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Design of oscillatory control systems

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Abstract

This paper presents a broad and rigorous framework for the analysis and design of control systems subject to oscillatory inputs, i.e., inputs of large amplitude and high frequency. The key analysis result is a series expansion characterizing the averaged system. This expansion leads to efficient algorithms for stabilization of systems with positive trace and for trajectory tracking for second-order underactuated systems. A second-order nonholonomic integrator and the PVTOL example provide insight into the trajectory tracking controller.

1 Introduction

This paper investigates the behavior of finite dimensional analytic systems subject to oscillatory controls described by a differential equation of the form

$$\dot{x} = f(t,x) + \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}, t, x\right),$$

where g is periodic in its first argument, $0 < \epsilon \ll 1$, and both f and g are analytic in x. We provide a rigorous framework that leads to (1) a coordinate-free expression of the averaged system; (2) control tools for stabilization and trajectory planning in underactuated systems. The recent conference submission [1] presented the analysis results, and here we focus on design. Various proofs and detailed discussions can be found in [2].

Motivation: The study of oscillations in nonlinear differential equations is a classic and widespread research topic. Related areas include nonlinear dynamical systems [3], geometric control [4], analysis of animal locomotion [5], design of robotic locomotion and manipulation devices [6], analysis of switching circuit models [7], control of quantum dynamics [8] and chemical reactions [9], etc. Moreover, averaging analysis seems a perfect fit for novel applications in the fields of microelectro mechanical systems [10, 11] and active control of fluids and separation control [12].

Literature review: This work has connections with numerous ongoing research efforts. First of all, our analysis complements the study of differential equations subject to high frequency, high amplitude forcing terms [13, 14]. A second set of related results deals with high frequency vibrations in mechanical systems [15, 16, 17, 18], and averaging analysis in locomotion and rectification [19, 20]. Within the context of control design, three related areas are: time-varying stabilizing laws for driftless systems [21, 22]; oscillatory controls for point stabilization in general nonlinear and mechanical control systems [23, 24] and for trajectory planning in driftless systems [25, 26].

Statement of contributions: We start by reviewing the results in [1]: we provide an explicit representation of the averaged system as an infinite sum of Lie brackets of the input vector fields with the drift and iterated integrals of the open-loop controls. Next, we focus on control design. Regarding stabilization problems, we recover and extend a number of previous results on (1) the equilibrium points of the averaged system, (2) the order of linearizing and averaging, and (3) the stabilization of systems with negative trace. These results and others are immediate consequences of the coordinate-free analysis. Furthermore, we present novel results on stabilization of systems with positive trace via nonlinear feedback (but we refer to the journal version [2] for the full details).

Regarding tracking problems, we consider the setting of second-order underactuated systems. Under a nonlinear controllability condiion, we exploit the novel

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two-time-scales analysis to design a trajectory tracking controller. We apply the strategy to a second-order nonholonomic integrator and to the PVTOL system. The simulations illustrate how the averaged system is steered along arbitrary reference paths.

The paper is organized as follows. Preliminary concepts are introduced in Section 2. Section 3 reviews the averaging analysis presented in [1]. Section 3 contains assumptions that simplify this analysis and pave the way to design. Section 4 and Section 5 discuss stabilization and tracking problems, respectively. Finally, we present our conclusions in Section 6.

2 Preliminaries and notation

Let $x, x_0 \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, and let ϵ vary in the range $(0, \epsilon_0]$ with $\epsilon_0 \ll 1$. Let $f, g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ be smooth time-varying vector fields. Define their Lie bracket by

$$[g, f] = \frac{\partial f(t,x)}{\partial x} g(t,x) - \frac{\partial g(t,x)}{\partial x} f(t,x) .$$

We use the notation $\mathrm{ad}_g^0 f = f$, $\mathrm{ad}_g f = [g, f]$ and $\mathrm{ad}_g^k f = \mathrm{ad}_g^{k-1}[g, f]$. Let $x(t) = \Phi^g_{0,T}(x_0)$ be the flow map describing the solution at time T to the initial value problem $\dot{x} = g(t, x)$, $x(0) = x_0$.

