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On Nonlinear Controllability of Homogeneous Systems Linear in Control*

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Abstract

This work considers small-time local controllability (STLC) of single and multiple-input systems, $\dot{x} = f_\circ(x) + \sum_{i=1}^{m} f_i u_i$ where $f_\circ(x)$ contains homogeneous polynomials and $f_1, \dots, f_m$ are constant vector fields. For single-input systems, it is shown that even-degree homogeneity precludes STLC if the state dimension is larger than one. This, along with the obvious result that for odd-degree homogeneous systems STLC is equivalent to accessibility, provides a complete characterization of STLC for this class of systems. In the multiple-input case, transformations on the input space are applied to homogeneous systems of degree two, an example of this type of system being motion of a rigid-body in a plane. Such input transformations are related via consideration of a tensor on the tangent space to congruence transformation of a matrix to one with zeros on the diagonal. Conditions are given for successful neutralization of bad type (1,2) brackets via congruence transformations.

1 Introduction

Various concepts of controllability for nonlinear systems were initially explored in [1, 2, 3]. In particular, [2] is primarily concerned with the property of accessibility of the analytic control system $\dot{x} = F(x, u)$, namely that the set of points attainable from a given initial point via application of feasible input is full in the sense of having a nonempty interior. Sussmann and Jurdjević demonstrated in [2] that a necessary and sufficient condition for accessibility of these systems is that the Lie algebra generated by the system have full rank, the so-called Lie Algebra Rank Condition (LARC).

In [4], Sussmann explored the property of small-time local controllability (STLC) for affine analytic single-input systems $\dot{x} = f_\circ(x) + f_1(x)u$ with $|u| \le 1$. A system is said to be STLC at a point $x_\circ$ if that initial point is in the interior of the set of points attainable from it in time T for all $T > 0$. In this case, the Lie algebra generated by the system, denoted by $L(\{f_\circ, f_1\})$, is the smallest involutive distribution containing $\{f_\circ, f_1\}$ or, similarly, the distribution spanned by iterated Lie brackets of $f_\circ$ and f_1. Sussmann gave various necessary and sufficient conditions for STLC in [4]. For example, a necessary condition for STLC is that the tangent vector $[f_1, [f_1, f_\circ]]_{x_\circ}$ be in the subspace spanned by all tangent vectors at $x_\circ$ generated by brackets with only one occurrence of f_1, which is denoted $\mathcal{L}^1(\{f_\circ, f_1\})_{x_\circ}$. More importantly, the conditions conjectured by Hermes were proved to be sufficient conditions for STLC. These Hermes Local Controllability Conditions (HLCC) consist of (1) $x_\circ$ is a (regular) equilibrium point, (2) the LARC is satisfied, and (3) $\mathcal{S}^k(\{f_\circ, f_1\})_{x_\circ} \subset \mathcal{S}^{k-1}(\{f_\circ, f_1\})_{x_\circ}$ for all even $k > 1$ where $\mathcal{S}^k(\{f_\circ, f_1\})_{x_\circ}$ denotes the span of all tangent vectors at $x_\circ$ generated by brackets with k or less occurrences of f_1. Stefani [5] provided an extension of Sussmann's necessary condition by demonstrating that STLC implies $(\mathrm{ad}_{f_1}^{2m} f_\circ)_{x_\circ} \in \mathcal{S}_{x_\circ}^{2m-1}$ for all $m \in \{1, 2, \dots\}$. For an excellent summary and tutorial of these as well as other results in the single-input case, the inquisitive reader is directed to Kawski [6].

STLC of multiple-input affine analytic control systems was addressed by Sussmann in [7], where a general sufficiency theorem was proven for analytic systems of the form $\dot{x} = f_\circ(x) + \sum_{i=1}^{m} f_i(x) u_i$ with the constraints $|u_i| \le 1$ for all $i \in \{1, \dots, m\}$. In order to understand this result, it is necessary to distinguish between *formal* brackets and *evaluated* brackets. On the one hand, a formal bracket is a pairwise parenthesized word (*i.e.*, an element of the free magma) with well-defined left and right factors, number of factors, and degree. On the other hand, an evaluated bracket is the vector field that results from an iterated Lie bracketing of particular vector fields. When we speak of the vector field *generated by* a formal bracket, we mean specifically

