On nonlinear controllability and series expansions for Lagrangian systems with dissipative forces

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Abstract— **This note presents series expansions and nonlinear controllability results for Lagrangian systems subject to dissipative forces. The treatment relies on the assumption of dissipative forces of linear isotropic nature. The approach is based on the affine connection formalism for Lagrangian control systems, and on the homogeneity property of all relevant vector fields.**

Keywords— **Nonlinear controllability, series expansions, mechanical control systems**

I. Introduction

This note presents novel controllability and perturbation analysis results for control systems with Lagrangian structure. The work belongs to a growing body of research devoted to the geometric control of mechanical systems. The objective is the development of coordinate-free analysis and design tools applicable in a unified manner to robotic manipulators, vehicle models, and systems with nonholonomic constraints. Contributions include results on modeling [1], [2], nonlinear controllability [3], [4], [5], series expansions [6], motion planning [7], [8], averaging [9], passivity-based stabilization [10], [11], [12], and optimal control [13], [14]. Notions from differential and Riemannian geometry provide the framework underlying these contributions: the formalism of affine connections plays a key role in modeling, analysis and control design for a large class of systems.

The motivation for this work is a standing limitation in the known results on controllability and series expansions. The analysis in [3], [4], [5], [6] applies only to systems subject to no external dissipation, i.e., the system's dynamics is fully determined by the Lagrangian function. With the aim of developing more accurate mathematical models for controlled mechanical systems, this note addresses the setting of dissipative or damping forces. It is worth adding that dissipation is a classic topic in Geometric Mechanics (see for example the work on dissipation induced instabilities [15] and the extensive literature on dissipation-based control [10], [11], [12]).

The contribution of this paper are controllability tests and series expansions that account for a linear isotropic model of dissipation. Remarkably, the same conditions guaranteeing a variety of local accessibility and controllability properties for systems without damping remain valid for the class of systems under consideration. This applies to small-time local controllability, local configuration controllability, and kinematic controllability. Furthermore, we develop a series expansion describing the evolution of the controlled trajectories starting from rest, thus generalizing the work in [6]. The technical approach exploits the homogeneity property of the affine connection model for mechanical control systems.

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II. Affine connections and mechanics

In this section we review the notion of affine connection; see [16] for a comprehensive treatment. We introduce a class of Lagrangian systems with dissipative forces and explore their homogeneity properties. All quantities are assumed analytic.

A. Affine connections

An *affine connection* on a manifold Q is a map that assigns to a pair of vector fields X, Y another vector field $\nabla_X Y$ such that

$$\begin{aligned}\nabla_{fX+Y} Z &= f\nabla_X Z + \nabla_Y Z \\ \nabla_X (fY + Z) &= (\mathcal{L}_X f)Y + f\nabla_X Y + \nabla_X Z\end{aligned} \tag{1}$$

for any function f and any vector field Z. Usually $\nabla_X Y$ is called the *covariant derivative* of Y with respect to X. Vector fields can also be covariantly differentiated along curves, and this concept will be instrumental in writing the Euler-Lagrange equations. Consider a curve $\gamma\colon [0,1] \to Q$ and a vector field along γ, i.e., a map $v\colon [0,1] \to TQ$ such that $\tau_Q(v(t)) = \gamma(t)$ for all $t \in [0,1]$ (where $\tau_Q : TQ \to Q$ denotes the tangent bundle projection). Take now a vector field V that satisfies $V(\gamma(t)) = v(t)$. The *covariant derivative of the vector field v along γ* is defined by

$$\frac{Dv(t)}{dt} = \nabla_{\dot\gamma(t)} v(t) = \nabla_{\dot\gamma(t)} V(q)\big|_{q=\gamma(t)}.$$

In particular, we may take $v(t) = \dot\gamma(t)$ and set up the equation $\nabla_{\dot\gamma(t)}\dot\gamma(t) = 0$. This equation is called the *geodesic equation*, and its solutions are termed the *geodesics* of ∇. The vector field Z on TQ describing this equation is called the *geodesic spray*.

