Y}$ is adapted at $q_0 \in Q$. Suppose that*

(i) $\mathsf{Sym}^{(\infty)}(\mathsf{Y})_{q_0} = \langle Z_{q_0}(\mathscr{Y}), \mathsf{Sym}^{(2)}(\mathsf{Y})\rangle$*, and that*

(ii) $B_{\mathsf{Y}_{q_0}}(\langle Z_{q_0}(\mathscr{Y}), \mathsf{Y}_{q_0}\rangle)$ *is essentially indefinite.*

Then $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ is properly STLC from q_0 if it is accessible from q_0, and is properly STLCC from q_0 if it is configuration accessible from q_0.

Proof. The proof essentially follows from Theorem 2.7. However, the extension to allow singular points for the input distribution Y does not follow directly from Theorem 2.7, but requires some manipulations with the variational cone that we will not go through here. The idea, in essence, is that if an input vector field vanishes at the reference point, then directions generated by symmetric products using these vector fields come "for free." Since these symmetric products are simply applications of a linear map, this explains the presence of the invariant subspace characterisations of the tangent space to the reachable set. We refer to [21, Lemma 7.2] for the details behind this, noting that the discussion in that paper builds on concepts presented in [36, 7]. A consequence of these discussions, once they are specialised to our setting, is the following result.

LEMMA 4.6. *Let $(Q, \nabla, \mathsf{D}, \mathscr{Y} = \{Y_1, \dots, Y_m\})$ be an analytic affine connection pre-control system for which $\mathscr{Y}$ is adapted at $q_0 \in Q$. Assume the following:*

(i) $\mathsf{Sym}^{(\infty)}(\mathsf{Y}) = \langle Z_{q_0}(\mathscr{Y}), \mathsf{Sym}^{(2)}(\mathsf{Y})_{q_0}\rangle$;

(ii) *there exists $\tilde{m} \geq m$ and a full rank matrix $\boldsymbol{T} \in \mathbb{R}^{m\times\tilde{m}}$ such that if $\tilde{Y}_\alpha = T^a_\alpha Y_\alpha$ then*

$$\sum_{a=1}^{\tilde{m}} \langle \tilde{Y}_\alpha : \tilde{Y}_\alpha\rangle(q_0) \in \langle Z_{q_0}(\mathscr{Y}), \mathsf{Y}_{q_0}\rangle.$$

Then $(Q, \nabla, \mathsf{D}, \mathscr{Y})$ is properly STLC from q_0 if it is accessible from q_0, and is properly STLCC from q_0 if it is configuration accessible from q_0.

We shall show that if the hypotheses of Theorem 4.5 are satisfied at q_0, then the hypotheses of Lemma 4.6 are satisfied for some possibly different collection of input vector fields. From this the conclusion of Theorem 4.5 will follow.

For brevity let us denote $S_{q_0} = \langle Z_{q_0}(\mathscr{Y}), \mathsf{Y}_{q_0}\rangle$ and $B = B_{\mathsf{Y}_{q_0}}(S_{q_0})$. First we need to find an appropriate collection of input vector fields. Choose $v_1, \dots, v_\ell \in \mathsf{Y}_{q_0}$ such that $0_{q_0} + S_{q_0} \in \mathsf{Sym}^{(\infty)}(\mathsf{Y})_{q_0}/S_{q_0}$ lies in the interior of the convex hull of the vectors $B(v_1, v_1), \dots, B(v_\ell, v_\ell)$. That this is possible is guaranteed by the hypotheses of Theorem 4.5 and by Lemma 3.2. If necessary, add vectors $v_{\ell+1}, \dots, v_{\tilde{k}}$ such that the vectors $v_1, \dots, v_{\tilde{k}}$ span Y_{q_0}. It now follows that the vectors $B(v_1, v_1), \dots, B(v_{\tilde{k}}, v_{\tilde{k}})$ contain $0_{q_0} + S_{q_0} \in \mathsf{Sym}^{(\infty)}(\mathsf{Y})_{q_0}/S_{q_0}$ in the interior of their convex hull. Thus the vectors $v_1, \dots, v_{\tilde{k}}$ may be rescaled by strictly positive constants (for simplicity, let us denote the rescaled vectors also by $v_1, \dots, v_{\tilde{k}}$) so that

$$\sum_{\alpha=1}^{\tilde{k}} B(v_\alpha, v_\alpha) = 0_{q_0} + S_{q_0} \in \mathsf{Sym}^{(\infty)}(\mathsf{Y})_{q_0}/S_{q_0}. \tag{4.2}$$

It is now possible to define vector fields $\tilde{\mathscr{Y}} = \{\tilde{Y}_1, \dots, \tilde{Y}_{\tilde{m}}\}$ such that, if $\dim(\mathsf{Y}_{q_0}) = k$, then

1. $\tilde{Y}_{\tilde{k}+a} = Y_{k+a}$, $a \in \{1, \dots, m-k\}$ and

2. $\tilde{Y}_\alpha = \sum_{a=1}^k \tilde{T}^a_\alpha Y_a$, $\alpha \in \{1, \ldots, \tilde{k}\}$, for a full-rank matrix $\tilde{\boldsymbol{T}} \in \mathbb{R}^{k\times\tilde{k}}$.

Clearly this then implies the existence of a full rank matrix $\boldsymbol{T} \in \mathbb{R}^{m\times\tilde{m}}$ such that $\tilde{Y}_\alpha = T^a_\alpha Y_a$, $\alpha \in \{1, \ldots, \tilde{m}\}$. From (4.2) it immediately follows that $(Q, \nabla, \mathrm{D}, \mathscr{Y})$ satisfies the hypotheses of Lemma 4.6, and so Theorem 4.5 follows. □

REMARK 4.7. Our use of the vector fields $Z_{q_0}(\mathscr{Y})$ from $\mathscr{Y}$ that vanish at q_0 is similar in spirit to how the vanishing of the drift vector appears in the work of Sussmann [36] and Bianchini and Stefani [7]. The idea is that brackets generated by such vanishing vector fields can be achieved "for free," without invoking bad brackets. •

A necessary condition for controllability is the following.

THEOREM 4.8. *Let $(Q, \nabla, \mathrm{D}, \mathscr{Y})$ be an analytic affine connection pre-control system for which $\mathscr{Y}$ is adapted at $q_0 \in Q$. Suppose that*

(i) *q_0 is a regular point for Y and that*
(ii) *$B_{\mathsf{Y}_{q_0}}(\mathsf{Y}_{q_0})$ is definite.*

Then $(Q, \nabla, \mathrm{D}, \mathscr{Y})$ is STLCUC from q_0.