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the vector field that results from evaluating the formal bracket with respect to particular vectors fields, an operation that is theoretically captured by the evaluation map in [7]. This distinction is captured by the following notation, which we employ both in the statement of Sussmann's general sufficiency theorem and throughout the sequel: if $B = (i_1, \ldots (i_{k-1}, i_k) \ldots)$ represents a formal bracket of indeterminates $\{0, \ldots, m\}$, then $f^B = [f_{i_1}, \ldots [f_{i_{k-1}}, f_{i_k}] \ldots]$ denotes the corresponding vector field generated by the evaluation map from the formal bracket B with respect to a particular set of vector fields $\{f_\circ, \ldots, f_m\}$. [1] Hence, for example, we have the formal bracket $B = (1, (1, 0))$ that generates the vector field resulting from evaluating the corresponding iterated Lie bracket $[f_1, [f_1, f_\circ]]$ for a particular pair of vector fields $f_\circ$ and f_1. Several results were presented by Sussmann in [7], but in the context of this paper the most appropriate result is based on the δ_θ degree of a formal bracket. For a given formal bracket B of indeterminates $\{i\}_{i=0}^m$, $|B|_i$ is used to denote the number of occurrences of i as a factor in B. For $\theta \in [0, 1]$, δ_θ is defined by $\delta_\theta(B) := \theta |B|_\circ + \sum_{i=1}^m |B|_i$. The general theorem states that systems that satisfy the LARC at $x_\circ$ and have $f_\circ(x_\circ) = 0$ are STLC if there exists a $\theta \in [0, 1]$ such that the tangent vector at $x_\circ$ generated by each formal bracket B with $|B|_\circ$ odd and $|B|_1, \ldots, |B|_m$ all even can be expressed as a linear combination of tangent vectors at $x_\circ$ generated by some set of brackets $\{B_k\}_{k=1}^N$ with $\delta_\theta(B_k) < \delta_\theta(B)$ for all $k \in \{1, \ldots, N\}$. It has become conventional to refer to the formal brackets with $|B|_\circ$ odd and $|B|_1, \ldots, |B|_m$ all even as *bad brackets*, and if these bad brackets have corresponding vector fields that are not contained in the span of vector fields generated by good brackets of lower δ_θ degree at $x_\circ$ then they are referred to as *potential obstructions* to STLC. The obstructions are only *potential* because they only obstruct the known sufficient conditions. In [6], Kawski presents several examples are given of systems with potential obstructions that are known to be STLC.

Both Sussmann in [7] and Kawski in [6] apply a generalized definition of homogeneity to STLC. This concept of homogeneity begins with definition of a dilation δ_ϵ as a parameterized map of $\mathbb{R}^n$ to $\mathbb{R}^n$ of the form $\delta_\epsilon(x) = (\epsilon^{r_1} x_1, \epsilon^{r_2} x_2, \ldots \epsilon^{r_n} x_n)$ where r_i are non-negative integers. A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ is then said to be homogeneous of degree k with respect to the dilation, symbolically $p \in H_k$ if $p(\delta_\epsilon(x)) = \epsilon^k p(x)$. Traditional homogeneity is recovered via the dilation with $r_1 = \cdots = r_n = 1$. The definition of homogeneity is then extended to vector fields in the following manner: a vector field f is said to be homogeneous of degree j if $fp \in H_{k-j}$ whenever $p \in H_k$ for all $k \geq 0$. A related area of research that capitalizes on this generalized concept of homogeneity is that of nilpotent and high-order approximation of control systems presented by Hermes for example in [9]. One pertinent outcome of Hermes's research is that a system is STLC if its Taylor approximation is STLC. However, the converse question of whether STLC can be determined from a finite number of differentiations is still open [10].

In the remainder of this paper, we restrict our attention to homogeneous nonlinear systems that are linear in control, using the traditional definition of homogeneity. We begin by addressing single-input systems. Building on Stefani's necessary condition [5] and the concepts of good and bad brackets of Sussmann's general sufficiency theorem [7], we demonstrate that such single-input homogeneous systems of even degree are STLC if and only if they have a scalar state. This result, combined with the obvious fact that for odd-degree homogeneous single-input systems STLC is equivalent to the LARC, completely characterizes STLC for these systems. Next, we address multiple-input systems with the additional restriction that they be homogeneous *of degree two*. In this case, we extend the applicability of Sussmann's general sufficiency theorem by incorporating a linear transformation on the multidimensional input in order to neutralize potential obstructions that arise from type (1,2) bad brackets (*i.e.*, brackets with $|B|_\circ = 1$ and $|B|_{!\circ} = 2$). In particular, we present a formal method for neutralizing these type (1,2) potential obstructions wherein the problem of finding the desired linear transformation on the input space is reduced to finding a particular matrix congruence transformation.

2 Problem Exposition

In this paper, we address STLC of systems of the form

$$\dot{x} = f_\circ(x) + \sum_{i=1}^m f_i u_i \tag{1}$$

where $|u_i| \leq 1$ and $x \in \mathbb{R}^n$. f_i for $i \in \{1, \ldots, m\}$ are assumed to be constant vector fields, *i.e.*, $f_i(x) \equiv f_i$, and the components of $f_\circ(x)$ are homogeneous polynomials of degree $k \geq 1$. We use the traditional definition of homogeneous polynomial p, namely that $p(\epsilon x) = \epsilon^k p(x)$. The set of such homogeneous vector fields is denoted by $\mathcal{H}_k$. Our definition of degree-k homogeneous vector fields is equivalent to that used in [6, 7, 9] in the following

[1]Readers interested in the rich detail of formal Lie Algebras may refer to [8].

manner: take $\delta_\epsilon : x \mapsto \epsilon x$ and then $\mathcal{H}_k$ is in the general framework the set of vector fields homogeneous of degree $1-k$. With this traditional definition, we have the following elementary facts: (i) $f(0) = 0$ for $f \in \mathcal{H}_k$ with $k \geq 1$, and (ii) $[f, g] \in \mathcal{H}_{j+k-1}$ for all $f \in \mathcal{H}_j$ and $g \in \mathcal{H}_k$, where $\mathcal{H}_{-1}$ is interpreted as the singleton containing the zero vector field.