In a system of local coordinates $(q^1, \dots, q^n)$, an affine connection is uniquely determined by its *Christoffel symbols* $\Gamma^i_{ij}(q)$, $\nabla_{\frac{\partial}{\partial q^i}}\left(\frac{\partial}{\partial q^j}\right) = \Gamma^k_{ij}\frac{\partial}{\partial q^k}$, and accordingly, the covariant derivative of a vector field is written using (1) as

$$\nabla_X Y = \Big(\frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k\Big)\frac{\partial}{\partial q^i}.$$

Taking natural coordinates (q^i, v^i) on TQ, the local expression of the geodesic spray reads

$$Z(v_q) = v^i \frac{\partial}{\partial q^i} - \Gamma^i_{jk}(q) v^j v^k \frac{\partial}{\partial v^i}\,.$$

B. Control systems described by affine connections

An *affine connection control system* consists of the following objects: an n-dimensional configuration manifold Q, with $q \in Q$ being the configuration of the system and $v_q \in T_qQ$ being the system's velocity; an affine connection ∇ on Q, with Christoffel symbols $\{\Gamma^i_{jk} : Q \to \mathbb{R} \,|\, i,j,k \in \{1,\dots,n\}\}$; and a family of input vector fields $\mathcal{Y} = \{Y_1, \dots, Y_m\}$ on Q. The corresponding equations of motion are written as

$$\nabla_{\dot q(t)}\dot q(t) = u^a(t) Y_a(q(t)), \tag{2}$$

or, equivalently, in coordinates as $\ddot q^i + \Gamma^i_{jk}(q)\dot q^j \dot q^k = u^a(t) Y^i_a(q)$, where the indexes $i,j,k \in \{1,\dots,n\}$. These equations are a generalization of the Euler-Lagrange equations. If ∇ is the *Levi-Civita affine connection* [16] associated with a kinetic energy metric, then the equations (2) are the forced Euler-Lagrange equations for the associated kinetic energy Lagrangian. If ∇ is the so called *nonholonomic affine connection* [2], the equations (2) represent the forced equations of motion for a nonholonomic system with a kinetic energy Lagrangian, and constraints linear in the velocities.

The systems described by equations (2) are subject to no damping force. However, in a number of situations, friction and dissipation play a relevant role. Consider, for instance, a blimp experiencing the resistance of the air or an underwater vehicle moving in the sea. We introduce a linear isotropic term of dissipation into equations (2), i.e., we consider

$$\nabla_{\dot{q}(t)}\dot{q}(t) = k_d\dot{q}(t) + u^a(t)Y_a(q(t)), \qquad (3)$$

where $k_d \in \mathbb{R}$. In local coordinates, $\ddot{q}^i + \Gamma^i_{jk}(q)\dot{q}^j\dot{q}^k = k_d\dot{q}^i + u^a(t)Y^i_a(q)$.

This second-order system can be written as a first-order differential equation on TQ. Using $\{\frac{\partial}{\partial q^i}, \frac{\partial}{\partial v^i}\}$ as a basis for vector fields on the tangent bundle of TQ, we define

$$L(v_q) = v^i\frac{\partial}{\partial v^i}, \quad Y^{\text{lift}}_a(v_q) = Y^i_a(q)\frac{\partial}{\partial v^i}, \quad a \in \{1,\dots,m\},$$

so that the control system becomes

$$\dot{v}(t) = Z(v(t)) + k_dL(v(t)) + u^a(t)Y^{\text{lift}}_a(v(t)), \qquad (4)$$

where $t \mapsto v(t)$ is now a curve in TQ describing the evolution of a first-order control affine system. We refer to [16] for coordinate-free definitions of the lifting operation $Y_a \to Y^{\text{lift}}_a$ and of the Liouville vector field L on TQ.

C. Homogeneity and Lie algebraic structure

One fundamental feature of the control systems (2) and (3) is the polynomial dependence of the vector fields Z, L and Y^{lift} on the velocity variables v^i. This structure leads to remarkable simplifications in the iterated Lie brackets between $\{Z, Y^{\text{lift}}_1, \dots, Y^{\text{lift}}_m\}$ (see e.g. [17]). As we see below, these simplifications also take place between the vector fields $\{Z + k_dL, Y^{\text{lift}}_1, \dots, Y^{\text{lift}}_m\}$.

We start by introducing the notion of *geometric homogeneity* [18]: given two vector fields X and X_E, X is *homogeneous with degree $m \in \mathbb{Z}$ with respect to X_E* if $[X_E, X] = mX$.

Lemma II.1: Let ∇ be an affine connection on Q with geodesic spray Z, and let Y be a vector field on Q. Then $[L, Z] = (+1)Z$ and $[L, Y^{\text{lift}}] = (-1)Y^{\text{lift}}$.

In the sequel, a vector field X on TQ is *homogeneous of degree $m \in \mathbb{Z}$* if it is homogeneous of degree m with respect to L. Let $\mathcal{P}_j$ be the set of vector fields on TQ of homogeneous degree j, so that $Z \in \mathcal{P}_1$ and $Y^{\text{lift}} \in \mathcal{P}_{-1}$. One can see that $[L, X] = 0$, for all $X \in \mathcal{P}_0$, and that $[\mathcal{P}_i, \mathcal{P}_j] \subset \mathcal{P}_{i+j}$.