Proof. We work locally. Therefore, we may assume that the vector fields $\{Y_1, \ldots, Y_k\}$ are linearly independent in a neighbourhood of q_0. First we show that the system is not STLC from q_0 using calculations of Hirschorn and Lewis [21]. We will not provide here a self-contained justification for all of our computations, since they take considerable space, but we refer to the paper [21]. The calculation uses the Chen–Fliess–Sussmann series [14, 17, 35]. For an analytic control-affine system

$$\xi'(t) = f_0(\xi(t)) + \sum_{a=1}^m u_a(t) f_a(\xi(t)), \qquad \xi(t) \in M$$

on a manifold M with a compact control set, and for an analytic function ϕ, the Chen–Fliess–Sussmann series gives the following formula for the value of ϕ along a controlled trajectory (ξ, u):

$$\phi(\xi(t)) = \sum_J U_J(t) f_J \phi(\xi(0)).$$

The sum is over multi-indices $J = (a_1, \ldots, a_k)$ in $\{0, 1, \ldots, m\}$,

$$U_J(t) = \int_0^t u_{a_k}(t_k) \int_0^{t_k} u_{a_{k-1}}(t_{k-1}) \ldots \int_0^{t_2} u_{a_1}(t_1)\, \mathrm{d}t_1 \ldots \mathrm{d}t_{k-1}\, \mathrm{d}t_k.$$

and

$$f_J\phi = f_{a_1} f_{a_2} \cdots f_{a_k}\phi.$$

We adopt the convention that $u_0 = 1$. We also regard an affine connection control system as a control-affine system in the usual manner by taking f_0 to be the geodesic spray for ∇ and $f_1, \ldots, f_m$ to be the vertical lifts of $Y_1, \ldots, Y_m$ [27].

The function we evaluate is defined as follows. We let λ be an analytic covector field defined in a neighbourhood of q_0 with the following properties:

1. λ annihilates the distribution Y;
2. $\lambda(q_0) B_{\mathsf{Y}_{q_0}} | \mathsf{Y}_{q_0}$ is negative-definite.

By a linear input transformation one can ensure that the input vector fields diagonalise $\lambda(q_0) B_{\mathsf{Y}_{q_0}}$ with the diagonal entries being -1. We assume this input transformation to have been made. We then define a function $\phi_\lambda \colon TQ \to \mathbb{R}$ by $\phi_\lambda(v_q) = \lambda(q) \cdot v_q$, and we also define

$$\Phi^+_\lambda = \{v_q \in TQ |\ \phi_\lambda(v_q) > 0\}, \quad \Phi^-_\lambda = \{v_q \in TQ |\ \phi_\lambda(v_q) < 0\}.$$

Note that, in any neighbourhood V of 0_{q_0} in TQ, the sets $V \cap \Phi_\lambda^-$ and $V \cap \Phi_\lambda^+$ will be nonempty, since ϕ_λ is linear on the fibres of TQ. Therefore, we can show that $(Q, \nabla, \mathrm{D}, \mathscr{Y})$ is STLUC from q_0 by showing that ϕ_λ has constant sign along any controlled trajectory. One may directly verify that ϕ_λ has the following properties:

1. $f_a\phi_\lambda$, $a \in \{1, \dots, m\}$, is zero in a neighbourhood of 0_{q_0};
2. $\mathrm{ad}_{f_0}^k f_a\phi_\lambda(0_{q_0}) = 0$, $a \in \{1, \dots, m\}$, $k \in \mathbb{N}$;
3. $[f_a, [f_0, f_a]]\phi_\lambda(0_{q_0}) = -1$, $a \in \{1, \dots, m\}$ (this and the next fact use the formula $[f_a, [f_0, f_b]] = \mathrm{verlift}(\langle Y_a : Y_b\rangle)$, $a, b \in \{1, , \dots, m\}$);
4. $[f_a, [f_0, f_b]]\phi_\lambda(0_{q_0}) = 0$, $a, b \in \{1, \dots, m\}$, $a \neq b$.

For an input $u\colon [0, T] \to U$, let us define

$$\|u\|_{2,t} = \max\Big\{\Big(\int_0^t |u_a(t)|^2\Big)^{1/2}\Big|\ a \in \{1, \dots, k\}\Big\}.$$

The calculations of Hirschorn and Lewis [21] now immediately give the following inequality for $\phi_\lambda(\gamma'(t))$ along a controlled trajectory (γ, u) for an affine connection control system like that under consideration here:

$$\phi_\lambda(\gamma'(t)) \geq \tfrac{1}{2}(\|u\|_{2,t})^2 - |E(t)|.$$

According to the analysis in Hirschorn and Lewis, the map $t \mapsto E(t)$ satisfies the bound $|E(t)| \leq tE_0(\|u\|_{2,t})^2$, for some $E_0 > 0$. For t sufficiently small, this shows that $\phi_\lambda(\gamma'(t))$ has constant sign. This shows that $(Q, \nabla, \mathrm{D}, \mathscr{Y})$ is STLCUC from q_0.

Now let us show that our above constructions also preclude the system from being locally *configuration* controllable. Choose a coordinate chart (V, χ) for Q around q_0 with the following properties: (1) $\chi(q_0) = \mathbf{0}$ and (2) $\mathrm{d}q^n(q_0) = \lambda(q_0)$. Let us define a function ψ_λ on the coordinate domain V by $\psi_\lambda(q) = q^n$ such that the sets

$$\Psi_\lambda^+ = \{q \in Q|\ \psi_\lambda(q) > 0\}, \quad \Psi_\lambda^- = \{q \in Q|\ \psi_\lambda(q) < 0\}$$

each intersect any neighbourhood of $q_0 \in Q$. Along any nontrivial trajectory $t \mapsto \gamma(t)$ we have

$$\frac{\mathrm{d}\psi_\lambda(\gamma(t))}{\mathrm{d}t}\Big|_{t=0} = \mathrm{d}\psi_\lambda(\gamma'(0)) = \phi_\lambda(\gamma'(0)) < 0,$$

Since $\psi_\lambda(q_0) = 0$, this means that, for sufficiently small t, $\psi_\lambda(\gamma(t)) < 0$, and this shows that the points in Ψ_λ^+ are not reachable in small time, and so Σ is not locally configuration controllable. □

Remark 4.9. The spirit of the preceding proof is that of the single-input necessary condition appearing as Proposition 6.3 in the paper of Sussmann [35]. However, the modifications to the multi-input case by Hirschorn and Lewis [21] require some care. •

Let us provide an example that nicely illustrates Theorems 4.5 and 4.8. This example is a slight modification of an example in [33].

Example 4.10. We take $Q = \mathbb{R}^2$ with (x, y) the usual Cartesian coordinates. We choose the affine connection on $\mathbb{R}^2$ with all vanishing Christoffel symbols except for $\Gamma_{xx}^y = x$. We choose the single input vector field $Y = \frac{\partial}{\partial x}$. We also take $\mathrm{D} = TQ$. One then readily computes

$$\langle Y : Y\rangle = 2x\frac{\partial}{\partial y}, \qquad \langle Y : \langle Y : Y\rangle\rangle = 2\frac{\partial}{\partial y}.$$

We consider two cases.

1. $q_0 = (0, y)$, $y \in \mathbb{R}$: We readily see that $B_{\mathsf{Y}_{q_0}}(\langle Z_{q_0}(\mathscr{Y}), \mathsf{Y}_{q_0}\rangle)$ is identically zero, and so essentially indefinite. We also have $\mathsf{Sym}^{(2)}(\mathsf{Y})_{q_0} = T_{q_0}Q$. Therefore, Theorem 4.5 shows that $(Q, \nabla, \mathsf{D}, \{Y\})$ is properly STLC from q_0.