Iterated integrals of multiple functions
Let $\mathcal{I}$ be the set of all multiindices $I = \{i_1, \ldots, i_k\}$, where $i_1, \ldots, i_k \in \{1, \ldots, m\}$. Given m bounded functions $u_i : \mathbb{R}_+ \to \mathbb{R}$, define their *iterated integrals* $\{U_I : \mathbb{R}_+ \to \mathbb{R}, I \in \mathcal{I}\}$ by

$$U_{\{i_1,\ldots,i_k\}}(t) = \int_0^t u_{i_k}(t_k) \int_0^{t_k} u_{i_{k-1}}(t_{k-1}) \ldots \int_0^{t_2} u_{i_1}(t_1) dt_1 \ldots dt_k .$$

Let $C_{k_1,\ldots,k_m}$ be the collection of all possible ways of taking m classes of $k_1 + \cdots + k_m$ different objects with k_i objects in the ith class. To each $\alpha \in C_{k_1,\ldots,k_m}$, associate a multiindex $I(\alpha)$ of length $k_1 + \cdots + k_m$ as follows: as $i \in \{1, \ldots, m\}$, place the index i in the k_i places corresponding to the ith class of α. Given m bounded functions $u_i : \mathbb{R}_+ \to \mathbb{R}$, define their *multinomial iterated integrals* $\{\mathcal{U}_{k_1,\ldots,k_m} : \mathbb{R}_+ \to \mathbb{R}, k_1, \ldots, k_m \in \mathbb{N}\}$,

$$\mathcal{U}_{k_1,\ldots,k_m}(t) = \sum_{\alpha \in C_{k_1,\ldots,k_m}} U_{I(\alpha)}(t) .$$

Lemma 2.1 *For $u_1, \ldots, u_m$ bounded functions,*

$$\mathcal{U}_{k_1,\ldots,k_m}(t) = \frac{1}{k_1! \ldots k_m!} \left(\int_0^t u_1(\tau) d\tau \right)^{k_1} \ldots \left(\int_0^t u_m(\tau) d\tau \right)^{k_m} .$$

The functions $\mathcal{U}_{k_1,\ldots,k_m}$ are T-periodic iff $\{u_i\}_{i=1}^m$ are T-periodic and zero-mean.

Given a T-periodic function $V(t)$, let $\overline{V} = \frac{1}{T} \int_0^T V(t) dt$. As an example, consider $u_i(t) = a_i \cos \omega t$, $\omega \in \mathbb{N}$. Then,

$$\mathcal{U}_{k_1,\ldots,k_m}(t) = \frac{a_1^{k_1} \ldots a_m^{k_m}}{k_1! \ldots k_m!} \left(\frac{1}{\omega} \sin \omega t \right)^{k_1 + \cdots + k_m} ,$$

and their averages are (note $k = \sum_{j=1}^m k_j$)

$$\overline{\mathcal{U}}_{k_1,\ldots,k_m} = \begin{cases} 0 & \text{if } k \text{ is odd} \\ \dfrac{a_1^{k_1} \ldots a_m^{k_m}}{k_1! \ldots k_m!} \left(\dfrac{1}{2\omega} \right)^k \dbinom{k}{k/2} & \text{if } k \text{ is even} \end{cases} .$$

3 Averaging under oscillatory controls

We summarize here our coordinate-free version of averaging under oscillatory controls. A detailed and thorough discussion of these results was presented in [1, 2].

Let $x : [0, T] \to \mathbb{R}^n$ be the solution to the problem

$$\frac{dx}{dt} = f(t, x) + \frac{1}{\epsilon} g\left(\frac{t}{\epsilon}, t, x \right) ,$$

where $g(\tau, t, x) = u_1(\tau, t) g_1(x) + \cdots + u_m(\tau, t) g_m(x)$.