III. Nonlinear controllability

In this section we investigate the controllability properties of systems with isotropic dissipation. We show how the basic observation contained in Lemma II.1 is very helpful in the analysis of local accessibility, controllability, and kinematic controllability.

A. Local accessibility and controllability

Here we study conditions for accessibility and controllability of mechanical systems with dissipation. A relevant notion is that of configuration controllability, which concerns the reachable set restricted to the configuration space Q and is weaker than full-state controllability; we refer the reader to [3] for the exact definitions.

Let $\overline{\text{Lie}}(\mathcal{Y})$ and $\overline{\text{Sym}}(\mathcal{Y})$ denote the involutive and the symmetric closure, respectively, of $\mathcal{Y} = \{Y_1, \dots, Y_m\}$. Let $\mathcal{Y}^{\text{lift}}$ denote the set of lifted vector fields $\{Y^{\text{lift}}_1, \dots, Y^{\text{lift}}_m\}$. The next result shows that the involutive closures of systems (2) and (3) at zero velocity coincide.

Proposition III.1: Consider the distributions

$$\mathcal{D}_{(1)} = \text{span}\{Z, \mathcal{Y}^{\text{lift}}\}, \quad \mathcal{D}^L_{(1)} = \text{span}\{Z + k_dL, \mathcal{Y}^{\text{lift}}\}.$$

Define recursively

$$\begin{aligned}\mathcal{D}_{(k)} &= \mathcal{D}_{(k-1)} + [\mathcal{D}_{(k-1)}, \mathcal{D}_{(k-1)}],\\ \mathcal{D}^L_{(k)} &= \mathcal{D}^L_{(k-1)} + [\mathcal{D}^L_{(k-1)}, \mathcal{D}^L_{(k-1)}],\ k \geq 2.\end{aligned}$$

Then, it holds that $\mathcal{D}_{(k)}(0_q) = \mathcal{D}^L_{(k)}(0_q)$, for all k. Consequently, the accessibility distributions $\mathcal{D}_{(\infty)}(0_q) = \overline{\text{Lie}}(Z, \mathcal{Y}^{\text{lift}})_q$ and $\mathcal{D}^L_{(\infty)}(0_q) = \overline{\text{Lie}}(Z + k_dL, \mathcal{Y}^{\text{lift}})_q$ coincide.

Proof: Obviously $\mathcal{D}_{(1)}(0_q) = \mathcal{D}^L_{(1)}(0_q)$. Moreover, we have $[\mathcal{D}_{(1)}, \mathcal{D}_{(1)}] \subset \mathcal{D}^L_{(2)}$ and $[\mathcal{D}^L_{(1)}, \mathcal{D}^L_{(1)}] \subset \mathcal{D}_{(2)}$, since $[Z + k_dL, Y^{\text{lift}}] = [Z, Y^{\text{lift}}] - k_dY^{\text{lift}}$. Let us assume that

$$\mathcal{D}_{(k)}(0_q) = \mathcal{D}^L_{(k)}(0_q), \qquad (5)$$

$$[\mathcal{D}_{(k)}, \mathcal{D}_{(k)}] \subset \mathcal{D}^L_{(k+1)}, \qquad (6)$$

$$[\mathcal{D}^L_{(k)}, \mathcal{D}^L_{(k)}] \subset \mathcal{D}_{(k+1)}. \qquad (7)$$

hold for k and let us show that (5-7) are valid for $k+1$. We have

$$\begin{aligned}\mathcal{D}_{k+1} = \mathcal{D}_{(k)} + [\mathcal{D}_{(k)}, \mathcal{D}_{(k)}] &\subset \mathcal{D}_{(k)} + \mathcal{D}^L_{(k+1)} \Longrightarrow\\ \mathcal{D}_{k+1}(0_q) &\subset \mathcal{D}_{(k)}(0_q) + \mathcal{D}^L_{(k+1)}(0_q)\\ &= \mathcal{D}^L_{(k)}(0_q) + \mathcal{D}^L_{(k+1)}(0_q) = \mathcal{D}^L_{(k+1)}(0_q).\end{aligned}$$