2. $q_0 \neq (0, y)$, $y \in \mathbb{R}$: Here we use $\mathrm{span}_{\mathbb{R}}\{\frac{\partial}{\partial y}\}$ as a model for $T_{q_0}Q/\mathsf{Y}_{q_0}$. Thus both Y_{q_0} and $T_{q_0}Q/\mathsf{Y}_{q_0}$ are one-dimensional, and so $B_{\mathsf{Y}_{q_0}}(\mathsf{Y}_{q_0})$ is essentially a quadratic function on $\mathbb{R}$. This quadratic function is then exactly $\xi \mapsto 2x\xi^2$. This function is definite, so Theorem 4.8 implies that the system is STLCUC from q_0.

Thus this example has the rather degenerate feature of being controllable on the y-axis but being uncontrollable at every point in a neighbourhood of the y-axis. Note that this example is also a counterexample to a single-input result of one of the authors [23]. There it was stated that a single-input affine connection control system is STLCC if and only if the dimension of the configuration space is one. We see here that this is false. However, what is true is that a single-input affine connection control system is STLCC at all points in an *open subset* of configuration space if and only if the configuration space has dimension one. •

5. Reductions of affine connection control systems. The controllability results of Section 4 turn out to apply to a great many examples. That is to say, many interesting physical examples may be shown to be controllable or uncontrollable using these results. What is not obvious is that many of these systems are describable, in some sense, by a driftless system. This effectively simplifies the system, making certain control design tasks, especially motion planning, considerably simpler. In this section we introduce the framework for discussing these simplifications.

The objective in this section is then to relate second-order systems to first-order systems. In order to do this, one must be aware that the allowable inputs for the two classes of systems cannot be the same. For example, a trajectory for a first-order system using a discontinuous input will be continuous in configuration, but not in velocity. These velocity discontinuities are not allowed for second-order systems with bounded inputs. Therefore, we need to fix a set of inputs to use in each case, and they need to differ, essentially, by one integration. To be specific, we let $\mathscr{U}_{\mathrm{kin}}$ be the collection of locally absolutely continuous controls and we let $\mathscr{U}_{\mathrm{dyn}}$ be the collection of locally integrable controls. The former will be used for first-order systems and the latter for second-order systems. In all cases, we allow controls to be defined on an arbitrary interval $I \subset \mathbb{R}$.

5.1. Kinematic reductions. In this section, in order to emphasise the difference between the two kinds of systems we are comparing, we shall denote an affine connection control system by $\Sigma_{\mathrm{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y}, \mathbb{R}^m)$. A *driftless system* is a triple $\Sigma_{\mathrm{kin}} = (Q, \mathscr{X} = \{X_1, \dots, X_{\tilde{m}}\}, U \subset \mathbb{R}^{\tilde{m}})$. The associated control system is then

$$\gamma'(t) = \sum_{\alpha=1}^{\tilde{m}} \tilde{u}^\alpha(t) X_\alpha(\gamma(t)), \tag{5.1}$$

so that a *controlled trajectory* is a pair $(\gamma, \tilde{u})$ where

1. $\gamma\colon I \to Q$ and $\tilde{u}\colon I \to U$ are both defined on the same interval $I \subset \mathbb{R}$,
2. $\tilde{u} \in \mathscr{U}_{\mathrm{kin}}$, and
3. $(\gamma, \tilde{u})$ together satisfy (5.1).

A driftless system $(Q, \mathscr{X}, U)$ is *STLC from* q_0 if the set of points reachable from q_0 contains q_0 in its interior, and a pair $(Q, \mathscr{X})$ is *properly STLC from* q_0 if $(Q, \mathscr{X}, U)$ is STLC from q_0 for every proper U. With our underlying assumption of analyticity, it is well-known that $(Q, \mathscr{X})$ is properly STLC from q_0 if and only if $\mathsf{Lie}^{(\infty)}(\mathsf{X})_{q_0} = T_{q_0}Q$.

First we define what we mean by a kinematic reduction.

DEFINITION 5.1. Let $\Sigma_{\text{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y} = \{Y_1, \dots, Y_m\}, \mathbb{R}^m)$ be an affine connection control system with Y having constant rank. A driftless system $\Sigma_{\text{kin}} = (Q, \mathscr{X} = \{X_1, \dots, X_{\tilde{m}}\}, \mathbb{R}^{\tilde{m}})$ is a *kinematic reduction* of Σ_{dyn} if

(i) X is a constant-rank subbundle of D and

(ii) for every controlled trajectory (γ, u_{kin}) for Σ_{kin} with $u_{\text{kin}} \in \mathscr{U}_{\text{kin}}$, there exists $u_{\text{dyn}} \in \mathscr{U}_{\text{dyn}}$ such that (γ, u_{dyn}) is a controlled trajectory for Σ_{dyn}.

The *rank* of the kinematic reduction Σ_{kin} is the rank of X. •

Thus kinematic reductions are driftless systems whose controlled trajectories, at least for controls in $\mathscr{U}_{\text{kin}}$, can be followed by controlled trajectories of Σ_{dyn}. Let us characterise kinematic reductions. To do so, recall that with our constant rank assumptions, given an affine connection ∇ and a family of vector fields $\mathscr{Y} = \{Y_1, \dots, Y_m\}$ on Q, we may globally define B_{Y} as at the end of Section 3.2. This also allows us to define a map $Q_{B_{\mathsf{Y}}} \colon \Gamma(TQ) \to \Gamma(TQ/\mathsf{Y})$ by $Q_{B_{\mathsf{Y}}}(X)(q) = B_{\mathsf{Y}}(q)(X(q), X(q))$. With this notation, we have the following result.

THEOREM 5.2. *Let $\Sigma_{\text{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y}, \mathbb{R}^m)$ be an affine connection control system with Y of constant rank and let $\Sigma_{\text{kin}} = (Q, \mathscr{X}, \mathbb{R}^{\tilde{m}})$ be a driftless system with X of constant rank. The following statements are equivalent:*

(i) *Σ_{kin} is a kinematic reduction of Σ_{dyn};*

(ii) *$\mathsf{Sym}^{(1)}(\mathsf{X}) \subset \mathsf{Y}$;*

(iii) *$\mathsf{X} \subset \mathsf{Y}$ and $Q_{B_{\mathsf{Y}}}|\mathsf{X} = 0$.*

Proof. (i) $\Longrightarrow$ (ii) Let $X \in \Gamma(\mathsf{X})$ such that $X = \phi^\alpha X_\alpha$ for some $\phi^1, \dots, \phi^{\tilde{m}} \in \mathcal{F}(Q)$. For $q \in Q$, define controls $\tilde{u}_1, \tilde{u}_2 \in \mathscr{U}_{\text{kin}}$ by $\tilde{u}_1 = (\phi^1(q), \dots, \phi^{\tilde{m}}(q))$ and $\tilde{u}_2 = (1+t)\tilde{u}_1$. Let $(\gamma_1, \tilde{u}_1)$ and $(\gamma_2, \tilde{u}_2)$ be the corresponding controlled trajectories of Σ_{kin} satisfying $\gamma_1(0) = \gamma_2(0) = q$. Thus $\gamma_i'(t) = \sum_{\alpha=1}^{\tilde{m}} \tilde{u}_i^\alpha(t) X_\alpha(\gamma_i(t))$, $i \in \{1, 2\}$. We compute

$$\begin{aligned}\nabla_{\gamma_1'(t)} \gamma_1'(t) &= \sum_{\alpha,\beta=1}^{\tilde{m}} \nabla_{\tilde{u}_1^\alpha(t) X_\alpha(\gamma_1(t))} \tilde{u}_1^\beta(t) X_\beta(\gamma_1(t)) \\ &= \sum_{\alpha,\beta=1}^{m} \tilde{u}_1^\alpha(t) \tilde{u}_1^\beta(t) \nabla_{X_\alpha(\gamma_1(t))} X_\beta(\gamma_1(t)) + \dot{\tilde{u}}_1^\beta(t) X_\beta(\gamma_1(t)).\end{aligned}$$