Theorem 3.1 (Coordinate-free averaging) *Let $(\tau, t) \mapsto u_1(\tau, t), \ldots, u_m(\tau, t)$ be bounded functions, T-periodic and zero-mean in τ, continuously differentiable in t. Let $g_1, \ldots, g_m$ be commuting vector fields. For $t \in \mathbb{R}_+$, we have*

$$x(t) = \Phi^{g(\tau,t,x)}_{0,t/\epsilon}(z(t)) ,$$

and, as $\epsilon \to 0$ on the time scale 1, we have $z(t) - y(t) = O(\epsilon)$, where z and y are the solutions to the problems

$$\frac{dz}{dt} = F\left(\frac{t}{\epsilon}, t, z \right), \quad z(0) = x_0, \tag{1}$$

$$\frac{dy}{dt} = \overline{F}(t, y), \quad y(0) = x_0 , \tag{2}$$

with the vector fields

$$F(\tau, t, x) = f(t, x) - \sum_{i=1}^m \frac{\partial U_{\{i\}}}{\partial t}(\tau, t) g_i(x) + \sum_{\substack{k=1,\ldots,+\infty \\ \{i_1 \ldots i_k\} \in \mathcal{I}}} U_{\{i_1,\ldots,i_k\}}(\tau, t) \, \mathrm{ad}_{g_{i_1}} \ldots \mathrm{ad}_{g_{i_k}} f(t, x) \tag{3}$$

$$\overline{F}(t, x) = f(t, x) - \sum_{i=1}^m \frac{d\overline{U}_{\{i\}}}{dt}(t) g_i(x) + \sum_{\substack{k=1,\ldots,+\infty \\ \{i_1 \ldots i_k\} \in \mathcal{I}}} \overline{U}_{\{i_1,\ldots,i_k\}}(t) \, \mathrm{ad}_{g_{i_1}} \ldots \mathrm{ad}_{g_{i_k}} f(t, x). \tag{4}$$

In addition, assume f and g do not depend explicitly on the slow time scale t, i.e. $f = f(x)$ and $g = g(t/\epsilon, x)$. If 0 is a hyperbolically stable critical point for $\overline{F} = \overline{F}(x)$, then $z(t) - y(t) = O(\epsilon)$ as $\epsilon \to 0$ for all $t \in \mathbb{R}_+$ and eq. (1) possesses a unique hyperbolically stable periodic orbit belonging to an $O(\epsilon)$ neighborhood of 0.

An equivalent expression in terms of multinomial iterated integrals for F and $\overline{F}$ are, respectively

$$F = \sum_{k_1,\dots,k_m=0}^{+\infty} \mathcal{U}_{k_1,\dots,k_m}(\tau,t)\,\mathrm{ad}_{g_1}^{k_1}\dots\mathrm{ad}_{g_m}^{k_m} f - \sum_{i=1}^{m} \frac{\partial U_{\{i\}}}{\partial t}(\tau,t) g_i$$

$$\overline{F} = \sum_{k_1,\dots,k_m=0}^{+\infty} \overline{\mathcal{U}}_{k_1,\dots,k_m}(t)\,\mathrm{ad}_{g_1}^{k_1}\dots\mathrm{ad}_{g_m}^{k_m} f - \sum_{i=1}^{m} \frac{d\overline{U}_{\{i\}}}{dt}(t) g_i .$$

On summable series
We present sufficient conditions under which the series for the averaged system (4) can be summed. In what follows, i, j, k take values in $\{1, \dots, m\}$.

Proposition 3.2 ([16]) *Consider the bilinear system*

$$\dot{x} = Ax + \frac{1}{\epsilon}\sum_{i=1}^{m} u_i\left(\frac{t}{\epsilon}, t\right) B_i x ,$$

and assume $B_i B_j = 0$, $\forall i, j$. Consider also the problem

$$\dot{y} = \left(A + \sum_{i=1}^{m} \overline{U}_{\{i\}}(t)\,\mathrm{ad}_{B_i} A + \sum_{i,j=1}^{m} \overline{U}_{\{i,j\}}(t)\,\mathrm{ad}_{B_i}\mathrm{ad}_{B_j} A - \sum_{i=1}^{m} B_i \frac{d}{dt}\overline{U}_{\{i\}}(t)\right) y ,$$

with initial condition $y(0) = x(0)$. Then,

$$x(t) = \mathrm{e}^{\sum_{i=1}^{m} B_i \int_0^{(t/\epsilon \mod T)} u_i(\tau,t) d\tau} y(t) + O(\epsilon) .$$

The result follows from equation (4) by noting that $B_i B_j = 0$ for all i, j implies $\mathrm{ad}_{B_i}\mathrm{ad}_{B_j}\mathrm{ad}_{B_k} A = 0$ for all i, j, k.