Similarly, $\mathcal{D}^L_{k+1}(0_q) \subset \mathcal{D}_{(k+1)}(0_q)$, and thus $\mathcal{D}_{(k+1)}(0_q) = \mathcal{D}^L_{(k+1)}(0_q)$. On the other hand,

$$\begin{aligned}[\mathcal{D}^L_{(k+1)}, \mathcal{D}^L_{(k+1)}] &= [\mathcal{D}^L_{(k)} + [\mathcal{D}^L_{(k)}, \mathcal{D}^L_{(k)}], \mathcal{D}^L_{(k)} + [\mathcal{D}^L_{(k)}, \mathcal{D}^L_{(k)}]]\\ &\subset [\mathcal{D}^L_{(k)} + \mathcal{D}_{(k+1)}, \mathcal{D}^L_{(k)} + \mathcal{D}_{(k+1)}] \subset \mathcal{D}_{(k+1)} + [\mathcal{D}^L_{(k)}, \mathcal{D}_{(k+1)}]\\ &\quad + \mathcal{D}_{(k+2)} = [\mathcal{D}^L_{(k)}, \mathcal{D}_{(k+1)}] + \mathcal{D}_{(k+2)}.\end{aligned}$$

Thus, it remains to be checked that $[\mathcal{D}^L_{(k)}, \mathcal{D}_{(k+1)}] \subset \mathcal{D}_{(k+2)}$. Observe that

$$\begin{aligned}[\mathcal{D}^L_{(k)}, \mathcal{D}_{(k+1)}] &= [\mathcal{D}^L_{(k-1)} + [\mathcal{D}^L_{(k-1)}, \mathcal{D}^L_{(k-1)}], \mathcal{D}_{(k+1)}]\\ &\subset [\mathcal{D}^L_{(k-1)}, \mathcal{D}_{(k+1)}] + \mathcal{D}_{(k+2)},\end{aligned}$$

where we have used the induction hypothesis on (7), i.e. $[\mathcal{D}^L_{(k-1)}, \mathcal{D}^L_{(k-1)}] \subset \mathcal{D}_{(k)}$. By a recursive argument, we find that what we must show is $[\mathcal{D}^L_{(1)}, \mathcal{D}_{(k+1)}] \subset \mathcal{D}_{(k+2)}$. Clearly, $[Y^{\text{lift}}_i, \mathcal{D}_{(k+1)}] \subset \mathcal{D}_{(k+2)}$, $i \in \{1, \dots, m\}$. In addition,

$$[Z + k_dL, \mathcal{D}_{(k+1)}] = [Z, \mathcal{D}_{(k+1)}] + [k_dL, \mathcal{D}_{(k+1)}] \subset \mathcal{D}_{(k+2)},$$

since $[L, X] \in \mathcal{D}_{(k+1)}$, for all $X \in \mathcal{D}_{(k+1)}$, by homogeneity. Finally, it can be similarly shown using (6) that $[\mathcal{D}_{(k+1)}, \mathcal{D}_{(k+1)}] \subset \mathcal{D}^L_{(k+2)}$. Thus, (5-7) are satisfied for all k. ■

Corollary III.2: Consider a mechanical control system of the form (3). Then

(i) the system is locally accessible (LA) at q starting with zero velocity if $\overline{\text{Sym}}(\mathcal{Y})_q = T_qQ$,

(ii) the system is locally configuration accessible (LCA) at $q \in Q$ if $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathcal{Y}))_q = T_qQ$.

Proof: The manifold Q can be identified with the set of zero vectors $Z(TQ)$ of TQ by the diffeomorphism $q \mapsto 0_q$. Hence, the tangent space to $Z(TQ)$ at 0_q is isomorphic to T_qQ. On the other hand, the projection $\tau_Q(v_q) = q$ defines $\mathcal{V} = \ker T\tau_Q$. One has that $\mathcal{V}_{0_q}$ is isomorphic to T_qQ for all $q \in Q$. Both parts yield the natural decomposition

$$T_{0_q}TQ = T_{0_q}(Z(TQ)) \oplus \mathcal{V}_{0_q} \simeq T_qQ \oplus T_qQ\,.$$

The first copy of T_qQ corresponds to configurations, the second one to velocities. The result follows from the former proposition and Proposition 5.9 in [3] which asserts that

$$\mathcal{D}_{(\infty)}(0_q) \cap \mathcal{V}_{0_q} = \overline{\mathrm{Sym}}(\mathcal{Y})_q^{\mathrm{lift}},$$
$$\mathcal{D}_{(\infty)}(0_q) \cap T_{0_q}(Z(TQ)) = \overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathcal{Y}))_q.$$

■

Next, we examine the small-time local controllability properties of the system (3). We shall use the following conventions. Given a set of vector fields $\mathcal{X} = \{X_0, X_1, \dots, X_m\}$, every Lie bracket B in $\mathcal{X}$ has a unique decomposition as $B = [B_1, B_2]$. In turn, each of B_1 and B_2 may be uniquely expressed as $B_1 = [B_{11}, B_{12}]$ and $B_2 = [B_{21}, B_{22}]$. This process may be continued until we obtain not decomposable elements. All such elements $B_{i_1\dots i_l}$, $i_b \in \{1, 2\}$, are called *components* of B. The *length* of a component $B_{i_1\dots i_l}$ is l. A Lie bracket B is *bad* if $\delta_0(B)$ is odd and $\delta_a(B)$ is even, $a \in \{1, \dots, m\}$, with $\delta_a(B)$ the number of occurrences of X_a in B. Otherwise, B is *good*. The *degree* of B is $\delta(B) = \sum_{b=0}^m \delta_b(B)$.

