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Controllable kinematic reductions for mechanical systems: concepts, computational tools, and examples

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Abstract

This paper introduces the novel notion of kinematic reductions for mechanical systems and studies their controllability properties. We focus on the class of simple mechanical control systems with constraints and model them as affine connection control systems. For these systems, a kinematic reduction is a driftless control system whose controlled trajectories are also solutions to the full dynamic model under appropriate controls. We present a comprehensive treatment of local controllability properties of mechanical systems and their kinematic reductions. Remarkably, a number of interesting reduction and controllability conditions can be characterized in terms of a certain vector-valued quadratic form. We conclude with a catalog of example systems and their kinematic reductions.

1 Introduction

The setting of affine connection control systems can be used to model a large class of mechanical systems from a Lagrangian point of view. It provides a particularly convenient viewpoint for systems with no external forces other than the applied control forces (e.g., no potential or dissipation forces). These are difficult control systems since they have unstabilizable linearizations, and so fail Brockett's necessary condition for the existence of continuous stabilizing feedback. What's more, many of the systems are not known to be flat, and cannot generally be put into a form where backstepping methods may be applied. Indeed, existing control methodologies will generally not apply to the class of mechanical systems we consider in this paper. Thus one must set about understanding these systems in their own right.

Kinematic reductions and hybrid models of motion control systems

An objective of this paper is to characterize mechanical control systems in terms of equivalent lower-dimensional kinematic (or driftless) systems. The interest in low-complexity representations of mechanical control systems can be related to numerous previous efforts, including work on hybrid models for motion control systems [1], motion description languages [2], oscillatory motion primitives [3], consistent control abstractions [4], hierarchical steering algorithms [5], and maneuver automata [6].

In Section 3, we introduce the notion of kinematic reduction as a model reduction technique adapted to mechanical control systems. This novel concept extends and unifies our previous results in [7, 8]. A kinematic model for a mechanical system is one such that every controlled trajectory for the kinematic model can be implemented as a trajectory of the full second-order system under some appropriate control input.

The key advantage of a low-complexity system representation is the subsequent simplification of various control problems including planning, stabilization, and optimal control. In general, a reduced-order representation of the system dynamics will be useful in any hierarchical control scheme. For example, when considering planning problems, motion along a kinematic reduction can be regarded as a motion primitive to be used in higher-level motion scripts. Given a rich family of motion primitives, planning can then be performed via a variety of analytical or numerical methods; e.g., see [9, 10, 11] on inverse kinematics, nonlinear programming, and randomized algorithms.

Local controllability and computational tools

An important obvious property to require of kinematic reductions is controllability. We therefore proceed to characterize locally controllable kinematic reductions and relate them to the current understanding on the matter of local controllability for mechanical control systems.

Initial accessibility results and some weak local controllability results for affine connection control systems were provided in [12]. This work also introduces a fundamental distinction between controllability and configuration controllability. Recently, progress has been made on the local controllability problem for such systems in [13], which provides first-order conditions for local controllability in terms of a vector-valued quadratic form.

Building on this body of knowledge, it is straightforward to define and characterize controllability for kinematic reductions. A mechanical system is locally kinematically controllable if it admits a kinematic reduction which is a locally controllable driftless system. A locally kinematically controllable system is therefore small-time locally configuration controllable.

One interesting outcome of our conditions for kinematic controllability is that they have a strong connection to the vector-valued quadratic form condition for local controllability in [13]. Indeed, it appears that many (but not all) systems satisfying the sufficient condition of [13] are also locally kinematically controllable. Physical examples of such systems include the planar rigid body with a single, variable-direction thruster [3], the spatial version of the same system [3], a three-link planar manipulator with various actuator configurations [8], a hopping robot while in flight phase [14], and the snakeboard [15]. We present all these systems and summarize their properties in a detailed catalog.

Organization

The paper is organized as follows. Section 2 presents a modeling framework for simple mechanical control systems with constraints. Section 3 introduces and characterizes the notion of a kinematic reduction. Section 4 presents controllability definitions and tests; Subsection 4.1 describes a set of inferences, counterexamples, and special results for low-dimensional systems. Finally, Section 5 presents a catalog of mechanical control systems.

2 Modeling mechanical control systems via affine connections

In this section we review some ideas on modeling of mechanical control systems. We consider the class of simple mechanical control systems with constraints. We model them as affine connection systems, and study their representations in various local bases of vector fields. In this way, we recover the controlled geodesic, Poincaré and Euler-Lagrange equations. We refer the reader to the more detailed presentations in [16, 17].

Simple mechanical control systems with constraints

A *simple mechanical control system with constraints* is a quintuple $(\mathsf{Q}, \mathbb{G}, V, \mathcal{D}, \mathcal{F})$ comprised of the following objects:

(i) an n-dimensional configuration manifold Q,

(ii) a Riemannian metric $\mathbb{G}$ on Q describing the kinetic energy,

(iii) a function V on Q describing the potential energy,

(iv) a distribution $\mathcal{D}$ of feasible velocities describing the linear velocity constraints, and

(v) a collection of m covector fields $\mathcal{F} = \{F^1, \dots, F^m\}$, linearly independent at each $q \in \mathsf{Q}$, defining the control forces.

Given the metric $\mathbb{G}$ and the distribution $\mathcal{D}$, we define the following objects. We let $P : T\mathsf{Q} \to T\mathsf{Q}$ be the orthogonal projection onto the distribution $\mathcal{D}$ with respect to the metric $\mathbb{G}$. We let ${}^{\mathbb{G}}\nabla$ be the Levi-Civita connection on Q induced by the metric $\mathbb{G}$. We let ∇ be the *constrained affine connection* defined by the metric $\mathbb{G}$ and the constraint distribution $\mathcal{D}$ according to

$$\nabla_X Y = {}^{\mathbb{G}}\nabla_X Y - \left({}^{\mathbb{G}}\nabla_X P\right)(Y),$$

for any vector fields X and Y. When the vector field Y takes value in $\mathcal{D}$, we have

$$\nabla_X Y = P({}^{\mathbb{G}}\nabla_X Y),$$

as shown in [18].