Systems of this form are theoretically interesting because their Lie algebra at $x_\circ = 0$ has a diagonal structure, as depicted in Figure 1. In particular, the only brackets B that generate vector fields with nonzero value at $x_\circ$ are those with $|B|_{!\circ} = (k-1)|B|_\circ + 1$, where $|B|_{!\circ} := \sum_{i=1}^m |B|_i$. This follows directly from the elementary facts given above. Keeping in mind that $f_\circ \in \mathcal{H}_k$ and $f_i \in \mathcal{H}_\circ$ for all $i \neq 0$, from fact (ii) it is clear that brackets above the diagonal have homogeneity degree greater than zero, and hence by fact (i) have zero value at $x_\circ$. Similarly, from fact (ii) we have that brackets below this line have homogeneity degree less than zero, and hence by definition are identically zero.

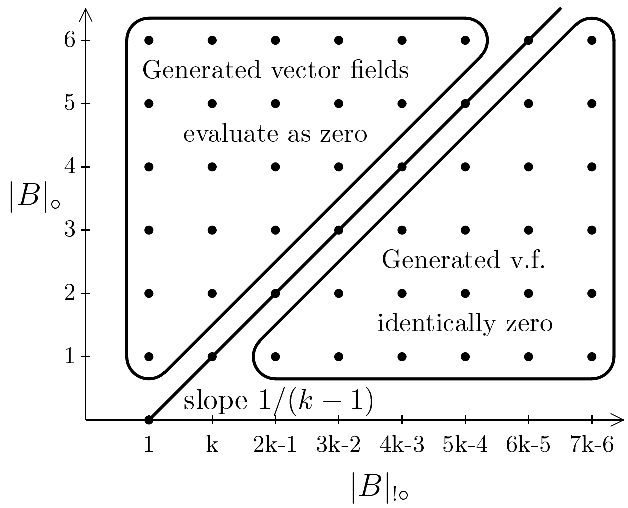

Figure 1: Graphical depiction of brackets that generate nonzero vector fields of polynomial system (3) with homogeneity degree k.

Furthermore, systems of the form (1) commonly arise in mechanics. An example of such a system is the motion of a rigid body in a plane expressed in body-fixed coordinates, as depicted in Figure 2. The equations of motion for this system are

$$\begin{aligned} \dot{\omega} &= u_2 + hu_1 \\ \dot{v}_x &= -\omega v_y \\ \dot{v}_y &= \omega v_x + u_1. \end{aligned} \tag{2}$$

The state consists of rotational velocity ω and the two body-fixed translation velocities v_x and v_y. The input consists of the torque u_2 and the force u_1 applied at a moment arm of h. Hence $f_\circ(x) = (0, -x_1x_3,\ x_1x_2)$, $f_1 = (h, 0, 1)$ for some constant h, $f_2 = (1, 0, 0)$, and the system is of the form (1) with homogeneity degree two. This provides a simple example of a system for which Sussmann's sufficient condition in [7] is not invariant with respect to input transformations. In particular, if a pure force ($h = 0$) and a torque are used as inputs, then the system satisfies Sussmann's sufficient condition for the multiple-input case.[2] However, if an offset force ($h \neq 0$) is used, then potential obstructions appear as vector fields generated from the type (1,2) brackets, *i.e.*, brackets with $|B|_\circ = 1$ and $|B|_{!\circ} = 2$. In general, we employ the phrase *type (k,ℓ) brackets* to refer to all formal brackets B of indeterminates $\{0, \ldots, m\}$ with $|B|_\circ = k$ and $|B|_{!\circ} = \ell$, and denote the distribution spanned by such brackets as $\mathcal{L}^{(k,\ell)}(\mathcal{F})$.[3] Using this system as a motivating example, we explore the neutralization via congruence transformation of potential obstructions generated by bad brackets of type (1,2) with vector fields generated by other brackets (perhaps also bad) of type (1,2).

[2] A generalized force on a rigid body consists of a pure force component which induces only a translational motion and a torque component which induces only a rotational motion.

[3] The notation $\mathcal{L}$ is used instead of L to emphasize that $\mathcal{L}^{(k,\ell)}(\mathcal{F})$ is not necessarily a Lie subalgebra.

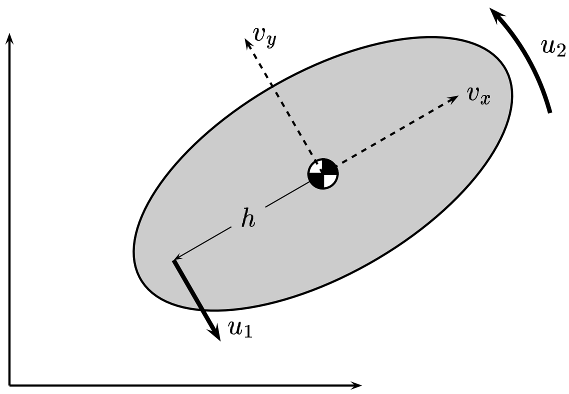

Figure 2: Motion of a rigid body in a plane expressed in body-fixed coordinates.

3 Single-Input Systems

In this section we consider the single-input system

$$\dot{x} = f_\circ(x) + f_1 u \tag{3}$$

where $x \in I\!R^n$, $u \in [-1,1]$, $f_1 \in I\!R^n$ and $f_\circ \in \mathcal{H}_k$. In light of HLCC and Sussmann's general sufficiency result [7], it is clear that for a system as in (3) with odd homogeneity degree, accessibility is equivalent to STLC, since there are no nonzero brackets with $|\cdot|_\circ$ odd and $|\cdot|_1$ even. In other words, the question of STLC reduces to the LARC. The following lemma asserts that if the vector fields generated by brackets of type (1,k) do not add to the Lie algebra rank, then neither do the vector fields generated by higher-degree brackets.