The results in [3], [19] include sufficient conditions for small-time local controllability (STLC) and small-time local configuration controllability (STLCC). Let the system be LA at $q \in Q$ starting with zero velocity (resp. LCA at $q \in Q$). The system in equation (3) is STLC at q starting with zero velocity (resp. STLCC) if:

(Sussmann's criterium on $\{Z + k_dL, \mathcal{Y}^{lift}\}$): Every bad bracket B in $\{Z + k_dL, \mathcal{Y}^{\mathrm{lift}}\}$ is a $\mathbb{R}$-linear combination of good brackets evaluated at 0_q of lower degree than B.

We shall show that if these conditions are satisfied for the set $\{Z, \mathcal{Y}^{\mathrm{lift}}\}$, then they are also verified for the set $\{Z + k_dL, \mathcal{Y}^{\mathrm{lift}}\}$. We illustrate this fact by considering two low order settings. First, every bracket B of order 1 or 2, i.e., $\delta(B) \leq 2$, is good. In addition, $[Z + k_dL, Y^{\mathrm{lift}}] = [Z, Y^{\mathrm{lift}}] - k_dY^{\mathrm{lift}}$, and therefore, every good bracket in $\{Z + k_dL, \mathcal{Y}^{\mathrm{lift}}\}$ of degree 2 is the sum of the corresponding good bracket in $\{Z, \mathcal{Y}^{\mathrm{lift}}\}$ plus some good brackets of lower degree in $\{Z + k_dL, \mathcal{Y}^{\mathrm{lift}}\}$.

Proposition III.3: Assume Sussmann's criterium on $\{Z, \mathcal{Y}^{\mathrm{lift}}\}$. Then

(i) every bad bracket B in $\{Z+k_dL, \mathcal{Y}^{\mathrm{lift}}\}$ of degree k, evaluated at 0_q, is a $\mathbb{R}$-linear combination of good brackets of lower degree,

(ii) every good bracket C in $\{Z + k_dL, \mathcal{Y}^{\mathrm{lift}}\}$ of degree k, evaluated at 0_q, is a $\mathbb{R}$-linear combination of the corresponding good bracket in $\{Z, \mathcal{Y}^{\mathrm{lift}}\}$ and of some brackets in $\{Z + k_dL, \mathcal{Y}^{\mathrm{lift}}\}$ of lower degree, and

(iii) every good bracket in $\{Z, \mathcal{Y}^{\mathrm{lift}}\}$ of degree k, evaluated at 0_q, is a $\mathbb{R}$-linear combination of good brackets in $\{Z+k_dL, \mathcal{Y}^{\mathrm{lift}}\}$ of degree $\leq k$.

Proof: First, note that (iii) is an immediate consequence of (i) and (ii). Next, we show (i) by induction. The result holds for $k = 2$. Suppose that it is valid for k and let us prove it for $k+1$. Let B be a bad bracket in $\{Z+k_dL, \mathcal{Y}^{\mathrm{lift}}\}$ of degree $k+1$. This means that $\delta_0(B)$ is odd and $\delta_i(B)$ is even, $i \in \{1, \dots, m\}$. Select a term of the form $Z + k_dL$ which is in one of the longest components of B. We then write B as the sum of two Lie brackets, $B = B_1 + B_2$, by expanding the chosen term. By the homogeneity properties, we have that $\delta_0(B_2) = \delta_0(B) - 1$, $\delta_i(B_2) = \delta_i(B)$. Consequently, B_2 is a good bracket in $\{Z + k_dL, \mathcal{Y}^{\mathrm{lift}}\}$ of degree k. Expanding now all the possible terms $Z+k_dL$ in B_1 as the sum of two Lie brackets, one with Z and the other with k_dL (going from the ones in the longest components of B_1 to those in the shortest ones), we finally obtain that B can be written as the sum of the corresponding bad bracket in $\{Z, \mathcal{Y}^{\mathrm{lift}}\}$, plus good/bad brackets in $\{Z, \mathcal{Y}^{\mathrm{lift}}\}$ of degree $\leq k$, plus B_2, which is a good bracket in $\{Z + k_dL, \mathcal{Y}^{\mathrm{lift}}\}$ of degree k. The induction hypothesis now implies (i).