Next, we consider the class of second-order control systems. Define the *symmetric product* between g_i, g_j by

$$\langle g_i : g_j \rangle = \frac{\partial g_i}{\partial x} g_j + \frac{\partial g_j}{\partial x} g_i .$$

Proposition 3.3 (Second-order systems)
Consider the control system

$$\ddot{x} + f_1(x)\dot{x} + f_0(x) = \frac{1}{\epsilon}\sum_{i=1}^{m} u_i\left(\frac{t}{\epsilon}, t\right) g_i(x) ,$$

and the initial value problem

$$\ddot{y} + f_1(y)\dot{y} + f_0(y) = \frac{1}{2}\sum_{i,j=1}^{m}\left(\overline{U}_{\{i\}}(t)\overline{U}_{\{j\}}(t) - \overline{U}_{\{i,j\}}(t) - \overline{U}_{\{j,i\}}(t)\right)\langle g_i : g_j\rangle (y)$$

with initial conditions $y(0) = x(0)$, $\dot{y}(0) = \dot{x}(0) + \sum_{i=1}^{m}\overline{U}_{\{i\}}(0) g_i(x(0))$. Then, we have

$$x(t) = y(t) + O(\epsilon)$$
$$\dot{x}(t) = \dot{y}(t) + \sum_{i=1}^{m} g_i(y(t))\left(\int_0^{(t/\epsilon \mod T)} u_i(\tau,t)d\tau - \overline{U}_{\{i\}}(t)\right) + O(\epsilon) .$$

4 Stabilization via oscillatory controls

Here we discuss the problem of stabilization of the nonlinear system $\dot{x} = f(x)$ by means of highly oscillatory controls $(1/\epsilon)u(t/\epsilon)\,g(x)$ making use of the result in Theorem 3.1. We shall prove either asymptotic stability for the original equilibrium point (t-stabilizability after [23]) or that the equilibrium bifurcates to an asymptotically stable periodic orbit contained in an $O(\epsilon)$-neighbor (v-stabilizability in [23]).

Lemma 4.1 *The origin is an equilibrium point of the averaged system, that is, $\overline{F}(0) = 0$, if either of the following conditions are satisfied:*

(i) $f(0) = g_1(0) = \cdots = g_m(0) = 0$,

(ii) $f(0) = 0$ and f is an odd function, g_j is an even function for all $1 \leq j \leq m$, there exists i such that $g_i(0) \neq 0$, and $\overline{\mathcal{U}}_{k_1,\dots,k_m} = 0$ whenever $\sum_j k_j$ is odd.

Proposition 4.2 *Assume $f(0) = g_1(0) = \cdots = g_m(0) = 0$. At the origin, the linearization of the averaged system equals the average of the linearized system.*

Proof: We prove it for the single input setting. Let $f = \sum_{i=1}^{+\infty} f^{[i]}$, $g = \sum_{i=1}^{+\infty} g^{[i]}$ be the Taylor expansions around $x = 0$ of f and g, with $f^{[i]}$, $g^{[i]}$ homogeneous polynomials of degree i. Then we have

$$\mathrm{ad}_g^k f = \sum_{\substack{j=1\\ i_1,\dots,i_k=1}}^{+\infty} \mathrm{ad}_{g^{[i_1]}}\dots\mathrm{ad}_{g^{[i_k]}} f^{[j]} = \mathrm{ad}_{g^{[1]}}^k f^{[1]} + h ,$$