Evaluating this at $t = 0$ gives

$$\nabla_{\gamma_1'(t)} \gamma_1'(t)\big|_{t=0} = \sum_{\alpha,\beta=1}^{\tilde{m}} \tilde{u}_1^\alpha(0) \tilde{u}_1^\beta(0) \nabla_{X_\alpha} X_\beta(q) + \dot{\tilde{u}}_1^\beta(0) X_\beta(q) = \nabla_X X(q).$$

Similarly, for γ_2 we have

$$\nabla_{\gamma_2'(t)} \gamma_2'(t)\big|_{t=0} = \nabla_X X(q) + X(q).$$

Therefore, since Σ_{kin} is a kinematic reduction of Σ_{dyn}, we have $\nabla_X X(q), \nabla_X X(q) + X(q) \in \mathsf{Y}_q$, or simply $X, \nabla_X X \in \Gamma(\mathsf{Y})$ since the above constructions can be performed for all $X \in \Gamma(\mathsf{X})$ and $q \in Q$. Therefore, for $X, Y \in \Gamma(\mathsf{X})$ we have the polarisation identity,

$$\langle X : Y \rangle = \frac{1}{2}\big(\langle X + Y : X + Y \rangle - \langle X : X \rangle - \langle Y : Y \rangle\big) \in \Gamma(\mathsf{Y}), \tag{5.2}$$

which gives (ii).

(ii) $\implies$ (iii) From the definition of B_{Y} we readily see that $Q_{B_{\mathsf{Y}}}|\mathsf{X} = 0$ exactly means that $\langle X : X\rangle = 2\nabla_X X \in \Gamma(\mathsf{Y})$ for each $X \in \Gamma(\mathsf{X})$. From this observation, the current implication follows easily by employing the formula for $\langle X : Y\rangle$ in (5.2).

(iii) $\implies$ (i) As in the preceding step, we saw that the condition $Q_{B_{\mathsf{Y}}}|\mathsf{X} = 0$ is equivalent to asserting that $\nabla_X X \in \Gamma(\mathsf{Y})$ for each $X \in \Gamma(\mathsf{X})$. By (5.2) this implies that $\langle X_\alpha : X_\beta\rangle \in \Gamma(\mathsf{Y})$ for $\alpha, \beta \in \{1, \dots, \tilde{m}\}$. Let $u_{\text{kin}} \in \mathscr{U}_{\text{kin}}$ and let (γ, u_{kin}) be the corresponding controlled trajectory for Σ_{kin}. We then have

$$\nabla_{\gamma'(t)}\gamma'(t) = u^\alpha_{\text{kin}}(t)u^\beta_{\text{kin}}(t)\nabla_{X_\alpha(\gamma(t))}X_\beta(\gamma(t)) + \dot{u}^\alpha_{\text{kin}}(t)X_\alpha(\gamma(t)).$$

We note that

$$u^\alpha_{\text{kin}}(t)u^\beta_{\text{kin}}(t)\nabla_{X_\alpha(\gamma(t))}X_\beta(\gamma(t)) = \frac{1}{2}u^\alpha_{\text{kin}}(t)u^\beta_{\text{kin}}(t)\langle X_\alpha(\gamma(t)) : X_\beta(\gamma(t))\rangle.$$

Since $X_\alpha, \langle X_\alpha : X_\beta\rangle \in \Gamma(\mathsf{Y})$ it now follows that $\nabla_{\gamma'(t)}\gamma'(t) \in \mathsf{Y}_{\gamma(t)}$, implying that there exists a control $u_{\text{dyn}} \in \mathscr{U}_{\text{dyn}}$ such that (γ, u_{dyn}) is a controlled trajectory for Σ_{dyn}. □

Of particular interest are kinematic reductions of rank one: $(Q, \{X_1\}, \mathbb{R})$. In this case, any vector field of the form $X = \phi X_1$, where $\phi \in \mathcal{F}(Q)$ is nowhere vanishing, is called a *decoupling vector field*. From Theorem 5.2 we have the following description of a decoupling vector field.

Corollary 5.3. *A vector field X is a decoupling vector field for $\Sigma_{\text{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y}, \mathbb{R}^m)$ if and only if $X, \nabla_X X \in \Gamma(\mathsf{Y})$.*

It is the notion of a decoupling vector field that was initially presented by Bullo and Lynch [13], and which is generalised by our idea of a kinematic reduction.

Remark 5.4. While in general, even when a kinematic reduction exists, it will not be easy to find, it turns out that in practice many examples exhibit kinematic reductions in a more or less obvious way. We shall see this in the examples below. Note that condition (iii) of Theorem 5.2 provides a set of algebraic equations that can, in principle, be solved to identify decoupling vector fields. This is discussed by Bullo and Lynch [13]. •

Next, let us consider affine connection control systems endowed with multiple kinematic reductions. It is interesting to characterise when the concatenation of controlled trajectories of the kinematic reductions gives rise to a controlled trajectory for the affine connection control system. Given two curves γ_1 and γ_2 on Q, let $\gamma_1 * \gamma_2$ be their concatenation. The following lemma follows immediately from the definition of a kinematic reduction.

Lemma 5.5. *Consider an affine connection control system $\Sigma_{\text{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y}, \mathbb{R}^m)$ with two kinematic reductions $\Sigma_{\text{kin},1} = (Q, \mathscr{X}_1, \mathbb{R}^{m_1})$ and $\Sigma_{\text{kin},2} = (Q, \mathscr{X}_2, \mathbb{R}^{m_2})$. For $i \in \{1, 2\}$, let $(\gamma_i, u_{\text{kin},i})$ be a controlled trajectory for $\Sigma_{\text{kin,i}}$ defined on the interval $[0, T_i]$ with $u_{\text{kin},i} \in \mathscr{U}_{\text{kin}}$. There exists a control $u_{\text{dyn}} \in \mathscr{U}_{\text{dyn}}$ such that $(\gamma_1 * \gamma_2, u_{\text{dyn}})$ is a controlled trajectory for Σ_{dyn} if and only if $\gamma_1'(T_1) = \gamma_2'(0)$.*

Motivated by this result we make the following definition.

Definition 5.6. An affine connection control system $\Sigma_{\text{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y}, \mathbb{R}^m)$ is *kinematically controllable from $q_0 \in Q$ (KC from $q_0 \in Q$)* if there exists a finite collection

$$\Sigma_{\text{kin},1} = (Q, \mathscr{X}_1, \mathbb{R}^{m_1}), \dots, \Sigma_{\text{kin},k} = (Q, \mathscr{X}_k, \mathbb{R}^{m_k})$$

of kinematic reductions for Σ_{dyn} such that $(Q, \mathscr{X}_1 \cup \dots \cup \mathscr{X}_k)$ is properly STLC from q_0. •

Remarks 5.7.