Given the Riemannian metric $\mathbb{G}$, we let $\mathbb{G}\colon T\mathsf{Q} \to T^*\mathsf{Q}$ and $\mathbb{G}^{-1}\colon T^*\mathsf{Q} \to T\mathsf{Q}$ denote the musical isomorphisms associated with $\mathbb{G}$. For $a \in \{1, \dots, m\}$, we define the input vector fields $Y_a = P(\mathbb{G}^{-1}(F^a))$, the family of *input vector fields* $\mathcal{Y} = \{Y_1, \dots, Y_m\}$, and the *input distribution* $\mathcal{Y}$ with $\mathcal{Y}_q = \operatorname{span}_{\mathbb{R}}\{Y_1(q), \dots, Y_m(q)\}$. Let $\mathcal{L}_X f$ be the Lie derivative of a scalar function f with respect to the vector field X. The *gradient* of the function V is the vector field $\operatorname{grad} V$ defined implicitly by

$$\mathbb{G}(\operatorname{grad} V, X) = \mathcal{L}_X V.$$

A *controlled trajectory* for the mechanical control system with constraints $(\mathsf{Q}, \mathbb{G}, V, \mathcal{D}, \mathcal{F})$ is a pair (γ, u) with $\gamma\colon [0, T] \to \mathsf{Q}$ and $u = (u_1, \dots, u_m)\colon [0, T] \to \mathbb{R}^m$ satisfying the *controlled geodesic equations*

$$\nabla_{\dot\gamma(t)}\dot\gamma(t) = -P(\operatorname{grad} V(\gamma(t))) + \sum_{a=1}^{m} Y_a(\gamma(t))u_a(t). \tag{2.1}$$

Here we assume that $\dot{\gamma}(0) \in \mathcal{D}_{\gamma(0)}$ and comment that this implies that $\dot{\gamma}(t) \in \mathcal{D}_{\gamma(t)}$ for all $t \in [0, T]$. Furthermore, we assume the input functions $u = (u_1, \ldots, u_m) \colon [0, T] \to \mathbb{R}^m$ to be Lebesgue measurable functions, and we write $u \in \mathcal{U}^m_{\mathrm{dyn}}$.

Coordinate representations

On an open subset $U \subset \mathsf{Q}$ let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a basis of vector fields. We write the covariant derivative of the vector fields in the basis $\mathcal{X}$ as

$$\nabla_{X_i} X_j = ({}^{\mathcal{X}}\Gamma)^k_{ij} X_k, \tag{2.2}$$

where the n^3 functions $\{({}^{\mathcal{X}}\Gamma)^k_{ij} |\ i, j, k \in \{1, \ldots, n\}\}$ are called the *generalized Christoffel symbols* with respect to $\mathcal{X}$. Given vector fields Y and Z on U, we can write $Y = Y^i X_i$ and $Z = Z^i X_i$. Accordingly, the covariant derivative of the vector field Z with respect to the vector field Y is

$$\nabla_Y Z = \left((\mathcal{L}_{X_i} Z^k) Y^i + ({}^{\mathcal{X}}\Gamma)^k_{ij} Z^i Y^j \right) X_k.$$

It is instructive to write the controlled Euler-Lagrange equations with respect to the basis $\mathcal{X}$. Let the velocity curve $\dot{\gamma} \colon I \to TU$ have components $(v^1, \ldots, v^n)$ with respect to $\mathcal{X}$, i.e.,

$$\dot{\gamma}(t) = v^i(t) X_i(\gamma(t)).$$

The pair (γ, u) is a controlled trajectory for the controlled geodesic equations (2.1) if and only if it solves the *controlled Poincaré equations*

$$\dot{v}^k + ({}^{\mathcal{X}}\Gamma)^k_{ij}(\gamma) v^i v^j = -\left(P \operatorname{grad} V\right)^k(\gamma) + \sum_{a=1}^m Y^k_a(\gamma) u_a. \tag{2.3}$$

Remark 2.1. If the distribution $\mathcal{D}$ has rank $p < n$, it is useful to construct a local basis for $T\mathsf{Q}$ by selecting the first p vector fields to generate $\mathcal{D}$, and the remaining $n - p$ to generate $\mathcal{D}^{\perp}$. In this case, one can see that $v^k(t) = 0$ for all time t and all $k \in \{p + 1, \ldots, n\}$.

Remark 2.2. Assume a Lie group G acts on the manifold Q, and assume the metric $\mathbb{G}$, and the distribution $\mathcal{D}$ are invariant. Then the constrained connection ∇ is invariant, and, selecting invariant vector fields $\{X_1, \ldots, X_n\}$, the generalized Christoffel symbols are invariant functions.

Let $(q^1, \ldots, q^n)$ be a coordinate system for the open subset $U \subset \mathsf{Q}$. The curve $\gamma \colon I \to U$ has therefore components $(\gamma^1, \ldots, \gamma^n)$. The coordinate system on U induces the natural coordinate basis $\{\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\}$ for the tangent bundle TU. With respect to this basis, we write the velocity curve $\dot{\gamma} \colon I \to TU$ as

$$\dot{\gamma}(t) = \dot{\gamma}^i(t) \frac{\partial}{\partial q^i}(\gamma).$$

In the coordinate system $(q^1, \ldots, q^n)$, we write $\gamma = (\gamma^1, \ldots, \gamma^n)$, $\dot{\gamma} = (\dot{\gamma}^1, \ldots, \dot{\gamma}^n)$, and the equations of motion read

$$\ddot{\gamma}^k + \Gamma^k_{ij} \dot{\gamma}^i \dot{\gamma}^j = -\left(P \operatorname{grad} V\right)^k(\gamma) + \sum_{a=1}^m Y^k_a u_a. \tag{2.4}$$

Here, the Christoffel symbols $\{\Gamma^k_{ij} |\ i, j, k \in \{1, \ldots, n\}\}$ and the terms in the right-hand side are computed with respect to the natural coordinate basis. We refer to these equations as the *controlled Euler-Lagrange equations*.

3 Kinematic reductions for mechanical control systems

In this section we relate (i) controlled trajectories for the (second-order) controlled geodesic equation (2.1) to (ii) controlled trajectories for driftless control systems on Q. The purpose is to establish relationships between the given mechanical control system and an appropriate low-complexity kinematic representation.

Remark 3.1. For the remainder of the paper, we restrict our attention to mechanical control systems subject to no potential energy, i.e., we set $V = 0$.