where h is an infinite sum of homogeneous polynomials of degree ≥ 2. Consequently, $\frac{\partial}{\partial x}\left(\mathrm{ad}_g^k f\right)(0) = \mathrm{ad}^k_{\frac{\partial g}{\partial x}(0)}\frac{\partial f}{\partial x}(0)$, where one adjoint operator is a Lie bracket and the other a matrix commutator. The linearization of the averaged system is then equal to

$$\frac{\partial \overline{F}}{\partial x}(0) = \frac{\partial f}{\partial x}(0) + \sum_{k=1}^{+\infty}\overline{U}_k\,\mathrm{ad}^k_{\frac{\partial g}{\partial x}(0)}\frac{\partial f}{\partial x}(0) ,$$

which is the average of the linearized system. ■

Note that the setting of bilinear systems (cf. Proposition 3.2) is very important as it represents the linearization of the average of any nonlinear system with $f(0) = g_1(0) = \cdots = g_m(0) = 0$.

Corollary 4.3 *Let $f(0) = g_1(0) = \cdots = g_m(0) = 0$. If the trace of the linearization of f is positive, the averaged system is unstable for any oscillatory control law.*

Proof: Since $\operatorname{tr}(\operatorname{ad}_C D) = 0$ for any matrix C, D,

$$\operatorname{tr}\left(\frac{\partial \overline{F}}{\partial x}(0)\right) = \operatorname{tr}\left(\sum_{k_1,\ldots,k_m \geq 0}^{+\infty} \overline{\mathcal{U}}_{k_1,\ldots,k_m} \operatorname{ad}^{k_1}_{\frac{\partial g_1}{\partial x}(0)} \cdots \operatorname{ad}^{k_m}_{\frac{\partial g_m}{\partial x}(0)} \frac{\partial f}{\partial x}(0)\right) = \operatorname{tr}\left(\frac{\partial f}{\partial x}(0)\right) > 0\,,$$

and therefore the averaged system is unstable. ■

The corollary is a twofold generalization of the result in [23] about stabilizability by linear multiplicative vibrations. First, we do not require $\frac{\partial f}{\partial x}(0)$ to be nonderogatory. Second, we consider general nonlinear systems and vibrations. Next, we present a classical result on stabilization by means of oscillatory controls.

Proposition 4.4 ([23]) *Consider the system $\dot{x} = f(x)$, with $f(0) = 0$. If $A = \partial f / \partial x(0)$ is nonderogatory and $\operatorname{tr} A < 0$, then there exist commuting linear vector fields $\{g_i\}_{i=1}^{n-1}$ and controls $\{u_i\}_{i=1}^{n-1}$ such that the equilibrium $x = 0$ is asymptotically stable for*

$$\dot{x} = f(x) + \frac{1}{\epsilon}\sum_{i=1}^{n-1} u_i\left(\frac{t}{\epsilon}\right) g_i(x)\,. \tag{5}$$

An interesting observation is that systems with positive trace may be stabilized by means of vibrations g with $g(0) \neq 0$ (see [27]). Here, we give the following result, whose constructive proof is omitted due to space limitations.

Proposition 4.5 ([2]) *Consider the system $\dot{x} = Ax$, with A nonderogatory and $\operatorname{tr} A > 0$. Then, there exist a nonlinear vector field g_{nl}, commuting vector fields $\{g_i\}_{i=1}^{n-1}$ and controls $\{u_i\}_{i=1}^{n-1}$ such that the equilibrium $x = 0$ becomes an asymptotically stable periodic orbit contained in an $O(\epsilon)$-neighborhood of 0 for the system*

$$\dot{x} = Ax + g_{\mathrm{nl}}(x) + \frac{1}{\epsilon}\sum_{i=1}^{n-1} u_i\left(\frac{t}{\epsilon}\right) g_i(x)\,.$$

5 Tracking via oscillatory controls

Here we apply our averaging analysis to the trajectory tracking problem via oscillatory controls for underactuated second-order systems. Let

$$\ddot{x} + f_1(x)\dot{x} + f_0(x) = \sum_i w_i\, g_i(x)\,, \tag{6}$$

and consider the tracking problem: given a smooth desired curve $x^d : [0,T] \to \mathbb{R}^n$ with initial conditions $x^d(0) = x(0)$, $\dot{x}^d(0) = \dot{x}(0)$, find controls laws $w_i : \mathbb{R}^{2n} \times [0,T] \to \mathbb{R}^m$ such that the solution x to equation (6) approximates x^d up to an order ϵ error.