1. For analytic systems, the condition that $(Q, \mathscr{X}_1 \cup \cdots \cup \mathscr{X}_k)$ be properly STLC from q_0 is equivalent to the condition that $\mathsf{Lie}^{(\infty)}(\mathsf{X}_1 + \cdots + \mathsf{X}_k)_{q_0} = T_{q_0}Q$, where $\mathsf{X}_1 + \cdots + \mathsf{X}_k$ denotes the fibrewise sum of the distributions $\mathsf{X}_1, \ldots, \mathsf{X}_k$.

2. If an affine connection control system $\Sigma_{\text{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y}, \mathbb{R}^m)$ is kinematically controllable from q_0, then it is STLCC from q_0. This fact is proved in Proposition 5.16 below, and we refer to Section 5.3 for a discussion of the relationships between the various notions of controllability introduced in this paper.

3. Suppose the affine connection control system $\Sigma_{\text{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y}, \mathbb{R}^m)$ is kinematically controllable from all $q \in Q$. A standard control problem is to find a controlled trajectory connecting two given configurations $q_1, q_2 \in Q$, starting and ending with zero velocity. Lemma 5.5 says that this can be done for Σ_{dyn} by concatenating integral curves of decoupling vector fields where each segment is reparameterised to start and end at zero velocity. This is the viewpoint of Bullo and Lynch [13]. •

EXAMPLE 5.8. We consider a planar rigid body with a variable-direction thruster as shown in Figure 5.1. The system has configuration manifold $SE(2)$. We use

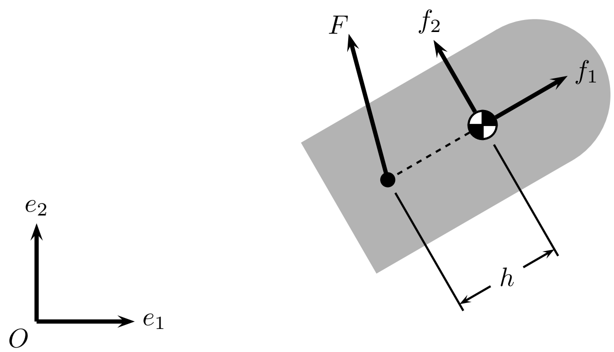

FIG. 5.1. *Planar rigid body with thruster*

coordinates (x, y, θ) defined as follows. Let $\{e_1, e_2\}$ be an orthonormal frame in E^2 fixed at $O \in E^2$, and let $\{f_1, f_2\}$ be a body orthonormal frame attached to the centre of mass and with the property that the vector f_1 points in the direction of the line connecting the centre of mass with the point of application of the force (see Figure 5.1). Then (x, y) denote the position of the centre of mass with respect to O, and θ is defined so that $f_1 = R(\theta)e_1$ with $R(\theta)$ the matrix giving a positive rotation by θ in E^2. With respect to these coordinates, the kinetic energy of the system is determined by the Riemannian metric

$$g = m\mathrm{d}x \otimes \mathrm{d}x + m\mathrm{d}y \otimes \mathrm{d}y + J\mathrm{d}\theta \otimes \mathrm{d}\theta,$$

where m is the mass of the body, and J is its inertia about the centre of mass. Since the coefficients of this Riemannian metric are independent of the coordinates, the Christoffel symbols for the corresponding Levi-Civita affine connection are zero. As shown by Lewis and Murray [27], Newton's law with the force F as shown in Figure 5.1 is equivalent to equation (2.1), if the affine connection ∇ is the Levi-Civita connection associated with g and if the vector fields $\{Y_1, Y_2\}$ are chosen as follows:

$$Y_1 = \frac{\cos\theta}{m}\frac{\partial}{\partial x} + \frac{\sin\theta}{m}\frac{\partial}{\partial y}, \qquad Y_2 = -\frac{\sin\theta}{m}\frac{\partial}{\partial x} + \frac{\cos\theta}{m}\frac{\partial}{\partial y} - \frac{h}{J}\frac{\partial}{\partial\theta}.$$

The system is unconstrained so we take $\mathsf{D} = TQ$.

We claim that the vector fields $X_1 = mY_1$ and $X_2 = mY_2$ are decoupling vector fields. Clearly, they are sections of Y. We also compute

$$\nabla_{X_1} X_1 = 0, \quad \nabla_{X_2} X_2 = \frac{mh\cos\theta}{J}\frac{\partial}{\partial x} + \frac{mh\sin\theta}{J}\frac{\partial}{\partial y}.$$

Therefore $\nabla_{X_1} X_1, \nabla_{X_2} X_2 \in \Gamma(\mathsf{Y})$, showing that X_1 and X_2 are indeed decoupling vector fields.

Let us explore the implications of the existence of these decoupling vector fields. Since X_1 and X_2 are decoupling vector fields, we may follow their integral curves. In Figure 5.2 we show motions of the body along sample integral curves of X_1 and

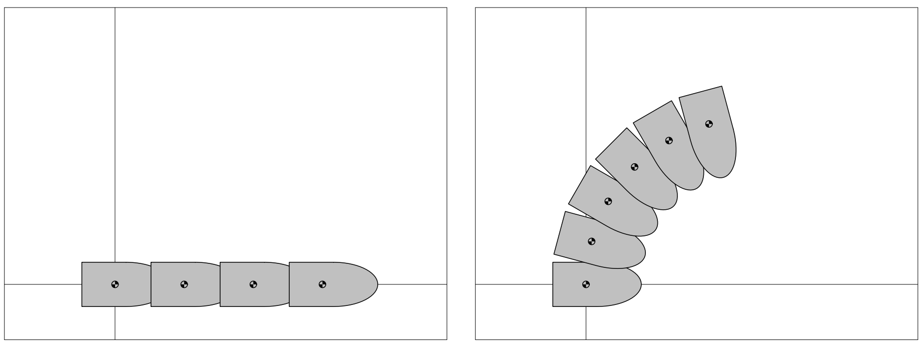

FIG. 5.2. *Decoupling motions for the planar rigid body: X_1 on the left and X_2 on the right*

X_2. In actuality, one can follow not only the integral curves of the decoupling vector fields, but any reparameterisation of these vector fields. With this in mind, one has the following possible methodology for moving the body around in the plane.

1. Given $q_1, q_2 \in Q$, find a concatenation of the integral curves of X_1 and X_2 that connects q_1 with q_2. (This is possible since $\mathsf{Lie}^{(\infty)}(\mathsf{X}) = TQ$.)
2. Reparameterise each segment of the preceding concatenated curve so that each segment has zero initial and final velocity.
3. Because of Lemma 5.5, the resulting reparameterised curve can be followed by controlled trajectories of Σ_{dyn}.

This method for motion planning is explained in detail in [11, Chapter 13]. •

5.2. Maximally reducible systems. If $\Sigma_{\mathrm{kin}} = (Q, \mathscr{X} = \{X_1, \dots, X_{\tilde{m}}\}, \mathbb{R}^{\tilde{m}})$ is a kinematic reduction of $\Sigma_{\mathrm{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y} = \{Y_1, \dots, Y_m, \mathbb{R}^m)$, then, by definition, any controlled trajectory of Σ_{kin} may be followed by a controlled trajectory of Σ_{dyn}. In this section we wish to consider the possibility of the converse statement. The following definition, and the attendant Theorem 5.11 below, are due to Lewis [25].