Let us start by establishing some nomenclature. We refer to second-order differential equations on Q of the form (2.1) as *dynamic models* of mechanical systems. In dynamic models the control inputs are accelerations. In contrast to this, we refer to first-order differential equations on Q as *kinematic models* of mechanical systems. In kinematic models the control inputs are velocity variables. Let $\mathcal{V} = \{V_1, \dots, V_\ell\}$ be a family of vector fields linearly independent at each $q \in \mathsf{Q}$. For curves $\gamma \colon [0, T] \to \mathsf{Q}$ and $w \colon [0, T] \to \mathbb{R}^\ell$, consider the differential equation

$$\dot{\gamma}(t) = \sum_{b=1}^{\ell} V_b(\gamma(t)) w_b(t). \tag{3.5}$$

We shall assume that the control inputs to kinematic systems are absolutely continuous, and we write $w \in \mathcal{U}^\ell_{\mathrm{kin}}$. We shall refer to the system as the kinematic model (or kinematic system) induced by $\mathcal{V}$.

Next, we establish relationships between controlled trajectories of kinematic and dynamic systems.

Kinematic reductions and decoupling vector fields

The kinematic model induced by $\mathcal{V} = \{V_1, \dots, V_\ell\}$ is said to be a *kinematic reduction* of the second-order system (2.1) if, for any control input $w \in \mathcal{U}^\ell_{\mathrm{kin}}$ and corresponding controlled trajectory (γ, w) for equation (3.5), there exists a control input $u \in \mathcal{U}^m_{\mathrm{dyn}}$ such that (γ, u) is a controlled trajectory for the second-order system (2.1). In other words, for any curve $\gamma \colon I \to \mathsf{Q}$ solving the equation (3.5) with $w \in \mathcal{U}^\ell_{\mathrm{kin}}$, there exists a control $u \in \mathcal{U}^m_{\mathrm{dyn}}$ such that (γ, u) is a controlled trajectory for the second-order system (2.1). Roughly speaking, the curve $\gamma \colon I \to \mathsf{Q}$ solving (3.5) can be lifted to a solution to the second-order system (2.1).

The *rank* of a kinematic reduction is the rank of the distribution generated by the vector fields $\mathcal{V}$. Rank-one kinematic reductions are particularly interesting. We shall call a vector field V *decoupling* if the rank-one kinematic system induced by $\mathcal{V} = \{V\}$ is a kinematic reduction. Hence, the second-order control system (2.1) can be steered along any time-scaled integral curve of a decoupling vector field. For a dynamic control system with a rank-m input distribution, there are at most m rank-one kinematic reductions linearly independent at each $q \in \mathsf{Q}$.

Before proceeding, we define the symmetric product of two vector fields X and Y as the vector field

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X.$$

The following theorem characterizes kinematic reductions in terms of the affine connection and the input distribution of the given dynamic model. A simplified version of this result is proved in [8].

Theorem 3.2. *A kinematic model induced by* $\{V_1, \dots, V_\ell\}$ *is a kinematic reduction of the second-order system* (2.1) *if and only if the distribution generated by the vector fields* $\{V_i, \langle V_j : V_k\rangle \mid i, j, k \in \{1, \dots, \ell\}\}$ *is a constant rank subbundle of the input distribution* $\mathcal{Y}$*.*

Mechanical systems fully reducible to kinematic systems

We are here interested in characterizing *when is a mechanical system kinematic?* That is, we are interested in when the largest possible kinematic reduction will be attained. By Theorem 3.2, any kinematic reduction must be contained in $\mathcal{Y}$, so one can do no better than have $\mathcal{Y}$ itself as a kinematic reduction. Formally, we say that the dynamic model (2.1) is *fully reducible to the kinematic system induced by* $\mathcal{V}$ if, $\mathcal{V}$ is a kinematic reduction of (2.1) and if, for any control input $u \in \mathcal{U}^m_{\text{dyn}}$, initial condition $\dot\gamma(0) \in \mathcal{V}$, and corresponding controlled trajectory (γ, u) for equation (2.1), there exists a control input $w \in \mathcal{U}^\ell_{\text{kin}}$ such that (γ, w) is a controlled trajectory for the kinematic system (3.5) induced by $\mathcal{V}$. A dynamic system (2.1) is *fully reducible to a kinematic system* is there exists one such collection of vector fields $\mathcal{V}$.

Before proceeding, we introduce a useful notion. A distribution $\mathcal{X}$ is said to be *geodesically invariant* if it is closed under operation of symmetric product, i.e., if for all vector fields X and Y taking values in $\mathcal{X}$, the vector field $\langle X : Y\rangle$ also takes value in $\mathcal{X}$. The *symmetric closure* of the distribution $\mathcal{X}$ is the smallest geodesically invariant distribution containing $\mathcal{X}$. The motivation for the term "geodesically invariant" is explained in [18].

The following theorem characterizes dynamic systems which are fully reducible to kinematic systems; it is proved in [7].

Theorem 3.3. *A mechanical control system* (2.1) *is fully reducible to a kinematic system if and only if*

(i) the kinematic system is induced by the input distribution $\mathcal{Y}$ *and*

(ii) the input distribution $\mathcal{Y}$ *is geodesically invariant.*

Bases of decoupling vector fields for the input distribution

According to Theorem 3.3, testing if a mechanical system is fully reducible to a kinematic system is a straightforward test. For such a mechanical control system, any vector field taking values in the input distribution is decoupling. For mechanical control systems which are not fully reducible to a kinematic system, we continue our investigation into kinematic reductions, and in particular into rank-one reductions, i.e., decoupling vector fields. *When is there a basis of decoupling vector fields for the input distribution?*

The material in this section, and some of that in the next, relies on the notion of a vector-valued bilinear map. For $\mathbb{R}$-vector spaces E and F, let $B\colon E \times E \to F$ be symmetric and bilinear. For $\lambda \in F^*$ we denote by $\lambda B\colon E \times E \to \mathbb{R}$ the map defined by $\lambda B(m_1, m_2) = \lambda \cdot B(m_1, m_2)$. B is *definite* if there exists $\lambda \in F^*$ so that λB is positive-definite. B is *indefinite* if for each $\lambda \in F^* \setminus \operatorname{ann}(\operatorname{image}(B))$, λB is neither positive nor negative semidefinite ($\operatorname{ann}(S) \subset F^*$ is the annihilator of $S \subset F$). The following result is proved in [19].