We make the following controllability assumption: (A) *the distribution* $\operatorname{span}\{g_i, \langle g_j : g_k\rangle\}$ *is full rank, and* $\langle g_j : g_j\rangle$ *belongs to* $\operatorname{span}\{g_i\}$. Accordingly, there exist functions $z_i^d, z_{jk}^d : [0,T] \to \mathbb{R}$, for $j < k$, such that

$$\ddot{x}^d + f_1(x^d)\dot{x}^d + f_0(x^d) = \sum_i z_i^d g_i(x^d) + \sum_{j<k} z_{jk}^d \langle g_j : g_k\rangle (x^d)\,,$$

and there exist smooth functions $\alpha_{ij} : \mathbb{R}^n \to \mathbb{R}$ with $\langle g_i : g_i\rangle(x) = \sum_j \alpha_{ij}(x) g_j(x)$, $\forall x \in \mathbb{R}^n$. There are $N = m(m-1)/2$ pairs of integers (j,k), with $j < k$. Let $(j,k) \mapsto a(j,k) \in \{1,\ldots,N\}$ be a enumeration of these pairs, and define the scalar functions $\psi_{a(j,k)}(t) = \sqrt{2}\, a(j,k)\cos(a(j,k)t)$.

Proposition 5.1 *Let $x^d : [0,T] \to \mathbb{R}^n$ be a curve with initial conditions $x^d(0) = x(0)$, $\dot{x}^d(0) = \dot{x}(0)$. The solution x to equation (6) equals x^d up to an error of order ϵ over the time scale 1 under the controls w_i,*

$$w_i = v_i(t,x) + \frac{1}{\epsilon} u_i\left(\frac{t}{\epsilon}, t\right),$$

$$v_i(t,x) = z_i^d(t) + \frac{1}{2}\sum_j \alpha_{ji}(x)\left(j - 1 + \sum_{\ell=j+1}^{m} (z_{j\ell}^d(t))^2\right),$$

$$u_i(\tau,t) = -\sum_{\ell=1}^{i-1} \psi_{a(\ell,i)}(\tau) + \sum_{\ell=i+1}^{m} z_{i\ell}^d(t)\psi_{a(i,\ell)}(\tau)\,.$$

Proof: The control system (6) is written as

$$\ddot{x} + f_1(x)\dot{x} + f_0(x) = \sum_i v_i(t,x)g_i(x) + \frac{1}{\epsilon}\sum_i u_i\left(\frac{t}{\epsilon}, t\right) g_i(x)\,,$$

and, following Proposition 3.3, its averaged system is

$$\ddot{y} + f_1(y)\dot{y} + f_0(y) = \sum_i v_i(t,y)g_i(y) + \sum_i\left(\frac{1}{2}\overline{U}^2_{\{i\}}(t) - \overline{U}_{\{i,i\}}(t)\right)\langle g_i : g_i\rangle(y)$$
$$+ \sum_{i<j}\left(\overline{U}_{\{i\}}(t)\overline{U}_{\{j\}}(t) - \overline{U}_{\{i,j\}}(t) - \overline{U}_{\{j,i\}}(t)\right)\langle g_i : g_j\rangle(y)\,,$$

with initial conditions $(y(0), \dot{y}(0)) = (x(0), \dot{x}(0) + \sum_i \overline{U}_{\{i\}}(0) g_i(x(0))$. The iterated integrals of $\{u_i\}$ are

$$\overline{U}_{\{i\}}(t) = \frac{1}{T}\int_0^T u_i(\tau, t) d\tau = 0\,,$$
$$\overline{U}_{\{i,j\}}(t) + \overline{U}_{\{j,i\}}(t) = \overline{U_{\{i\}} U_{\{j\}}}(t) = -z^d_{ij}(t)\,,$$

for $i < j$, so that the averaged system reads

$$\ddot{y} + f_1(y)\dot{y} + f_0(y) = \sum_i v_i(t, y) g_i(y)$$
$$- \sum_i \overline{U}_{\{i,i\}}(t) \langle g_i : g_i \rangle (y) + \sum_{i<j} z^d_{ij}(t) \langle g_i : g_j \rangle (y)\,.$$