Let us prove (ii). Let C be a good bracket in $\{Z + k_dL, \mathcal{Y}^{\mathrm{lift}}\}$ of degree $k+1$. Expanding the terms $Z + k_dL$ as before, we find that C can be written as the sum of the corresponding good bracket in $\{Z, \mathcal{Y}^{\mathrm{lift}}\}$, plus brackets in $\{Z, \mathcal{Y}^{\mathrm{lift}}\}$ of degree $\leq k$, plus brackets in $\{Z + k_dL, \mathcal{Y}^{\mathrm{lift}}\}$ of degree k. The induction hypothesis implies then (ii). ■

If P is a symmetric product of vector fields in $\mathcal{Y}$, we let $\gamma_a(P)$ denote the number of occurrences of Y_a in P. The *degree* of P will be $\gamma(P) = \sum_{a=1}^m \gamma_a(P)$. We say that P is *bad* if $\gamma_a(P)$ is even for each $a \in \{1, \dots, m\}$. We say that P is *good* if it is not bad.

Corollary III.4: Consider a mechanical control system as in (3). Then, we have

(i) the system is STLC at $q \in Q$ starting with zero velocity if $\overline{\mathrm{Sym}}(\mathcal{Y})_q = T_qQ$ and every bad symmetric product B in $\overline{\mathrm{Sym}}(\mathcal{Y})_q$ is a linear combination of good symmetric products of lower degree, and

(ii) the system is STLCC at $q \in Q$ if $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathcal{Y}))_q = T_qQ$ and every bad symmetric product B in $\overline{\mathrm{Sym}}(\mathcal{Y})_q$ is a linear combination of good symmetric products of lower degree.

Proof: It follows from the fact that there is a 1-1 correspondence between bad (resp. good) Lie brackets in $\{Z, \mathcal{Y}^{\mathrm{lift}}\}$ and bad (resp. good) symmetric products in $\mathcal{Y}$; see [3]. ■

B. Kinematic controllability

Kinematic controllability [4] has direct relevance to the trajectory planning problem for mechanical systems of the form (2). Here, we present a generalized notion of kinematic controllability for affine connection systems with isotropic dissipation. Consider a mechanical system as in (3), and let $\mathcal{I}$ is the distribution generated by the input vector fields $\{Y_1, \dots, Y_m\}$. A controlled solution to equations (3) is a curve $t \mapsto q(t) \in Q$ satisfying

$$\nabla_{\dot{q}}\dot{q} - k_d\dot{q} \in \mathcal{I}_{q(t)}\,. \tag{8}$$

Let $s : [0, T] \to [0, 1]$ be a twice-differentiable function such that $s(0) = 0, s(T) = 1, \dot{s}(0) = \dot{s}(T) = 0$, and $\dot{s}(t) > 0$ for all $t \in (0, T)$. We call such a curve s a *time scaling*. A vector field V is a *decoupling vector field* for the mechanical system (3) if, for any time scaling s and for any initial condition q_0, the curve $t \mapsto q(t)$ on Q solving

$$\dot{q}(t) = \dot{s}(t)V(q(t)), \qquad q(0) = q_0, \tag{9}$$

satisfies the conditions in (8). Additionally, the integral curves of V defined on the time interval $[0, 1]$ are called *kinematic motions*.

Lemma III.5: The vector field V is decoupling for system (3) iff $V \in \mathcal{I}$ and $\langle V : V\rangle \in \mathcal{I}$.

Proof: Given a curve $\gamma : [0, T] \to Q$ satisfying equation (9), we compute $\nabla_{\dot\gamma}\dot\gamma = \ddot{s}V + \dot{s}\nabla_{\dot\gamma}V = \ddot{s}V + \dot{s}^2\nabla_V V$, where we used (1) for vector fields along curves [16]. Now, γ is a kinematic motion if, for all time scalings s, the constraints (8) are satisfied. Thus

$$\nabla_{\dot\gamma}\dot\gamma - k_d\dot\gamma = (\ddot{s} - k_d\dot{s})V + \frac{\dot{s}^2}{2}\langle V : V\rangle \in \mathcal{I}\,.$$

Since s is an arbitrary time scaling and q_0 is an arbitrary point, V and $\langle V : V\rangle$ must separately belong to the input distribution $\mathcal{I}$. The other implication is trivial. ∎

We shall say that the system (3) is *locally kinematically controllable* if for any $q \in Q$ and any neighborhood U_q of q, the set of reachable configurations from q by kinematic motions remaining in U_q contains q in its interior.

Proposition III.6: The system (3) is locally kinematically controllable if there exist $p \in \{1, \dots, m\}$ vector fields $\{V_1, \dots, V_p\} \subset \mathcal{I}$ such that
(i) $\langle V_c : V_c\rangle \in \mathcal{I}$, for all $c \in \{1, \dots, p\}$, and
(ii) $\overline{\mathrm{Lie}}(V_1, \dots, V_p)$ has rank n at all $q \in Q$.