Lemma 1
For system (3) with homogeneity degree $k > 0$, if $(\mathrm{ad}^k_{f_1} f_\circ)_{x_\circ} \in \mathrm{span}\{f_1\}$, then $L(\{f_\circ, f_1\})_{x_\circ} = \mathrm{span}\{f_1\}$.
Proof 1
Since the Lie algebra structure is invariant to (analytic) coordinate transformations, without loss of generality we can take f_1 to be the basis vector e_1. Let $\mathcal{H}^{11}_k$ be the set of vector fields of homogeneity degree k with the form $(Cx_1^k + \eta_1(x), \eta_2(x), \dots, \eta_n(x))$ where C is any scalar constant and the power of x_1 in η_i is less than k for all $i \in \{1, \dots, m\}$. Then $(\mathrm{ad}^k_{f_1} f_\circ)_{x_\circ} \in \mathrm{span}\{e_1\}$ implies that $f_\circ \in \mathcal{H}^{11}_k$, since $\mathrm{ad}^k_{f_1}$ corresponds to the partial derivative operator $\partial^k/\partial x_1^k$. Furthermore, if $g \in \mathcal{H}^{11}_m$ for some $m \geq 1$, then $\mathrm{ad}_{f_1} g \in \mathcal{H}^{11}_{m-1}$ and $\mathrm{ad}_{f_\circ} g \in \mathcal{H}^{11}_{m+k-1}$. Since the Lie subalgebra of (3) is spanned by vector fields of the form $[f_{i_1}, [f_{i_2}, \cdots [f_{i_{r-1}}, f_{i_r}] \cdots]]$ (apply for example Proposition 3.8 of [11]), all vector fields in $L(\{f_\circ, f_1\})_{x_\circ}$ are a multiple of e_1.

Turning our attention to systems with even homogeneity degree k, we see that in general $L(\{f_\circ, f_1\})_{x_\circ}$ does include potential obstructions generated from bad brackets, *i.e.*, there are bad brackets along the diagonal of Figure 1. In particular, type $(m, (k-1)m+1)$ brackets are (odd, even) when m is odd. We make use of the necessary condition of Stefani restated here for convenience.

Theorem 2 (Stefani [5])
If the system $\dot{x} = f_\circ(x) + f_1(x)u_1$ is STLC, then $(\mathrm{ad}^{2k}_{f_1} f_\circ)_{x_\circ} \in \mathcal{S}^{2k-1}(\{f_\circ, f_1\})_{x_\circ}$.

When applied to even systems, Theorem 2 states that $(\mathrm{ad}^{2k}_{f_1} f_\circ)_{x_\circ} \in \mathrm{span}\{f_1\}$ is necessary for STLC. But if $(\mathrm{ad}^{2k}_{f_1} f_\circ)_{x_\circ} \in \mathrm{span}\{f_1\}$, then Lemma 1 asserts $\dim L(\{f_\circ, f_1\}) = 1$, and systems with $n \geq 2$ cannot be STLC (for otherwise LARC is violated). This reasoning and the fact that the system $\dot{x} = x^{2k} + u$ with $x \in I\!R$ is STLC provides the following result.

Proposition 3
If the system in (3) has odd homogeneity degree, then it is STLC if and only if it satisfies the LARC. On the other hand, if the system has even homogeneity degree $2k > 0$, then it is STLC if and only if the state x is scalar.

4 Multiple-Input Systems

We now return to consideration of the system in (1) where $f_\circ \in \mathcal{H}_2(x)$. An extension of the above results to this multiple-input case is problematic. In particular, the necessary condition of Stefani in [5] runs into the problem of a possibility of balancing between potential obstructions generated from bad brackets of the same degree. This consideration, along with the motivating example of planar rigid-body motion lead us to investigate neutralization of potential obstructions by vector fields generated from brackets of the same type.

Of course, for the system in (1) the general sufficient condition of Sussmann [7] can be applied to determine STLC. Since the Lie algebra has a diagonal structure, the choice of $\theta \in [0,1]$ in the theorem is immaterial. Using Sussmann's concepts of good and bad brackets, the sufficient condition allows us to neutralize potential obstructions from bad brackets with vector fields generated by good brackets of lower degree. Our goal with this section is to address the case where there are potential obstructions that cannot be neutralized in this manner, and to neutralize these potential obstructions with vector fields generated by other brackets of the *same* degree via appropriate choice of linear transformation on the input space. In this endeavor, the diagonal structure of the Lie algebra is particularly useful.