DEFINITION 5.9. An affine connection control system $\Sigma_{\mathrm{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y} = \{Y_1, \dots, Y_m\}, \mathbb{R}^m)$ with Y constant-rank is *maximally reducible* to $\Sigma_{\mathrm{kin}} = (Q, \mathscr{X} = \{X_1, \dots, X_{\tilde{m}}\}, \mathbb{R}^{\tilde{m}})$ if Σ_{kin} is a kinematic reduction of Σ_{dyn} and if for every controlled trajectory $(\gamma, u_{\mathrm{dyn}})$ for Σ_{dyn} satisfying $\gamma'(0) \in \mathsf{X}_{\gamma(0)}$, there exists a control $u_{\mathrm{kin}} \in \mathscr{U}_{\mathrm{kin}}$ such that $(\gamma, u_{\mathrm{kin}})$ is a controlled trajectory for Σ_{kin}. •

Before we proceed to characterise maximally reducible systems, let us illustrate that a system may not be maximally reducible to a given kinematic reduction.

EXAMPLE 5.10 (Example 5.8 cont'd). We claim that the affine connection control system corresponding to the planar rigid body with a thruster is not maximally reducible to either of the kinematic reductions $\Sigma_{\mathrm{kin},1} = (Q, \mathscr{X}_1 = \{X_1\}, \mathbb{R})$

or $\Sigma_{\mathrm{kin},2} = (Q, \mathscr{X}_2 = \{X_2\}, \mathbb{R})$ exhibited in Example 5.8. We shall exhibit this explicitly for $\Sigma_{\mathrm{kin},1}$, and leave the other case to the reader.

Consider the control $t \mapsto u(t) = (0,1) \in \mathscr{U}_{\mathrm{dyn}}$, along with the initial condition $\gamma'(0) = ((0,0,0),(1,0,0)) \in TQ$. We have $\gamma'(0) \in \mathsf{X}_{1,\gamma(0)}$, where X_1 is the distribution generated by the vector field X_1. If Σ_{dyn} is to be maximally reducible to $\Sigma_{\mathrm{kin},1}$, then we should have $\gamma'(t) \in \mathsf{X}_{1,\gamma(t)}$ for each $t > 0$. To show that this is not the case, consider the governing equations for the system with the given control:

$$\ddot{x} = -\frac{\sin\theta}{m}, \quad \ddot{y} = \frac{\cos\theta}{m}, \quad \ddot{\theta} = -\frac{h}{J}.$$

Clearly the solution to this ordinary differential equation is not a reparameterisation of the integral curve for X_1 through $\gamma(0)$ since the latter is given by $t \mapsto (t,0,0)$. Thus it cannot be that $\gamma'(t) \in \mathsf{X}_{1,\gamma(t)}$ for each $t > 0$. •

Now let us establish when an affine connection control system is in fact maximally reducible to *some* driftless system. Note that in the statement of the following theorem, the driftless systems to which Σ_{dyn} is maximally reducible are characterised sharply.

THEOREM 5.11. *An affine connection control system* $\Sigma_{\mathrm{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y} = \{Y_1, \dots, Y_m, \mathbb{R}^m)$*, with* Y *constant rank, is maximally reducible to* $\Sigma_{\mathrm{kin}} = (Q, \mathscr{X} = \{X_1, \dots, X_{\tilde{m}}\}, \mathbb{R}^{\tilde{m}})$ *if and only if the following two conditions hold:*

(i) $\mathsf{X} = \mathsf{Y}$*;*

(ii) $\mathsf{Sym}^{(\infty)}(\mathsf{Y}) = \mathsf{Y}$.

Proof. In the proof it is convenient to understand that the second-order system (2.1) on Q is equivalent to the first-order system on TQ given

$$\Upsilon'(t) = Z(\Upsilon(t)) + \sum_{a=1}^{m} u^a(t)\mathrm{verlift}(Y_a)(\Upsilon(t)), \tag{5.3}$$

for a curve Υ on TQ, where Z is the geodesic spray for ∇ and $\mathrm{verlift}(Y_a) \in \Gamma(TTQ)$ denotes the vertical lift of Y_a. This is discussed in Lewis and Murray [27]. Further, one may easily verify that a vector field X is a section of a distribution D if and only if $\mathrm{verlift}(X)$ is tangent to $\mathsf{D} \subset TQ$. Also, Lewis [24] shows that condition (ii) is equivalent to the assertion that Y be geodesically invariant, by which we mean that geodesics $\gamma\colon I \to Q$ satisfying $\gamma'(t_0) \in \mathsf{Y}_{\gamma(t_0)}$ for some $t_0 \in I$ satisfy $\gamma'(t) \in \mathsf{Y}_{\gamma(t)}$ for all $t \in I$. Clearly, geodesic invariance of Y is equivalent to Y being an invariant submanifold for Z.

First suppose that Σ_{dyn} is maximally reducible to a driftless system Σ_{kin}. Let $\gamma\colon [0,T] \to Q$ be a geodesic so that $(\gamma', 0)$ is a controlled trajectory for Σ_{dyn}. If we ask that $\gamma'(0) \in \mathsf{X}$, then Definition 5.9 implies that there exists $u_{\mathrm{kin}} \in \mathscr{U}_{\mathrm{kin}}$ such that $(\gamma, u_{\mathrm{kin}})$ is a controlled trajectory of Σ_{kin}. Indeed, u_{kin} is defined by

$$\gamma'(t) = \sum_{\alpha=1}^{\tilde{m}} u^{\alpha}_{\mathrm{kin}}(t) X_\alpha(\gamma(t)),$$

and so is smooth. Further, this implies that X is geodesically invariant. The remainder of this part of the proof will be directed towards showing that $\mathsf{X} = \mathsf{Y}$.

Let $\boldsymbol{e}_a$ be the ath standard basis vector for $\mathbb{R}^m$ and let $u_a\colon [0,T] \to \mathbb{R}^m$ be the control defined by $u_a(t) = \boldsymbol{e}_a$. Let $\Upsilon\colon [0,T] \to TQ$ be an integral curve for the vector field $Z + \mathrm{verlift}(Y_a)$, so that (Υ, u_a) satisfies (5.3). By Definition 5.9, Υ must be tangent to X. Since X is geodesically invariant, Z is tangent to X, therefore verlift(Y_a) must be tangent to X. This implies that $\mathsf{Y} \subset \mathsf{X}$.

To show that $\mathsf{X} \subset \mathsf{Y}$ we employ the following lemma.

LEMMA 5.12. *If a distribution* D *is geodesically invariant for an affine connection* ∇*, then for each* $q \in Q$ *and each* $X \in \mathsf{D}_q$ *there exists* $T > 0$ *and a smooth curve* $\gamma\colon [0,T] \to Q$ *with the following properties:*
(i) $\gamma'(t) \in \mathsf{D}_{\gamma(t)}$ *for* $t \in (0,T]$*;*
(ii) $\nabla_{\gamma'(0)}\gamma'(0) = X$.