Proposition 3.4. *For a symmetric bilinear map* $B\colon E \times E \to F$ *and for* $\lambda \in F^* \setminus \operatorname{ann}(\operatorname{image}(B))$*, the following statements are equivalent:*

(i) λB *is indefinite;*

(ii) *there exists a basis for* E *so that the diagonal entries for the matrix of* λB *sum to zero;*

(iii) *there exists a basis for* E *so that all diagonal entries in the matrix for* λB *are zero.*

Now define $B_{\mathcal{Y}} : \mathcal{Y} \times \mathcal{Y} \to T\mathsf{Q}/\mathcal{Y}$ as the $T\mathsf{Q}/\mathcal{Y}$-valued symmetric, bilinear bundle mapping given by

$$B_{\mathcal{Y}}(q)(v_1, v_2) = \pi_{\mathcal{Y}}(\langle X_1 : X_2 \rangle(q)),$$

where $\pi_{\mathcal{Y}}$ is the canonical projection onto $T\mathsf{Q}/\mathcal{Y}$, and where X_1 and X_2 are vector fields extending v_1 and v_2, respectively (one readily shows that $B_{\mathcal{Y}}(q)$ is independent of these extensions). If V is a decoupling vector field, then $B_{\mathcal{Y}}(V, V) = 0$. If $V_1, \dots, V_m$ are decoupling, and if we write the vector-valued bilinear form with respect to this basis, then its matrix representation has zeros along the diagonal. Vice-versa, assume we can find a basis such that all elements in the diagonal are zero, then that basis would be a basis of decoupling vector fields.

From Proposition 3.4 we immediately have the following result which summarizes the relationship between $B_{\mathcal{Y}}$ and the existence of a basis for the input distribution of decoupling vector fields.

Proposition 3.5. *If the input distribution* $\mathcal{Y}$ *for a simple mechanical system admits a (local) basis of decoupling vector fields, then* $B_{\mathcal{Y}}(q)$ *is indefinite for each* $q \in \mathsf{Q}$*. Furthermore, if* $\mathcal{Y}$ *is codimension one, then* $\mathcal{Y}$ *admits a (local) basis of decoupling vector fields if and only if* $B_{\mathcal{Y}}(q)$ *is indefinite for each* $q \in \mathsf{Q}$*.*

4 Accessibility and controllability notions

Let $[X, Y]$ be the Lie bracket between the vector fields X and Y. Given a collection of vector fields $\mathscr{X} = \{X_1, \dots, X_\ell\}$, consider the associated distribution $\mathcal{X}$ defined by $\mathcal{X}_q = \text{span}_{\mathbb{R}}\{X_1(q), \dots, X_\ell(q)\}$. The distribution $\mathcal{X}$ is said to be *involutive* if it is closed under operation of Lie bracket, i.e., if for all vector fields X and Y taking values in $\mathcal{X}$, the vector field $[X, Y]$ also takes value in $\mathcal{X}$. The *involutive closure* of the distribution $\mathcal{X}$ is the smallest involutive distribution containing $\mathcal{X}$, and is denoted $\overline{\text{Lie}}\{\mathcal{X}\}$.

Controllable kinematic systems

We start by defining accessibility and controllability for general kinematic systems. Here we let Q be an analytic manifold and we let $\mathcal{V} = \{V_1, \dots, V_\ell\}$ be analytic vector fields giving rise to the driftless nonlinear control system (3.5). For $q_0 \in \mathsf{Q}$ we denote

$$\mathcal{R}^{\mathcal{V}}(q_0, T) = \{\gamma(T) \mid \ (\gamma, u) \text{ is a controlled trajectory for (3.5) defined on } [0, T] \text{ with } \gamma(0) = q_0\},$$

and $\mathcal{R}^{\mathcal{V}}(q_0, \leq T) = \bigcup_{t \in [0,T]} \mathcal{R}^{\mathcal{V}}(q_0, t)$. We make the basic controllability definitions.

Definition 4.1. The system (3.5) is

(i) *locally accessible* from q_0 if there exists $T > 0$ so that $\text{int}(\mathcal{R}^{\mathcal{V}}(q_0, \leq t)) \neq \emptyset$ for $t \in (0, T]$, is

(ii) *small-time locally controllable* (*STLC*) from q_0 if there exists $T > 0$ so that $q_0 \in \text{int}(\mathcal{R}^{\mathcal{V}}(q_0, \leq t))$ for $t \in (0, T]$, and is

(iii) *controllable* if for every $q_1, q_2 \in \mathsf{Q}$ there exists a controlled trajectory (γ, u) defined on $[0, T]$ for some $T > 0$ with the property that $\gamma(0) = q_1$ and $\gamma(T) = q_2$.

Let us state some well-known results concerning the various types of controllability of (3.5).

Theorem 4.2. *The system* (3.5) *is STLC (and therefore accessible) from q_0 if and only if $\overline{\mathrm{Lie}}\{\mathcal{V}\}_{q_0} = T_{q_0}\mathsf{Q}$. Furthermore, if Q is connected and if $\overline{\mathrm{Lie}}\{\mathcal{V}\}_q = T_q\mathsf{Q}$ for each $q \in \mathsf{Q}$, then* (3.5) *is controllable.*

Kinematically controllable dynamic systems

A dynamic mechanical system (2.1) described by $(\mathsf{Q}, \mathbb{G}, V, \mathcal{D}, \mathcal{F})$ is *kinematically controllable* if there exists a sequence of kinematic reductions $\{\mathcal{V}_i |\ i \in \{1, \dots, k\}, \operatorname{rank} \mathcal{V}_i = \ell_i\}$ so that for every $q_1, q_2 \in \mathsf{Q}$ there are corresponding controlled trajectories $\{(\gamma_i, w_i)|\ \gamma_i\colon [T_{i-1}, T_i] \to \mathsf{Q}, w_i\colon [T_{i-1}, T_i] \to \mathbb{R}^{\ell_i}, i \in \{1, \dots, k\}\}$ such that $\gamma_1(T_0) = q_1$, $\gamma_k(T_k) = q_2$, and $\gamma_i(T_i) = \gamma_{i+1}(T_i)$ for all $i \in \{1, \dots, k-1\}$. In other words, any $q_2 \in \mathsf{Q}$ is reachable from any $q_1 \in \mathsf{Q}$ by concatenating motions on Q corresponding to kinematic reductions of (2.1). The dynamic system (2.1) is *locally kinematically controllable* from q_0 if, for any neighborhood of q_0 on Q, the set of reachable configurations by trajectories remaining in the neighborhood and following motions of its kinematic reductions contains q_0 in its interior.

By assembling the discussion from the preceding section, and surrounding Proposition 3.5, we arrive at the following conditions for local kinematic controllability.

Proposition 4.3. *Consider a dynamic mechanical system* (2.1).