Returning to the motivating example of planar motion in equation (2), it is clear that this system is STLC. (For example, use the feedback transformation $u_1 = \omega v_x + \bar{u}_1$ and $u_2 = \bar{u}_2 - h u_1$ to obtain the system described by $\dot{\omega} = \bar{u}_2$, $\dot{v}_x = -\omega v_y$, and $\dot{v}_y = \bar{u}_1$.) However, in attempting to apply Sussmann's general sufficiency result directly on the unchanged equations of motion, we have the potential obstruction $[f_1,[f_1,f_\circ]](0) = (0,-2h,0) \notin \text{span}\{f_1(0), f_2(0)\}$. This potential obstruction prevents the use of Sussmann's sufficient condition for any $h \neq 0$. On the other hand, if $h = 0$ (corresponding to a force input u_1 through the center of mass) then we can apply the sufficient condition, for we have the vector field $[f_1,[f_2,f_\circ]]$ generated by the good bracket $(1,(2,0))$ is $(0,-1,0)$, bringing the Lie algebra to full rank, and STLC is demonstrated. Furthermore, the system for $h \neq 0$ can be transformed into the pure-force system ($h = 0$) via the input transformation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -h & 1 \end{pmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}.$$

Since STLC is clearly invariant to full-rank input transformations, we see that a particular choice of input transformation may provide a means of removing a potential obstruction, thus extending the applicability of Sussmann's general theorem for this class of systems.

Remark. It is worth noting that Kawski in [6] has considered techniques for neutralizing and balancing bad brackets. However, this technique is not clearly related to ours. Kawski's technique applies in the single-input setting, and neutralizes brackets possibly with brackets of different degree via parameterized families of controls carefully tailored to the system and the brackets in question. In some cases these families of controls involve switching between control limits, with the parameter affecting the switching times. Our technique utilizes the freedom of multiple independent inputs to enforce a linear relation between the inputs in order to neutralize brackets of the same type.

4.1 Neutralization via congruence transform

Moving to the generic multiple-input homogeneous system of degree two, suppose that there is some bad bracket of the form $(i,(i,0))$ that generates a potential obstruction, *i.e.*, $[f_i,[f_i,f_\circ]]_{x_\circ} \notin \text{span}\{f_k(x_\circ)\}_{k=1}^m$. We would like to find a full-rank, linear transformation T on the input space such that the transformed system

$$\dot{x} = f_\circ(x) + \sum_{j=1}^{m} \overbrace{(\sum_{i=1}^{m} f_i T_{ij})}^{\bar{f}_j}\, \bar{u}_j$$

has the corresponding potential obstruction removed. Of course, the full-rank input transformation will not affect $\text{span}\{f_k\}_{k=1}^m$. We make the restriction $|\bar{u}_j| \leq 1/\bar{\lambda}$ where $\bar{\lambda}$ is the spectral radius of T in order to have the resulting u_i satisfy the bounds $|u_i| \leq 1$. [4]

Suppose that there is at least one type (1,2) potential obstruction, *i.e.*, there is some $\bar{\jmath} \in \{1,\ldots,m\}$ such that $[[f_{\bar{\jmath}},[f_{\bar{\jmath}},f_\circ]]_{x_\circ} \notin \mathcal{L}^{(0,1)}(\mathcal{F})_{x_\circ}$. Consider the codistribution Ker $\mathcal{L}^{(0,1)}(\mathcal{F})$ that annihilates the distribution $\mathcal{L}^{(0,1)}(\mathcal{F})$. Then there must exist some differential one-form $\gamma \in$ Ker $\mathcal{L}^{(0,1)}(\mathcal{F})$ such that $(\gamma[f_{\bar{\jmath}},[f_{\bar{\jmath}},f_\circ]])_{x_\circ} \neq 0$.

[4] While modification of the control bound can result in difficulties in the balancing of brackets in [6], this is not a concern in our case, since the neutralization that we achieve is independent of the relative magnitudes of $\bar{u}_i$.

Let $\mathfrak{B}^{(1,2)}$ be the codistribution containing all covectors $\gamma \in \text{Ker}\,\mathcal{L}^{(0,1)}(\mathcal{F})$ with the property $(\gamma[f_j,[f_j,f_\circ]])_{x_\circ} \neq 0$ for some j. Let us suppose that $\mathfrak{B}^{(1,2)}_{x_\circ}$ has exactly dimension one. Choosing any nonzero $\beta \in \mathfrak{B}^{(1,2)}$, we define the map ψ_β from $T\mathbb{R}^n \times T\mathbb{R}^n$ to $\mathbb{R}$ by $\psi_\beta : (f,g) \mapsto (\beta[f,[g,f_\circ]])_{x_\circ}$. This map inherits bilinearity from the Lie bracket, and hence is a tensor of covariant order two at $x_\circ$. Next we derive a matrix $\Psi_\beta \in \mathbb{R}^{m\times m}$ from ψ_β via

$$(\Psi_\beta)_{ij} := \psi_\beta(f_i, f_j) \tag{4}$$

for $i,j \in \{1,\ldots,m\}$ and where f_i, f_j are the input vector fields of the system in (1). By employing the Jacobi identity and the fact that the input vector fields commute, it is clear that Ψ_β is also symmetric. Denoting by $\hat{\Psi}_\beta$ the corresponding matrix for the transformed system, it is easy to see that $\hat{\Psi}_\beta = T^T \Psi_\beta T$. In this manner, the question of whether the obstructing brackets can be neutralized is reduced to the linear algebra question:

> Given a symmetric matrix $\Psi_\beta \neq 0$, is there a full rank, square matrix T such that the congruence transformation of Ψ_β, $\hat{\Psi}_\beta = T^T\Psi_\beta T$, has all zeros along the diagonal?