IV. Series expansion for the forced evolution starting from rest

The result in this section extends the treatment in [6] and sets the basis for the design of motion planning strategies [7] and the sharpening of the controllability tests [20]. Consider the system (3), with initial condition $\dot q(0) = 0$.

Proposition IV.1: Given any integrable input vector field $(q, t) \mapsto Y(q, t)$, consider

$$V_1(q,t) = \int_0^t \mathrm{e}^{k_d(t-\tau)} Y(q,\tau)d\tau\,,$$

$$V_k(q,t) = -\frac{1}{2}\sum_{j=1}^{k-1}\int_0^t \mathrm{e}^{k_d(t-\tau)}\langle V_j(q,\tau) : V_{k-j}(q,\tau)\rangle\, d\tau\,,\ k \geq 2\,.$$

There exists a $T > 0$ such that the series $(q,t) \mapsto \sum_{k=1}^{+\infty} V_k(q,t)$ converges absolutely and uniformly for $t \in [0,T]$ and for q in an appropriate neighborhood of q_0. Over the same interval, the solution $\gamma\colon [0,T] \to Q$ to the system (3) with $\dot\gamma(0) = 0$ satisfies

$$\dot\gamma = \sum_{k=1}^{+\infty} V_k(\gamma, t)\,. \tag{10}$$

Proof: *Step I.* A time-varying vector field $(q,t) \mapsto X(q,t)$ gives rise to the initial value problem on Q, $\dot q(t) = X(q,t)$, $q(0) = q_0$. We denote its solution at time T via $q(T) = \Phi^X_{0,T}(q_0)$, and we refer to it as the flow of X. Consider the initial value problem

$$\dot q(t) = X(q,t) + Y(q,t), \qquad q(0) = q_0,$$

where X and Y are analytic (in q) time-varying vector fields. Regarding X as a perturbation to Y, we can describe the flow of $X + Y$ in terms of a nominal and perturbed flow. The following relationship (*variation of constants* formula after [21]) describes this flow:

$$\Phi^{X+Y}_{0,t} = \Phi^Y_{0,t} \circ \Phi^{(\Phi^Y_{0,t})^* X}_{0,t}\,, \tag{11}$$

where $(\Phi^Y_{0,t})^* X$ is the pull-back of X along $\Phi^Y_{0,t}$. This pull-back admits the series expansion representation [21]

$$(\Phi^Y_{0,t})^* X(q,t) = X(q,t) \tag{12}$$
$$+ \sum_{k=1}^{+\infty}\int_0^t \cdots \int_0^{s_{k-1}} \left(\mathrm{ad}_{Y(q,s_k)} \dots \mathrm{ad}_{Y(q,s_1)} X(q,t)\right) ds_k \dots ds_1.$$

Step II. In equation (4), let the Liouville vector field play the role of the perturbation to the vector field $Z + Y^{\mathrm{lift}}$. Then the application of (11) yields $\Phi^{Z+k_dL+Y^{\mathrm{lift}}} = \Phi^{k_dL} \circ \Phi^{\Delta}$, where we compute $\Phi^{k_dL}(q_0, v_0) = (q_0, \mathrm{e}^{k_dt}v_0)$, and where the homogeneity leads to

$$\begin{aligned}\Delta &= \sum_{k=0}^{+\infty}\frac{t^k}{k!}\,\mathrm{ad}^k_{k_dL}(Z + Y^{\mathrm{lift}}) = \sum_{k=0}^{+\infty}\frac{(k_dt)^k}{k!}\,\mathrm{ad}^k_L(Z + Y^{\mathrm{lift}}) \\ &= \sum_{k=0}^{+\infty}\frac{(k_dt)^k}{k!}(Z + (-1)^k Y^{\mathrm{lift}}) \\ &= \sum_{k=0}^{+\infty}\left(\frac{(k_dt)^k}{k!}Z + \frac{(-k_dt)^k}{k!}Y^{\mathrm{lift}}\right) = \mathrm{e}^{k_dt}Z + \mathrm{e}^{-k_dt}Y^{\mathrm{lift}}.\end{aligned}$$

Let $Z' = \mathrm{e}^{k_dt}Z$, and accordingly $\langle X_1 : X_2\rangle' = \mathrm{e}^{k_dt}\langle X_1 : X_2\rangle$. The initial value problem associated with Δ is therefore,

$$\dot y = Z'(y) + \mathrm{e}^{-k_dt}Y(y,t)^{\mathrm{lift}}\,. \tag{13}$$

Step III. Let $k \in \mathbb{N}$ and consider the differential equation

$$\dot y_k = \left(Z' + [X^{\mathrm{lift}}_k, Z'] + Y^{\mathrm{lift}}_k\right)(y_k, t)\,. \tag{14}$$