(i) The system is locally kinematically controllable if and only if it possesses a collection of decoupling vector fields (i.e., rank-one kinematic reductions) whose involutive closure has maximal rank everywhere in Q.

(ii) If the system is locally kinematically controllable then there is a subbundle $\tilde{\mathcal{Y}}$ of $\mathcal{Y}$ with the property that $B_{\mathcal{Y}}(q)|\tilde{\mathcal{Y}}$ is indefinite and $\overline{\mathrm{Lie}}\{\tilde{\mathcal{Y}}\}_q = T_q\mathsf{Q}$ for each $q \in \mathsf{Q}$.

(iii) If the input distribution $\mathcal{Y}$ is codimension one, $B_{\mathcal{Y}}(q)$ is indefinite and $\overline{\mathrm{Lie}}\{\mathcal{Y}\}_q = T_q\mathsf{Q}$ for each $q \in \mathsf{Q}$, then the system is locally kinematically controllable.

Controllable dynamic systems

We consider again a dynamic mechanical system (2.1) derived from $(\mathsf{Q}, \mathbb{G}, V, \mathcal{D}, \mathcal{F})$. For $q_0 \in \mathsf{Q}$ we denote

$$\mathcal{R}_{T\mathsf{Q}}(q_0, T) = \{\dot{\gamma}(T) \mid \ (\gamma, u) \text{ is a controlled trajectory of (2.1) defined on } [0, T] \text{ and satisfying } \dot{\gamma}(0) = 0_{q_0}\}.$$

Here $0_{q_0} \in T_{q_0}\mathsf{Q}$ is the zero vector. We also define $\mathcal{R}_{T\mathsf{Q}}(q_0, \leq T) = \bigcup_{t \in [0,T]} \mathcal{R}_{T\mathsf{Q}}(q_0, t)$. With these notions of reachable sets, we have the following definitions of controllability.

Definition 4.4. Consider a dynamic mechanical system (2.1) described by $(\mathsf{Q}, \mathbb{G}, V, \mathcal{D}, \mathcal{F})$ and let $q_0 \in \mathsf{Q}$. Suppose that the controls for (2.1) are restricted to take their values in a compact set of $\mathbb{R}^m$ which contains 0 in the interior of its convex hull. The system (2.1) is

(i) *locally accessible* from q_0 if there exists $T > 0$ so that $\operatorname{int}(\mathcal{R}_{T\mathsf{Q}}(q_0, \leq t)) \neq \emptyset$ for $t \in (0, T]$, and is

(ii) *small-time locally controllable* (*STLC*) from q_0 if there exists $T > 0$ so that $0_{q_0} \in \operatorname{int}(\mathcal{R}_{T\mathsf{Q}}(q_0, \leq t))$ for all $t \in (0, T]$.

To present the results in [12] we need some notation concerning iterated symmetric products in the vector fields $\{Y_1, \ldots, Y_m\}$. Such a symmetric product is *bad* if it contains an even number of each of the vector fields $Y_1, \ldots, Y_m$, and otherwise is *good*. Thus, for example, $\langle\langle Y_a : Y_b\rangle : \langle Y_a : Y_b\rangle\rangle$ is bad for all $a, b \in \{1, \ldots, m\}$ and $\langle Y_a : \langle Y_b : Y_c\rangle\rangle$ is good for any $a, b, c \in \{1, \ldots, m\}$. The *degree* of a symmetric product is the total number of input vector fields comprising the symmetric product. For example, our given bad symmetric product has degree 4 and the given good symmetric product has degree 3. If P is a symmetric product in the vector fields $\{Y_1, \ldots, Y_m\}$ and if $\sigma \in S_m$ is an element of the permutation group on $\{1, \ldots, m\}$, $\sigma(P)$ denotes the symmetric product obtained by replacing each occurrence of Y_a with $Y_{\sigma(a)}$.

We now state the main result concerning controllability in state space of dynamic mechanical systems.

Theorem 4.5. *Consider an analytic dynamic mechanical system* (2.1) *described by* $(\mathsf{Q}, \mathbb{G}, V, \mathcal{D}, \mathcal{F})$ *and let* $q_0 \in \mathsf{Q}$*. The dynamic mechanical system* (2.1) *is*

(i) locally accessible from q_0 *if and only if* $\overline{\mathrm{Sym}}\{\mathcal{Y}\}_{q_0} = T_{q_0}\mathsf{Q}$*, and is*

(ii) STLC from q_0 *if* $\overline{\mathrm{Sym}}\{\mathcal{Y}\}_{q_0} = T_{q_0}\mathsf{Q}$ *and if for every bad symmetric product* P *we have*

$$\sum_{\sigma \in S_m} \sigma(P)(q_0) \in \operatorname{span}_{\mathbb{R}}\{P_1(q_0), \ldots, P_k(q_0)\},$$

where $P_1, \ldots, P_k$ *are good symmetric products of degree less than* P*.*

The condition stated for STLC is derived from a result of Sussmann [20]. Hirschorn and Lewis [13] state the following low-order condition for controllability that is related to kinematic controllability.

Theorem 4.6. *Consider an analytic dynamic mechanical system* (2.1) *described by* $(\mathsf{Q}, \mathbb{G}, V, \mathcal{D}, \mathcal{F})$ *and let* $q_0 \in \mathsf{Q}$*. The dynamic mechanical system* (2.1) *is*

(i) STLC from q_0 *if*

(a) $\overline{\mathrm{Sym}}\{\mathcal{Y}\}_{q_0} = T_{q_0}\mathsf{Q}$ *with* $\overline{\mathrm{Sym}}\{\mathcal{Y}\}_{q_0}$ *being spanned by at most degree* 2 *symmetric products and*

(b) $B_{\mathcal{Y}}(q_0)$ *is indefinite, and is*

(ii) not STLC from q_0 *if* $B_{\mathcal{Y}}(q_0)$ *is definite.*

Configuration controllable dynamic systems

The preceding discussion concerned the set of reachable *states* for a dynamic mechanical system. Let us now restrict, as in [12], to descriptions of the set of reachable configurations. We define

$$\mathcal{R}_{\mathsf{Q}}(q_0, T) = \tau(\mathcal{R}_{T\mathsf{Q}}(q_0, T)), \quad \mathcal{R}_{\mathsf{Q}}(q_0, \leq T) = \bigcup_{t \in [0,T]} \mathcal{R}_{\mathsf{Q}}(q_0, t).$$

This gives the following notions of controllability relative to configurations.