Since $\overline{U}_{\{j,j\}}(t) = \frac{1}{2}\left(j - 1 + \sum_{\ell=j+1}^m (z^d_{j\ell}(t))^2\right)$, then

$$\sum_i v_i(t, y) g_i(y) = \sum_i z^d_i(t) g_i(y) + \sum_i \overline{U}_{\{i,i\}}(t) \langle g_i : g_i \rangle (y)$$

where we have exploited the property of the functions α_{ij}. In summary, we have shown that

$$\ddot{y} + f_1(y)\dot{y} + f_0(y) = \sum_i z^d_i g_i(y) + \sum_{i<j} z^d_{ij} \langle g_i : g_j \rangle (y)\,,$$

with initial conditions $y(0) = x^d(0)\,, \dot{y}(0) = \dot{x}^d(0)$. Since y and x^d solve the same initial value problem, they are identical. Finally, from Proposition 3.3, we conclude $x(t) = y(t) + O(\epsilon) = x^d(t) + O(\epsilon)$. ∎

A second-order nonholonomic integrator

There are many interesting dynamical extensions of Brockett's nonholonomic integrators; see the discussion in [26]. We consider

$$\ddot{x}_1 = w_1\,, \quad \ddot{x}_2 = w_2\,, \quad \ddot{x}_3 = w_1 x_2 + w_2 x_1\,,$$

and note that it fulfills the controllability assumption (A). We design control inputs to track a desired trajectory, $(x^d_1(t), x^d_2(t), x^d_3(t))$, after Proposition 5.1,

$$w_1 = \ddot{x}^d_1 + \frac{1}{\sqrt{2}\epsilon}\left(\ddot{x}^d_3 - \ddot{x}^d_1 x^d_2 - \ddot{x}^d_2 x^d_1\right)\cos\left(\frac{t}{\epsilon}\right)$$
$$w_2 = \ddot{x}^d_2 - \frac{\sqrt{2}}{\epsilon}\cos\left(\frac{t}{\epsilon}\right) \tag{7}$$

Figure 1 illustrates the performance of these controls.

A PVTOL model

We consider the model of a simple planar vertical take-off and landing aircraft model based upon that of [28] with added viscous damping forces; see Figure 2. We parametrize its configuration and velocity space via the state variables $\{x, z, \theta, v_x, v_z, \omega\}$. The angular velocity is ω and the linear velocities in the body-fixed x (resp. z) axis are v_x (resp. v_z). The equations are written as:

$$\dot{x} = \cos\theta v_x - \sin\theta v_z\,,\ \dot{z} = \sin\theta v_x + \cos\theta v_z\,,\ \dot{\theta} = \omega\,,$$
$$\dot{v}_x = (-k_1/m)v_x - g\sin\theta + v_z\omega + (1/m)w_2\,, \tag{8}$$
$$\dot{v}_z = (-k_2/m)v_z - g(\cos\theta - 1) - v_x\omega + (1/m)w_1\,,$$
$$\dot{\omega} = (-k_3/J)\omega + (h/J)w_2\,.$$

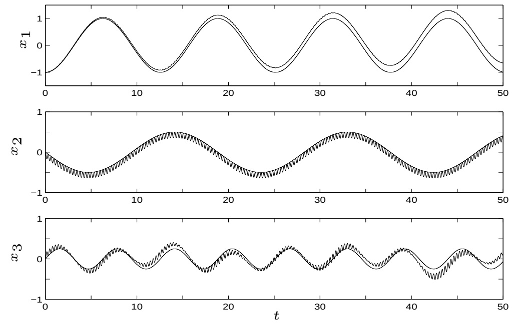

Figure 1: Tracking for the modified nonholonomic integrator with the controls in (7) and $\epsilon = .05$.