Supposing for a moment that such a congruence transform exists, since it is full rank it must be true that there is some particular $\tilde{\imath}$ and $\tilde{\jmath}$ with $\tilde{\imath} \neq \tilde{\jmath}$ such that $(\hat{\Psi}_\beta)_{\tilde{\imath}\tilde{\jmath}} \neq 0$. In simplified terms, such an input transformation not only neutralizes the potential obstructions along $\mathfrak{B}^{(1,2)}_{x_\circ}$ but also replaces them with vector fields generated by good brackets along $\mathfrak{B}^{(1,2)}_{x_\circ}$. Furthermore, the input transformation will not create type (1,2) potential obstructions that are annihilated by $\mathfrak{B}^{(1,2)}_{x_\circ}$. (This would be tantamount to $T^T 0 T \neq 0$.) Of course, the input transformation will also affect the vector fields generated by higher-degree brackets, possibly creating potential obstructions.

Recalling that a symmetric matrix is called indefinite if it has at least one positive eigenvalue and at least one negative eigenvalue, we have the following answer to the posed question.

Lemma 4
Given a matrix $\Psi_\beta = \Psi_\beta^T \neq 0$, there exists a full rank matrix T such that $\hat{\Psi}_\beta = T^T\Psi_\beta T$ has all zeros on the diagonal if and only if Ψ_β is indefinite.

Proof 2
First recall that by virtue of the symmetry of the matrix Ψ_β, there exists a choice of orthonormal eigenvectors $V := (v_1,\ldots,v_m)$ such that $V^T\Psi_\beta V = \text{diag}(\lambda_1,\ldots,\lambda_m)$ where λ_i are the real eigenvalues of Ψ_β. Expressing the columns t_i of T in terms of the orthonormal eigenvectors, we have $(\hat{\Psi}_\beta)_{ii} = \sum_{j=1}^m \lambda_j (t_i^T v_j)^2$. If $\Psi_\beta \neq 0$ is semidefinite, then without loss of generality, we can take $\Psi_\beta \geq 0$, and hence $(\hat{\Psi}_\beta)_{ii} = \sum_{j=1}^m \lambda_j (t_i^T v_j)^2 \geq 0$. For necessity, we must show that $(\hat{\Psi}_\beta)_{ii} > 0$. For any full rank T there is some column t_i and some eigenvalue λ_j such that $\lambda_j(t_i^T v_j)^2 > 0$, and since $\lambda_j \geq 0$ for all j, we have $(\hat{\Psi}_\beta)_{ii} > 0$.

Suppose Ψ_β is indefinite, and group the eigenvalues into those which are positive $\{\lambda_i^+\}_{i=1}^{m_+}$, those which are negative $\{\lambda_j^-\}_{j=1}^{m_-}$, and those which are zero $\{\lambda_k^\circ\}_{k=1}^{m_\circ}$. The eigenvectors are similarly grouped into $\{v_i^+\}_{i=1}^{m_+}$, $\{v_j^-\}_{j=1}^{m_-}$, and $\{v_k^\circ\}_{k=1}^{m_\circ}$. We proceed by constructing the matrix T. The first $m_\circ$ columns t_i of T are chosen so that $t_i = v_i^\circ$, achieving $t_i^T\Psi_\beta t_i = 0$ for $i \in \{1,\ldots,m_\circ\}$. The next m_+ columns t_j are chosen according to $t_{j+m_\circ} = v_j^+/(\lambda_j^+)^{1/2} - v_1^-/(\lambda_1^-)^{1/2}$ for all $j \in \{1,\ldots,m_+\}$. For this choice, $t_i \perp t_j$ for all $i \in \{1,\ldots,m_\circ\}$ and all $j \in \{1,\ldots,m_+\}$, and $t_{j+m_\circ}^T \Psi_\beta t_{j+m_\circ} = 0$ for all $j \in \{1,\ldots,m_+\}$. The final m_- columns t_k are chosen to be $t_{k+m_\circ+m_+} = v_1^+/(\lambda_1^+)^{1/2} + v_k^-/(\lambda_k^-)^{1/2}$ for all $k \in \{1,\ldots,m_-\}$. Similarly, this final group of columns is orthogonal to $\{t_i\}_{i=1}^{m_\circ}$ and has the property $t_{k+m_\circ+m_+}^T\ \Psi_\beta\ t_{k+m_\circ+m_+} = 0$ for all $k \in \{1,\ldots,m_-\}$. Furthermore, $\{t_\ell\}_{\ell=m_\circ+1}^m$ is linearly independent. This completes the construction of T.

4.2 The planar vehicle example revisited

Applying this line of reasoning to the planar vehicle example presented above, the codistribution $\mathfrak{B}^{(1,2)}_{x_\circ}$ is spanned by $\beta = (0,-2h,0)$. In this example, if we use the canonical isomorphisms $T\mathbb{R}^n \sim T^*\mathbb{R}^n \sim \mathbb{R}^n$ and furthermore if we endow $\mathbb{R}^n$ with the natural inner product based on a particular choice of basis, we can interpret $\mathfrak{B}^{(1,2)}_{x_\circ}$ as the projection of the vector fields generated by the type (1,2) brackets onto the orthogonal complement of $\mathcal{L}^{(0,1)}(\mathcal{F})$. It represents the direction in which the vector fields generated by the type (1,2) brackets cannot be neutralized by vector fields generated by the type (0,1) brackets, *i.e.*, the direction of potential obstruction. The tensor ψ_β in coordinates is $(0,0,2h;0,0,0;2h,0,0)$. The associated matrix $\Psi_\beta = (4h^2, 2h; 2h, 0)$ has eigenvalues