Definition 4.7. Consider a dynamic mechanical system (2.1) described by $(\mathsf{Q}, \mathbb{G}, V, \mathcal{D}, \mathcal{F})$ and let $q_0 \in \mathsf{Q}$. The dynamic mechanical system (2.1) is

(i) *locally configuration accessible* from q_0 if there exists $T > 0$ so that $\operatorname{int}(\mathcal{R}_{\mathsf{Q}}(q_0, \leq t)) \neq \emptyset$ for all $t \in (0, T]$, and is

(ii) *small-time locally configuration controllable* (*STLCC*) from q_0 if there exists $T > 0$ so that $q_0 \in \operatorname{int}(\mathcal{R}_{\mathsf{Q}}(q_0, \leq t))$ for all $t \in (0, T]$ with the controls restricted to take their values in a compact subset of $\mathbb{R}^m$ that contains the origin in its convex hull.

The following results were proved by Lewis and Murray [12].

Theorem 4.8. *Consider an analytic dynamic mechanical system* (2.1) *described by* $(\mathsf{Q}, \mathbb{G}, V, \mathcal{D}, \mathcal{F})$ *and let* $q_0 \in \mathsf{Q}$*. The dynamic mechanical system* (2.1) *is*

(i) locally configuration accessible from q_0 *if and only if* $\overline{\operatorname{Lie}}\{\overline{\operatorname{Sym}}\{\mathcal{Y}\}\}_{q_0} = T_{q_0}\mathsf{Q}$*, and is*

(ii) STLCC from q_0 *if* $\overline{\operatorname{Lie}}\{\overline{\operatorname{Sym}}\{\mathcal{Y}\}\}_{q_0} = T_{q_0}\mathsf{Q}$ *and if for every bad symmetric product* P *we have*

$$\sum_{\sigma \in S_m} \sigma(P)(q_0) \in \operatorname{span}_{\mathbb{R}}\{P_1(q_0), \ldots, P_k(q_0)\},$$

where $P_1, \ldots, P_k$ *are good symmetric products of degree less than* P*.*

We also have the following minor extension of Theorem 4.6.

Theorem 4.9. *Consider an analytic dynamic mechanical system* (2.1) *described by* $(\mathsf{Q}, \mathbb{G}, V, \mathcal{D}, \mathcal{F})$ *and let* $q_0 \in \mathsf{Q}$*. The dynamic mechanical system* (2.1) *is*

(i) STLCC from q_0 *if*

(a) $\overline{\operatorname{Lie}}\{\overline{\operatorname{Sym}}\{\mathcal{Y}\}\}_{q_0} = T_{q_0}\mathsf{Q}$ *with* $\overline{\operatorname{Sym}}\{\mathcal{Y}\}_{q_0}$ *being spanned by at most degree* 2 *symmetric products and*

(b) $B_{\mathcal{Y}}(q_0)$ *is indefinite, and is*

(ii) not STLCC from q_0 *if* $B_{\mathcal{Y}}(q_0)$ *is definite.*

From part (ii) follows the single-input result of Lewis [21].

Corollary 4.10. *If* $m = 1$ *and if* $\dim(\mathsf{Q}) > 1$ *then* (2.1) *is not STLCC from* q_0*.*

4.1 Controllability inferences and counter-examples

In this subsection we summarize the relationships between the various controllability concepts described previously. In particular, Figure 1 illustrates the relationships between small-time locally controllable (STLC), small-time locally configuration controllable (STLCC), locally kinematically controllable (LKC), and fully reducible, locally kinematically controllable (FR-LKC) systems. All implications in figure are clear from the theoretical treatment. Without further assumptions on the dimension of the configuration space n and on the dimension of the input distribution m, no further implications can be added to Figure 1. To prove this statement, we present the following counter-examples.

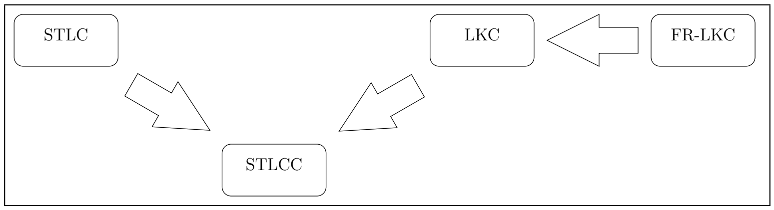

Figure 1: Inference between controllability notions for mechanical control systems.

(i) STLC does not imply LKC nor FR-LKC — Consider the example system:

$$\begin{aligned}\ddot{q}^1 &= u_1\\ \ddot{q}^2 &= u_2\\ \ddot{q}^3 &= \dot{q}^1\dot{q}^2.\end{aligned}$$

The input vector fields are $Y_1 = \frac{\partial}{\partial q^1}$, $Y_2 = \frac{\partial}{\partial q^2}$. This system is STLC since $\langle Y_1 : Y_2\rangle = 2\frac{\partial}{\partial q^3}$. It is not LKC since Y_1 and Y_2 are the only decoupling vector fields (note $\langle Y_1 : Y_1\rangle = 0 = \langle Y_2 : Y_2\rangle$) but their Lie bracket vanishes identically. Additionally, the system is not fully reducible since the input distribution is not geodesically invariant.

(ii) FR-LKC does not imply STLC — Consider the example system in Poincaré format:

$$\begin{aligned}\dot{q}^1 &= v^1 & \dot{v}^1 &= u_1\\ \dot{q}^2 &= \cos(q^1)v^2 - \sin(q^1)v^3, & \dot{v}^2 &= u_2\\ \dot{q}^3 &= \sin(q^1)v^2 + \cos(q^1)v^3 & \dot{v}^3 &= 0.\end{aligned}$$

The input vector fields are $Y_1 = \frac{\partial}{\partial q^1}$ and $Y_2 = \cos(q^1)\frac{\partial}{\partial q^2} + \sin(q^1)\frac{\partial}{\partial q^3}$. This system is not STLC, since $\overline{\mathrm{Sym}}\{Y_1, Y_2\}_q = \mathrm{span}_{\mathbb{R}}\{Y_1(q), Y_2(q)\}$ for each $q \in \mathsf{Q}$. In particular, along any solution of this mechanical control system starting from rest, $v^3(t) = 0$ for all time t. However, both input vector fields are decoupling and $\overline{\mathrm{Lie}}\{Y_1, Y_2\}$ is full rank. Hence the system is fully reducible and locally kinematically controllable (FR-LKC), but not STLC.