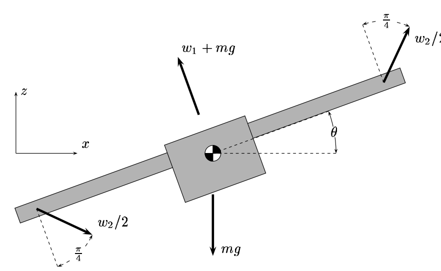

Figure 2: Diagram of the PVTOL model.

The distance from the center of mass to the wingtip is h, while m and J are mass and moment of inertia. Equations (8) can be written as a second-order system in (x, z, θ) and the model fulfills the controllability assumption (A). The simulations are run with $m = 20$, $J = 10$, $h = 5$, $k_1 = 12$, $k_2 = 11$, $k_3 = 10$, $g = 9.8$. Fig. 3 shows an example of the tracking behavior. Fig. 4 illustrates the linear decay of the error.

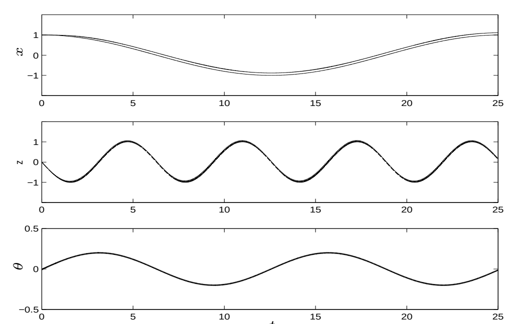

Figure 3: Tracking with $\epsilon = .01$.

Lagrangian systems on manifolds

Proposition 5.1 can be extended to a large class of Lagrangian control systems within the so-called affine connection formalism; e.g., see [18]. Let q be the system's configuration on the n-dimensional manifold Q, and let $\{\Gamma^a_{bc}, a, b, c \in \{1, \ldots, n\}\}$ be the n^3 Christoffel functions associated to the system's kinetic energy.

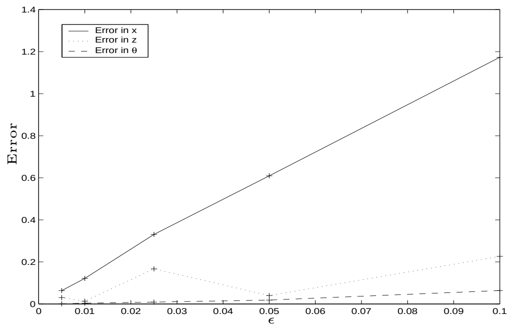

Figure 4: Illustration of the tracking errors at $t = 10$.

Define the operation of symmetric product between the vector fields g_i, g_j on Q according to

$$\langle g_i : g_j \rangle^a = \frac{\partial g_i^a}{\partial q^b} g_j^b + \frac{\partial g_j^a}{\partial q^b} g_i^b + \Gamma_{bc}^a \left(g_i^b g_j^c + g_i^c g_j^b \right) ,$$

and define the quantity $(\nabla_{\dot{q}} \dot{q})^a = \ddot{q}^a + \Gamma_{bc}^a(q) \dot{q}^b \dot{q}^c$. Then, the Euler-Lagrange equations read

$$\nabla_{\dot{q}} \dot{q} + f_1(q) \dot{q} + f_0(q) = \sum_i w_i g_i(q) .$$

Under the controllability assumption (A), the result in Proposition 5.1 holds verbatim.

6 Conclusions

Based on a coordinate-free averaging analysis of oscillatory control systems, we have developed design tools for stabilization and trajectory tracking in certain classes of nonlinear systems. Avenues for future research include extending these results to the setting of higher-order averaging, distributed parameter systems, time-delayed systems, and systems with resonances.

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