Proof. Let (U,χ) be a normal coordinate chart [22, Proposition 8.4] with $\chi(q) = \mathbf{0}$. In such a chart the Christoffel symbols for ∇ satisfy $\Gamma^i_{jk}(\mathbf{0}) + \Gamma^i_{kj}(\mathbf{0}) = 0$, $i,j,k \in \{1,\dots,n\}$. Let $\tilde{T} > 0$ be small if necessary and let $\tilde{\gamma}\colon [0,\tilde{T}] \to Q$ be the geodesic satisfying $\tilde{\gamma}'(0) = X$. Let us denote the local representative of $\tilde{\gamma}$ in our normal coordinate chart by $t \mapsto (\tilde{q}^1(t),\dots,\tilde{q}^n(t))$. We must then have $\ddot{\tilde{q}}^i(0) = 0$, $i \in \{1,\dots,n\}$, since $\tilde{\gamma}$ is a geodesic and we are using normal coordinates. Since D is geodesically invariant, $\tilde{\gamma}'(t) \in \mathsf{D}_{\tilde{\gamma}(t)}$ for $t \in (0,\tilde{T}]$. Now define $\tau\colon [0,\tilde{T}] \to [0,\frac{1}{2}\tilde{T}^2]$ by $\tau(t) = \frac{1}{2}t^2$. Let $T = \frac{1}{2}\tilde{T}^2$, define $\gamma\colon [0,T] \to Q$ by $\gamma = \tilde{\gamma}\circ\tau$, and denote by $t \mapsto (q^1(t),\dots,q^n(t))$ the local representative of γ. Then we have

$$\dot{q}^i(t) = \frac{2t\dot{\tilde{q}}^i(t)}{T}, \qquad i \in \{1,\dots,n\},$$
$$\ddot{q}^i(0) = \dot{\tilde{q}}^i(0), \qquad i \in \{1,\dots,n\}.$$

Since $\tilde{\gamma}'(0) = X$ the result follows. ▽

Now let $q \in Q$ and $X \in \mathsf{X}_q$. Choose a curve $\gamma\colon [0,T] \to Q$ as in the lemma. Define a smooth map $u_{\text{kin}}\colon [0,T] \to \mathbb{R}^{\tilde{m}}$ by asking that it satisfy

$$\gamma'(t) = \sum_{\alpha=1}^{\tilde{m}} u^\alpha_{\text{kin}}(t)X_\alpha(\gamma(t)).$$

Then (γ, u_{kin}) is a controlled trajectory for Σ_{kin}. Therefore, by Definition 5.9, there exists a map $u_{\text{dyn}}\colon [0,T] \to \mathbb{R}^m$ such that $(\gamma', u_{\text{dyn}})$ is a controlled trajectory for $(TQ, \mathscr{X}_{\Sigma_{\text{dyn}}}, \mathbb{R}^m)$. Indeed, since γ' is smooth, u_{dyn} will also be smooth. Furthermore, we have

$$X = \nabla_{\gamma'(0)}\gamma'(0) = \sum_{a=1}^{m} u^a_{\text{dyn}}(0)Y_a(\gamma(0)).$$

This shows that $\mathsf{X} \subset \mathsf{Y}$ which completes the proof of the "only if" part of the theorem.

Now suppose that parts (i) and (ii) of the theorem hold. Let us work locally, so we may as well assume that the vector fields $\{Y_1,\dots,Y_m\}$ and $\{X_1,\dots,X_{\tilde{m}}\}$ are linearly independent (and so $\tilde{m} = m$). First, part (ii) implies Y is an invariant submanifold for the system $(TQ, \mathscr{X}_{\Sigma_{\text{dyn}}}, \mathbb{R}^m)$, since verlift$(Y_a)$, $a \in \{1,\dots,m\}$, is tangent to Y. If $(\Upsilon, u_{\text{dyn}})$ is a controlled trajectory of $(TQ, \mathscr{X}_{\Sigma_{\text{dyn}}}, \mathbb{R}^m)$, then $\Upsilon\colon [0,T] \to TQ$ is absolutely continuous, and so $\gamma \triangleq \tau_Q \circ \Upsilon$ is also absolutely continuous. In fact, $\Upsilon = \gamma'$ and so not only is γ absolutely continuous, but γ' is absolutely continuous. If we further suppose that $\gamma'(0) \in \mathsf{Y}_{\gamma(0)}$, then $\gamma'(t) \in \mathsf{Y}_{\gamma(t)}$ for $t \in [0,T]$. We may then define $u_{\text{kin}}\colon [0,T] \to \mathbb{R}^{\tilde{m}}$ by $\gamma'(t) = u^\alpha_{\text{kin}}(t)X_\alpha(\gamma(t))$ which uniquely defines u_{kin} since $(TQ, \mathscr{X}_{\Sigma_{\text{dyn}}}, \mathbb{R}^m)$ leaves Y, and hence X, invariant. It is clear that u_{kin} is absolutely continuous.

Finally, let (γ, u_{kin}) be a controlled trajectory for Σ_{kin}. Thus γ' is absolutely continuous. Since Y, and therefore X, is geodesically invariant, $\nabla_{\gamma'(t)}\gamma'(t) \in \mathsf{Y}_{\gamma(t)}$ for $t \in [0, T]$. Thus we may write

$$\nabla_{\gamma'(t)}\gamma'(t) = \sum_{a=1}^{m} u^a_{\text{dyn}}(t) Y_a(\gamma(t)),$$

which defines $u_{\text{dyn}}\colon [0, T] \to \mathbb{R}^m$. It is clear that u is locally integrable, and this completes the proof. □

Remark 5.13. Note that all driftless systems to which a given affine connection control system $\Sigma_{\text{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y} = \{Y_1, \ldots, Y_m\}, \mathbb{R}^m)$ is maximally reducible are essentially the same, by which we mean that for two such driftless systems, $\Sigma_{\text{kin}} = (Q, \mathscr{X} = \{X_1, \ldots, X_m\}, \mathbb{R}^m)$ and $\tilde{\Sigma}_{\text{kin}} = (Q, \tilde{\mathscr{X}} = \{\tilde{X}_1, \ldots, \tilde{X}_{\tilde{m}}\}, \mathbb{R}^{\tilde{m}})$, we have $\mathsf{X} = \tilde{\mathsf{X}}$. Thus, without loss of generality, we may take $(Q, \{Y_1, \ldots, Y_m\}, \mathbb{R}^m)$ as the system to which Σ_{dyn} is maximally reducible. For this reason, it makes sense to simply say that Σ_{dyn} is *maximally reducible* if it is maximally reducible to *some* driftless system. •

Let us give an example of a system that is maximally reducible.