$2h^2 \pm 2\sqrt{h^4+h^2}$. Of course $h \neq 0$ is assumed, for otherwise there is no potential obstruction. It is easy to see that Ψ_β is sign indefinite, and hence the construction in the proof of Lemma 4 provides the transformation

$$T_\beta = \begin{pmatrix} \sigma_1(h)h - \sigma_2(h)\sqrt{1+h^2} & \sigma_2(h)h - \sigma_1(h)\sqrt{1+h^2} \\ \sigma_1(h) & \sigma_2(h) \end{pmatrix}$$

where σ_1 and σ_2 are continuous functions of $h > 0$ with $\sigma_i > 0$ for all finite $h > 0$. This transformation yields $\hat{\Psi}_\beta = (0, -2; -2, 0)$, and the resulting type (1,2) brackets of the transformed system generate $[\bar{f}_1, [\bar{f}_1, f_\circ]] = (0, 0, 0)$, $[\bar{f}_1, [\bar{f}_2, f_\circ]] = (0, 1/h, 0)$, and $[\bar{f}_2, [\bar{f}_2, f_\circ]] = (0, 0, 0)$. Thus the potential obstruction to STLC is removed, and the Lie subalgebra generated by the system at $x_\circ$ is spanned by vector fields corresponding to good brackets, namely $\{\bar{f}_1, \bar{f}_2, [\bar{f}_1, [\bar{f}_2, f_\circ]]\}$. Hence application of Sussmann's general result [7] demonstrates STLC. It is interesting to note that while T_β is not equal to T determined above, it does transform the system into one with a pure force and a torque input for any $h > 0$.

4.3 An example of balancing two bad brackets

Not only can this technique neutralize a potential obstruction from a bad type (1,2) bracket with a vector field generated by a good type (1,2) bracket, but it may also balance two potential obstructions generated by two type (1,2) brackets. Consider the two input example with $f_\circ(x) = (x_2x_3,\ x_1x_3,\ x_1^2 - x_2^2)$, $f_1 = (1, 0, 0)$, and $f_2 = (0, 1, 0)$. This system has two potential obstructions of type (1,2), namely $[f_1, [f_1, f_\circ]] = (0, 0, 2)$ and $[f_2, [f_2, f_\circ]] = (0, 0, -2)$. Furthermore, the good bracket $(1, (2, 0))$ evaluates as $[f_1, [f_2, f_\circ]] = (0, 0, 0)$. For this example, $\Psi_\beta = (4, 0; 0, -4)$ is clearly indefinite, and hence we have the desired transformation $T = (0.5, -0.5; 0.5, 0.5)$. The resulting type (1,2) brackets for the transformed system evaluate as $[\bar{f}_1, [\bar{f}_1, f_\circ]] = (0, 0, 0)$, $[\bar{f}_1, [\bar{f}_2, f_\circ]] = (0, 0, 1)$, and $[\bar{f}_2, [\bar{f}_2, f_\circ]] = (0, 0, 0)$, and again STLC is achieved. An interesting variation on this example is obtained if we replace $f_\circ(x)$ above with $(x_2x_3,\ x_1x_3,\ x_1^2 + x_2^2 + \alpha x_1x_2)$ where $\alpha \in I\!R$. This system has $\Psi_\beta = (4, 2\alpha; 2\alpha, 4)$ indefinite for $|\alpha| > 2$. This condition has the interpretation that even when the vector fields generated by the two bad brackets have values along β *with the same sign*, they can still be neutralized with the a vector field generated by a good bracket provided that its value along β is large enough.

4.4 Effect of neutralization on other directions

Next we consider the effect of neutralization of potential obstructions along one direction within $\mathfrak{B}_{x_\circ}^{(1,2)}$ on the value of the brackets along another direction within $\mathfrak{B}_{x_\circ}^{(1,2)}$. We consider the system that evolves on $x \in I\!R^4$ described by $f_\circ(x) = (x_2x_4,\ 0,\ x_1^2 + x_1x_2,\ x_1x_2)$, $f_1 = (1, 0, 0, 0)$, and $f_2(0, 1, 0, 0)$. The type (1,2) brackets for this system are $[f_1, [f_1, f_\circ]] = (0, 0, 2, 0)$, $[f_2, [f_2, f_\circ]] = (0, 0, 0, 0)$, and $[f_1, [f_2, f_\circ]] = (0, 0, 1, 1)$, and hence $\mathfrak{B}_{x_\circ}^{(1,2)}$ is spanned by the covectors $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. If we concentrate on neutralizing the bad bracket in the direction $\beta = (0, 0, 2, 0)$, then we have $\Psi_\beta = (4, 2; 2, 0)$ with eigenvalues $\lambda = 2 \pm 2\sqrt{2}$. The constructed transformation matrix $T = (-2^{-1/4}, 2^{-1/4}; 0, 2^{1/4})$ neutralizes the bad bracket along β. However, the transformation produces a potential obstruction along the covector $(0, 0, 0, 1)$, as evidenced by the resulting vector fields $[\bar{f}_1, [\bar{f}_1, f_\circ]] = (0, 0, 0, -\sqrt{2})$, $[\bar{f}_1, [\bar{f}_2, f_\circ]] = (0, 0, -1, -1)$, and $[\bar{f}_2, [\bar{f}_2, f_\circ]] = (0, 0, 0, 0)$. In fact, this system is known to be not STLC, as can be seen by applying the coordinate transformation $y_3 := x_3 - x_4$ and $y_i = x_i$ for $i \neq 3$.