(iii) LKC does not imply FR-LKC nor STLC — Consider the example system in Poincaré format:

$$\dot{q} = \sum_{i=1}^{4} X_i v_i, \qquad \begin{aligned}\dot{v}^1 &= u_1\\ \dot{v}^2 &= u_2\\ \dot{v}^3 &= v^1 v^2\\ \dot{v}^4 &= a(v^3)^2,\end{aligned} \tag{4.6}$$

where $\mathcal{X} = \{X_1, \dots, X_4\}$ is a basis for $T\mathbb{R}^4$. These equations are controlled Poincaré equations with respect to the basis $\mathcal{X}$. All generalized Christoffel symbols vanish except for $({}^{\mathcal{X}}\Gamma)^3_{12} = ({}^{\mathcal{X}}\Gamma)^3_{21} = 1$, and $({}^{\mathcal{X}}\Gamma)^4_{33} = a$. According to equation (2.2) the input vector fields X_1 and X_2 are decoupling. If the basis $\mathcal{X}$ is chosen so that $\overline{\mathrm{Lie}}\{X_1, X_2\}$ is full rank, then the system is locally kinematically controllable. It is not fully reducible to a kinematic system, since $\overline{\mathrm{Sym}}\{X_1, X_2\}$ is at least dimension 3. If $a = 0$, the system is not locally accessible. If $a = 1$, the system is locally accessible but not STLC.

(iv) STLCC does not imply STLC nor LKC nor FR-LKC — Consider the example system in Poincaré format:

$$\dot{q} = \sum_{i=1}^{4} X_i v_i, \qquad \begin{aligned} \dot{v}^1 &= u_1 \\ \dot{v}^2 &= u_2 \\ \dot{v}^3 &= v^1 v^2 \\ \dot{v}^4 &= 0. \end{aligned}$$

As previously, these equations are controlled Poincaré equations. As previously, the input vector fields X_1 and X_2 are decoupling. We now suppose the basis $\{X_1, \dots, X_4\}$ is chosen so that $\overline{\text{Lie}}\{X_1, X_2\}_q = \text{span}_{\mathbb{R}}\{X_1(q), X_2(q)\}$ for each $q \in \mathsf{Q}$ and so that $\overline{\text{Lie}}\{X_1, X_2, X_3\}$ is full rank. Note that the system is not LKC since the Lie closure of the input distribution is not full rank. Note that $\langle X_1 : X_2 \rangle = X_3$, and that $\overline{\text{Sym}}\{X_1, X_2\}_q = \text{span}_{\mathbb{R}}\{X_1(q), X_2(q), X_3(q)\}$ for each $q \in \mathsf{Q}$; therefore the system is neither fully reducible, nor STLC. It is STLCC, since $\overline{\text{Lie}}\{\overline{\text{Sym}}\{X_1, X_2\}\}$ is full rank.

Analysis of low-dimensional systems

We here study how the dimensions of the configuration space n and of the input distribution m affect the modeling and controllability analysis in the previous sections. If $n = m$, the system is STLC because one control input is available for each degree of freedom. Hence, we restrict our following analysis to the underactuated setting $m < n$.

- Assume $m = 1$ and $n \geq 2$, and let Y be the single input vector field. If $\overline{\text{Sym}}\{Y\}_q = \text{span}_{\mathbb{R}}\{Y(q)\}$ for each $q \in \mathsf{Q}$, then the system has one decoupling vector field, and since $\overline{\text{Lie}}\{\overline{\text{Sym}}\{Y\}\} = \overline{\text{Sym}}\{Y\}$, the system will not be locally accessible, nor locally configuration accessible. If $\overline{\text{Sym}}\{Y\}$ has rank 2, then the system is possibly accessible or configuration accessible, but never STLC nor STLCC. In terms of the quadratic form $B_{\mathcal{Y}}$, note that its domain and codomain have dimension 1. Accordingly, $B_{\mathcal{Y}}$ is either identically vanishing (fully reducible system) or sign definite (possibly accessible, but never STLC).

- If $m = 2$, $n = 3$, then LKC implies either the system is fully reducible, or the system is STLC. To prove it, consider the input distribution: either it is geodesically invariant (rank $\overline{\text{Sym}}\{\mathcal{Y}\} = 2$) or not (rank $\overline{\text{Sym}}\{\mathcal{Y}\} = 3$). In the first case, the system is fully reducible to a kinematic system. In the second case, the dynamic system is locally accessible and, because of the good properties of decoupling vector fields, the system satisfies the bad symmetric product test and it is STLC. This statement does not hold anymore at $m = 2$ $n = 4$ as proved by example system (4.6). In terms of the quadratic form $B_{\mathcal{Y}}$, note that its domain has dimension 2 and its codomain has dimension 1. Accordingly, $B_{\mathcal{Y}}$ is either identically vanishing (fully reducible system), or indefinite (STLC system) or sign definite (accessible, but never STLC dynamic system).

5 A catalog of affine connection control systems

In this section we consider a number of instructive examples and present a detailed description of their kinematic reductions and of their controllability properties. The catalog is presented in tabular form on page 14.

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References

[1] R. W. Brockett, "Hybrid models for motion control systems," in *Essays in Control: Perspectives in the Theory and its Applications*, pp. 29–53, Boston, MA: Birkhäuser, 1993.

[2] V. Manikonda, P. S. Krishnaprasad, and J. Hendler, "Languages, behaviors, hybrid architectures and motion control," in *Mathematical Control Theory* (J. Baillieul and J. C. Willems, eds.), New York, NY: Springer Verlag, 1998.

[3] F. Bullo, N. E. Leonard, and A. D. Lewis, "Controllability and motion algorithms for underactuated Lagrangian systems on Lie groups," *IEEE Transactions on Automatic Control*, vol. 45, no. 8, pp. 1437–1454, 2000.

[4] G. J. Pappas, G. Lafferriere, and S. S. Sastry, "Hierarchically consistent control systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 6, pp. 1144–60, 2000.

[5] K. McIsaac, and J. P. Ostrowski, "Steering algorithms for dynamic robotic locomotion systems," in *Algorithmic and Computational Robotics: New Directions* (B. R. Donald and K. M. Lynch and D. Rus, eds.), Natick, MA: A. K. Peters, pp. 221-231, 2001.

[6] E. Frazzoli, M. A. Daleh, and E. Feron, "Real-time motion planning for agile autonomous vehicles," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 25, no. 1, pp. 116–129, 2002.