Example 5.14. We consider the robotic leg system depicted in Figure 5.3. The

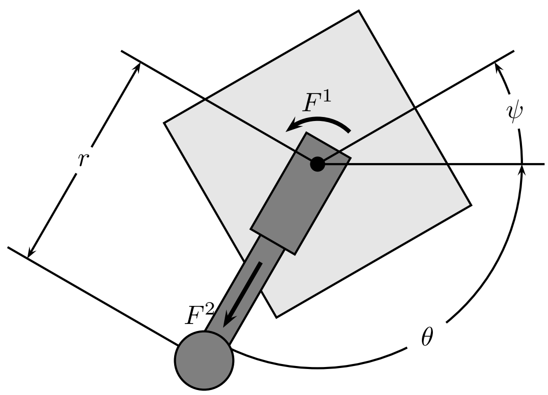

Fig. 5.3. *The robotic leg*

configuration space for the system is $Q = \mathbb{R}_+ \times \mathbb{S}^1 \times \mathbb{S}^1$, and the coordinates we use are (r, θ, ψ) as indicated in Figure 5.3. The Riemannian metric for the system is

$$g = m(\mathrm{d}r \otimes \mathrm{d}r + r^2 \mathrm{d}\theta \otimes \mathrm{d}\theta) + J \mathrm{d}\psi \otimes \mathrm{d}\psi,$$

where m is the mass of the particle on the end of the extensible massless leg, and J is the moment of inertia of the base rigid body about the pivot point. The nonzero Christoffel symbols for the associated affine connection are $\Gamma^r_{\theta\theta} = -r$ and $\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}$. Lewis and Murray [27] show that if we define Y_1 and Y_2 by

$$Y_1 = \frac{1}{mr^2}\frac{\partial}{\partial\theta} - \frac{1}{J}\frac{\partial}{\partial\psi}, \quad Y_2 = \frac{1}{m}\frac{\partial}{\partial r},$$

then the equations of motion for the system are of the form (2.1), where ∇ is the Levi-Civita connection associated with g. There are no constraints on the system so we take $\mathsf{D} = TQ$.

One readily computes

$$\langle Y_1 : Y_1 \rangle = -\frac{2}{m^2 r^3}\frac{\partial}{\partial r}, \quad \langle Y_1 : Y_2 \rangle = 0, \qquad \langle Y_2 : Y_2 \rangle = 0.$$

This shows that Y is geodesically invariant. Thus the corresponding affine connection control system Σ_{dyn} is maximally reducible to $(Q, \{Y_1, Y_2\}, \mathbb{R}^2)$. •

Since $\mathsf{Sym}^{(\infty)}(\mathsf{Y}) = \mathsf{Y}$ for an affine connection control system that is maximally reducible to a driftless system, by Remark 5.7–2 such an affine connection control system, if analytic, is STLCC from $q \in Q$ if and only if $\mathsf{Lie}^{(\infty)}(\mathsf{Y})_q = T_q Q$. Thus we make the following definition.

Definition 5.15. A maximally reducible affine connection control system $\Sigma_{\mathrm{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y}, \mathbb{R}^m)$ is *maximally reducibly kinematically controllable from* $q_0 \in Q$ (*MR-KC from* q_0) if $(Q, \mathscr{Y})$ is properly STLC from q_0. •

5.3. Relationships to controllability. The appearance in Theorem 5.2 of the vector-valued quadratic form B_{Y} raises questions about how the notion of kinematic reductions are related to the low-order controllability results of Section 4. In this section we describe the proper relationships. In [12] counterexamples are provided to show that one cannot generally improve on the relationships presented here.

Let $\Sigma_{\mathrm{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y}, \mathbb{R}^m)$ be an affine connection control system. First let us list the various types of controllability we have at hand for Σ_{dyn} from a point $q_0 \in Q$:

1. small-time local controllability (STLC);
2. small-time local configuration controllability (STLCC);
3. kinematic controllability (KC);
4. maximal reducible kinematic controllability (MR-KC).

The relationships between these concepts are demonstrated in Figure 5.4. Let us show

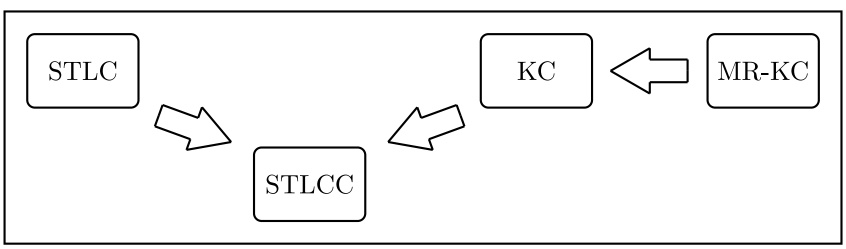

Fig. 5.4. *Relationships between various forms of controllability for affine connection control systems*

that these implications do indeed hold.

Proposition 5.16. *For an analytic affine connection control system* $\Sigma_{\mathrm{dyn}} = (Q, \nabla, \mathsf{D}, \mathscr{Y}, \mathbb{R}^m)$ *and for* $q_0 \in Q$*, the implications of Figure 5.4 hold.*

Proof. The implications STLC $\Longrightarrow$ STLCC and MR-KC $\Longrightarrow$ KC follow directly from the definitions of the various notions of controllability involved. Thus we need only show that KC $\Longrightarrow$ STLCC. We let

$$\Sigma_{\mathrm{kin},1} = (Q, \mathscr{X}_1, \mathbb{R}^{m_1}), \ldots, \Sigma_{\mathrm{kin},k} = (Q, \mathscr{X}_k, \mathbb{R}^{m_k})$$

be a collection of kinematic reductions for which $\mathsf{Lie}^{(\infty)}(\mathsf{X}_1 + \cdots + \mathsf{X}_k)_{q_0} = T_{q_0} Q$, where $\mathsf{X}_1 + \cdots + \mathsf{X}_k$ denotes the fiberwise sum of the distributions $\mathsf{X}_1, \ldots, \mathsf{X}_k$. Let $\mathscr{X} = \mathscr{X}_1 \cup \cdots \cup \mathscr{X}_k$. Note that since $\mathsf{X}_i \subset \mathsf{Y}$, Σ_{dyn} is STLCC from q_0 if $(Q, \nabla, \mathsf{D}, \mathscr{X})$

is properly STLCC from q_0. Select vector fields $X_{a_1}, \dots, X_{a_\ell}$ from the family $\mathscr{X}$ such that $\{X_{a_1}(q_0), \dots, X_{a_\ell}(q_0)\}$ is a basis for X_{q_0}. For brevity, let us denote by $B \in \Sigma_2(\mathsf{Y}_{q_0}; T_{q_0}Q/\mathsf{Y}_{q_0})$ the vector-valued quadratic form $B_{\mathsf{Y}}(q_0)$. By Theorem 5.2 we know that $Q_B|\mathsf{X}_{i,q_0} = 0$, $i \in \{1, \dots, k\}$. It therefore follows that, for each $\lambda \in \operatorname{ann}(\mathsf{Y}_{q_0})$, $\lambda B(X_{a_j}(q_0), X_{a_j}(q_0)) = 0$, $j \in \{1, \dots, \ell\}$. From Lemma 3.3 this means that λB is essentially indefinite, and since this holds for every $\lambda \in \operatorname{ann}(\mathsf{Y}_{q_0})$, B is itself essentially indefinite. Therefore, by Theorem 4.5, $(Q, \nabla, \mathsf{D}, \mathscr{X})$ is properly STLCC if $\mathsf{Lie}^{(\infty)}(\mathsf{X})_{q_0} = T_{q_0}Q$. The result now follows directly. □

REMARK 5.17. Note that all implications in Figure 5.4 are local. There are implications for global notions of controllability that follow from the local notions, but we do not consider this in a systematic way, as the understanding of global controllability of affine connection control systems is, as yet, poorly understood. •

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