4.5 Interpretation and impact

We have developed a methodology for neutralizing potential obstructions generated by bad brackets of type (1,2) for homogeneous degree-two systems that requires indefiniteness of the matrix Ψ_β defined in (4). To interpret this requirement, we recall that a matrix is positive definite if and only if all of its principal minors are positive definite. On the other hand, a matrix is negative definite if and only if all of its principal minors are negative definite when of odd dimension and positive definite when of even dimension. Hence Ψ_β will be indefinite if some principal minor is itself indefinite. Recalling that the ij^{th} entry of Ψ_β is the value of the evaluated bracket $[f_i, [f_j, f_\circ]]$ along the covector β, the implications of the indefiniteness test become intuitively clear. For the moment let us restrict our attention to cases where $\mathfrak{B}_{x_\circ}^{(1,2)}$ has dimension one, say $\mathfrak{B}_{x_\circ}^{(1,2)} = \mathrm{span}\{\beta_{x_\circ}\}$ for some β. If there is a potential obstruction from a single, type (1,2) bad bracket and a type (1,2) good bracket is not annihilated at $x_\circ$ by β, then the obstruction can always be removed since the principal minor corresponding to these two brackets is always indefinite (*i.e.*, the matrix $(2a, b; b, 0)$ has eigenvalues $a \pm \sqrt{a^2 + 4b^2}$.) If two or more evaluated bad brackets are not annihilated at $x_\circ$ by β, then they can all be simultaneously neutralized so long as a pair of the evaluated bad brackets provide opposite signs when operated on by β. On the other hand, if one or more evaluated bad brackets all have the same sign at $x_\circ$ along the covector β and all good brackets are annihilated by β at $x_\circ$, then the technique fails. Notice that while the examples all had just two inputs, the technique applies without modification to homogeneous degree-two systems with more than two inputs ($m > 2$).

Table 1: Applicability of neutralization via congruence transformation.

$\dim(\mathfrak{B}_{x_\circ}^{(1,2)})$	# of good brackets	# of bad brackets	outcome
0	n.a.	n.a.	no obstr.
1	1	0	no obstr.
1	1	1	neutralized
1	≥ 0	2	possible neut.
≥ 2	________	open question	________

When other directions are involved, the neutralization may encounter difficulties. Supposing that $\mathfrak{B}_{x_\circ}^{(1,2)}$ is spanned by $\{\beta_i\}_{i=1}^k$ with $k \geq 2$, the question of neutralization of potential obstructions becomes one of simultaneously transforming the matrices Ψ_{β_i} so that they all have zeros on their diagonals. If the ranges of the matrices Ψ_{β_i} are orthogonal, then the problem can be solved with a block diagonal transformation T, where each block appropriately transforms each Ψ_{β_i}. This procedure requires a straightforward modification of the construction of T. These interpretations are summarized in Table 1.

Finally, notice that the homogeneity of $f_\circ$ is not essential to the development of neutralization via congruence transform, the construction of the matrix Ψ_β being sufficiently general that it applies to any nonlinear system that is linear in control. For example, neutralization of potential obstructions from type (1,2) brackets for the system $f_\circ(x) = (x_2x_3,\, x_1x_3,\, \sin^2 x_1 - \sin^2 x_2)$, $f_1 = (1,0,0)$, and $f_2 = (0,1,0)$ proceeds identically to that of the previous example with $f_\circ(x) = (x_2x_3,\, x_1x_3,\, x_1 - x_2)$. Clearly the proposed technique provides for neutralization of potential obstructions from type (1,2) brackets for these more general systems. A generalization of neutralization via congruence transformation to inhomogeneous nonlinear systems would involve incorporation of the rich differential geometry of nilpotent and higher-order approximations and foliations described for example by Hermes in [9].

5 Conclusion

We have presented a complete characterization of STLC for the class of single-input, homogeneous polynomial systems linear in control, where homogeneous is used in the traditional sense. Specifically for odd-degree systems, STLC is equivalent to the Lie Algebra Rank Condition, while even-degree systems are never STLC except for the degenerate case of a scalar state. For multiple-input homogeneous systems linear in control, we have investigated neutralization of bad brackets with brackets of the same type. The methodology presented in this paper provides a means of neutralizing bad brackets of type (1,2). By consideration of the tensor generated from the bracket structure $[\cdot, [\cdot, f_\circ]]$ applied to the direction containing a potential obstruction, we have reduced the question of neutralizing an obstruction to that of finding a congruence transform that results in a matrix with all zeros along its diagonal. It is shown that such a transformation exists if and only if the matrix in question is indefinite. When this test is translated back to type (1,2) brackets, it has intuitive implications, which are illustrated with several simple examples. The methodology presented is limited in its effectiveness by the fact that it removes a potential obstruction *only along a particular direction in the tangent space*, although an extension to multiple directions appears attainable. Although the neutralization via congruence transformation result has been presented in the context of homogeneous systems, its development does not rely on the homogeneity of the drift vector field, and hence applies to neutralization of type (1,2) brackets for any nonlinear system that is linear in control.

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