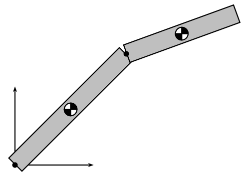
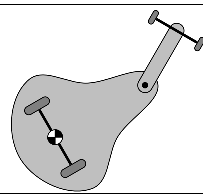
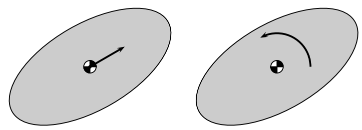
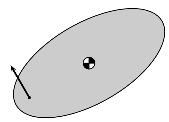
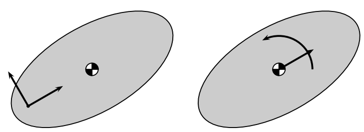
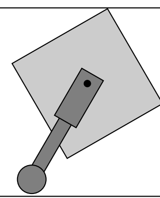
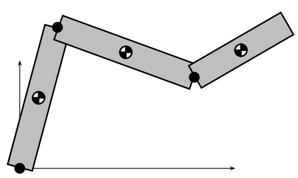
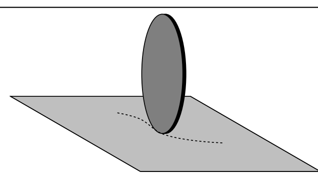
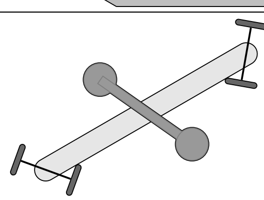
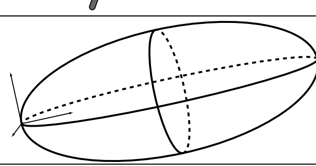
[7] A. D. Lewis, "When is a mechanical control system kinematic?," in *IEEE Conf. on Decision and Control*, (Phoenix, AZ), pp. 1162–1167, Dec. 1999.

[8] F. Bullo and K. M. Lynch, "Kinematic controllability for decoupled trajectory planning in underactuated mechanical systems," *IEEE Transactions on Robotics and Automation*, vol. 17, no. 4, pp. 402–412, 2001.

[9] D. Manocha and J. F. Canny, "Efficient inverse kinematics for general 6R manipulators," *IEEE Transactions on Robotics and Automation*, vol. 10, no. 5, pp. 648–657, 1994.

[10] J. T. Betts, "Survey of numerical methods for trajectory optimization," *AIAA Journal of Guidance, Control, and Dynamics*, vol. 21, no. 2, pp. 193–207, 1998.

[11] S. M. LaValle and J. J. Kuffner, "Randomized kinodynamic planning," *International Journal of Robotics Research*, vol. 20, no. 5, pp. 378–400, 2001.

System	Picture	Reducibility & Controllability	Ref
planar 2R robot single torque at either joint: $(1,0),(0,1)$ $n=2, m=1$		$(1,0)$: no reductions, accessible $(0,1)$: decoupling v.f., fully reducible, not accessible or STLCC	[22, 8]
roller racer single torque at joint $n=4, m=1$		no kinematic reductions, accessible, not STLCC	[16, 17]
planar body with single force or torque $n=3, m=1$		decoupling v.f., reducible, not accessible	
planar body with single generalized force $n=3, m=1$		no kinematic reductions, accessible, not STLCC	
planar body with two forces $n=3, m=2$		two decoupling v.f., LKC, STLC	[3, 8]
robotic leg $n=3, m=2$		two decoupling v.f., fully reducible and LKC	[7]
planar 3R robot, two torques: $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ $n=3, m=2$		$(1,0,1)$ and $(1,1,0)$: two decoupling v.f., LKC and STLC $(0,1,1)$: two decoupling v.f., fully reducible and LKC	[22, 8]
rolling penny $n=4, m=2$		fully reducible and LKC	[7]
snakeboard $n=5, m=2$		two decoupling v.f., LKC, STLCC	[16, 23]
3D vehicle with 3 generalized forces $n=6, m=3$		three decoupling v.f., LKC, STLC	[3, 8]

[12] A. D. Lewis and R. M. Murray, "Configuration controllability of simple mechanical control systems," *SIAM Journal on Control and Optimization*, vol. 35, no. 3, pp. 766–790, 1997.

[13] R. M. Hirschorn and A. D. Lewis, "Geometric first-order controllability conditions for affine connection control systems," in *IEEE Conf. on Decision and Control*, (Orlando, FL), pp. 4216–4211, Dec. 2001.

[14] Z. Li, R. Montgomery, and M. Raibert, "Dynamics and optimal control of a legged robot in flight phase," in *IEEE Int. Conf. on Robotics and Automation*, (Scottsdale, AZ), pp. 1816–1821, May 1989.

[15] A. D. Lewis, J. P. Ostrowski, R. M. Murray, and J. W. Burdick, "Nonholonomic mechanics and locomotion: the snakeboard example," in *IEEE Int. Conf. on Robotics and Automation*, (San Diego, CA), pp. 2391–2400, May 1994.

[16] A. D. Lewis, "Simple mechanical control systems with constraints," *IEEE Transactions on Automatic Control*, vol. 45, no. 8, pp. 1420–1436, 2000.

[17] F. Bullo and M. Žefran, "On mechanical control systems with nonholonomic constraints and symmetries," *Systems & Control Letters*, vol. 45, no. 2, pp. 133–143, 2002.

[18] A. D. Lewis, "Affine connections and distributions with applications to nonholonomic mechanics," in *Reports on Mathematical Physics*, vol. 42, no. 1/2. pp. 135-164, 1998.

[19] R. M. Hirschorn and A. D. Lewis, "Geometric local controllability: Second-order conditions." Preprint, Feb. 2002.

[20] H. J. Sussmann, "A general theorem on local controllability," *SIAM Journal on Control and Optimization*, vol. 25, no. 1, pp. 158–194, 1987.

[21] A. D. Lewis, "Local configuration controllability for a class of mechanical systems with a single input," in *European Control Conference*, (Brussels, Belgium), July 1997.

[22] K. M. Lynch, N. Shiroma, H. Arai, and K. Tanie, "Collision-free trajectory planning for a 3-DOF robot with a passive joint," *International Journal of Robotics Research*, vol. 19, no. 12, pp. 1171–1184, 2000.

[23] F. Bullo and A. D. Lewis, "Kinematic controllability and motion planning for the snakeboard," *IEEE Transactions on Robotics and Automation*, Jan. 2002. Submitted.