We recover (13) by setting $k = 1$, $X_1 = 0$, $Y_1 = \mathrm{e}^{-k_dt}Y(q,t)$, and accordingly $y(t) = y_1(t)$. We can now see the vector field $Z' + [X^{\mathrm{lift}}_k, Z']$ as the perturbation to Y^{lift}_k. Using equations (11) and (12), we set $y_k(t) = \Phi^{Y^{\mathrm{lift}}_k}_{0,t}(y_{k+1}(t))$. Some manipulations lead to

$$\dot y_{k+1}(t) = \left(\left(\Phi^{Y^{\mathrm{lift}}_k}_{0,t}\right)^*\left(Z' + [X^{\mathrm{lift}}_k, Z']\right)\right)(y_{k+1}(t))$$
$$= Z' + [X^{\mathrm{lift}}_k + \overline{Y}^{\mathrm{lift}}_k, Z'] - \mathrm{e}^{-k_dt}\left\langle \overline{Y}_k : X_k\right\rangle^{\mathrm{lift}} - \frac{\mathrm{e}^{k_dt}}{2}\left\langle \overline{Y}_k : \overline{Y}_k\right\rangle^{\mathrm{lift}}.$$

Therefore, the differential equation for $y_{k+1}(t)$ is of the same form as (14), where

$$X_{k+1} = X_k + \overline{Y}_k\,, \quad Y_{k+1} = -\mathrm{e}^{k_dt}\left\langle \overline{Y}_k : X_k + \frac{1}{2}\overline{Y}_k\right\rangle.$$

We easily compute $X_k = \sum_{m=1}^{k-1}\overline{Y}_m$ and set

$$Y_{k+1} = -\mathrm{e}^{k_dt}\left\langle \overline{Y}_k : \sum_{m=1}^{k-1}\overline{Y}_m + \frac{1}{2}\overline{Y}_k\right\rangle.$$

One can iterate this procedure for an infinite number of times as in the case of no dissipation [6] to obtain the formal expansion (note $y = (r, \dot r)$),

$$\dot r = \sum_{k=1}^{+\infty} V'(r,t)\,, \quad V'_1(r,t) = \int_0^t \mathrm{e}^{-k_d\tau}Y(r,\tau)d\tau\,,$$

$$V'_k(r,t) = -\frac{1}{2}\sum_{j=1}^{k-1}\int_0^t \mathrm{e}^{k_d\tau}\left\langle V'_j(r,\tau) : V'_{k-j}(r,\tau)\right\rangle d\tau.$$

To obtain the flow of $Z + k_dL + Y^{\mathrm{lift}}$, we compose the flow of Δ with that of k_dL to compute

$$\dot q = \mathrm{e}^{k_dt}\dot r = \sum_{k=1}^{+\infty} V(q,t)\,, \quad V_1(q,t) = \int_0^t \mathrm{e}^{k_d(t-\tau)}Y(q,\tau)d\tau\,,$$

$$V_k(q,t) = -\frac{1}{2}\sum_{j=1}^{k-1}\int_0^t \mathrm{e}^{k_d(\tau-2\tau+t)}\langle V_j(q,\tau) : V_{k-j}(q,\tau)\rangle\, d\tau.$$

Step IV. Select a coordinate chart at q_0. In this way, we locally identify Q with $\mathbb{R}^n$. Let $B_\sigma(q_0) = \{z \in \mathbb{C}^n : \|z - q_0\| < \sigma\}$. Resorting to the analysis in [6], one can see that there exists a $L > 0$ such that $\|V_k\|_{\sigma'} \leq L^{1-k} \|Y\|_\sigma \left(t\mathrm{e}^{k_d t}\right)^{2k-1}$, where $\sigma' < \sigma$, $\|\cdot\|_\sigma$ denotes $\|Y\|_\sigma = \max_{s\in[0,t]} \max_{i\in\{1,\dots,n\}} \max_{z\in B_\sigma(q_0)} |Y^i(q,s)|$, and Y^i is the ith component of Y with respect to the coordinate basis. As a consequence, for $\|Y\|_\sigma T^2 \mathrm{e}^{2k_d T} < L$, the previous expansion converges absolutely and uniformly in $t \in [0,T]$ and $q \in B_{\sigma'}(q_0)$. ■

V. Conclusions

This paper extends previous important results on nonlinear controllability in mechanical systems. The main limitation of the approach is the assumption of isotropic dissipation.

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