On mechanical control systems with nonholonomic constraints and symmetries [1]

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Abstract

This paper presents a computationally efficient method for deriving coordinate representations for the equations of motion and the affine connection describing a class of Lagrangian systems. We consider mechanical systems endowed with symmetries and subject to nonholonomic constraints and external forces. The method is demonstrated on two robotic locomotion mechanisms known as the snakeboard and the roller racer. The resulting coordinate representations are compact and lead to straightforward proofs of various controllability results.

Key words: nonlinear control, mechanical systems, differential geometric methods, modeling, nonholonomic constraints

1 Introduction

Over the past few years a wealth of geometric structure of Lagrangian systems subject to symmetries and constraints was uncovered through the study of robotic locomotion and manipulation [2]. For example, a mechanical device called the snakeboard illustrates the dynamical interplay between the nonholonomic constraints and symmetries [13,15]. A system that portrays similar dynamical issues is the roller racer described in [8,9]. Other related works on nonholonomic systems include [3,18,16,7].

Systems with constraints, external forces and symmetries can be described by a so-called *constrained affine connection*. An early contribution in this direction is the work of Synge [17]. Vershik [19], Bloch and Crouch [1] and Lewis [10,11] present various versions of constrained affine connections, investigate Lagrangian reduction of the equations of motion, and provide a coordinate-free treatment of various proper-

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ties of this object. We base this paper on the treatment in [10,11]. The formalism of affine connections is particularly useful for nonlinear controllability analysis, studies in vibrational control, and motion planning. In particular, Lewis *et al.* [12] characterize a variety of controllability notions, including controllability and configuration controllability, whereas Bullo *et al.* [4,5] present a perturbation analysis for systems subject to small amplitude or oscillatory forces and apply this analysis to design motion planning algorithms.

This paper provides novel, computationally efficient tools for analyzing systems with constraints, external forces and symmetries. We present efficient formulas (1) to compute the Christoffel symbols of constrained affine connections, and (2) to determine the effect of external forces while properly taking into account the system's symmetries. These formulas lead to simplified versions of the equations of motion and of the controllability computations. In particular, we present a concise, complete and straightforward treatment of the snakeboard and roller racer examples.

2 Simple mechanical control systems

A robotic manipulator with generalized forces applied at its joints is an example of a *simple mechanical control system*. More generally, a simple mechanical control system can be formally described by the following objects:

(i) an n-dimensional configuration manifold Q with coordinate system $\{q^1, \dots, q^n\}$,
(ii) an inertia tensor $M = \{M_{ij}\}$ describing the kinetic energy and defining an inner product $\langle\langle \cdot\,, \cdot \rangle\rangle$ between vector fields on Q, and
(iii) m one-forms $F_1, \dots, F_m$, describing m external control forces.

The Christoffel symbols $\{\Gamma^i_{jk} : i, j, k \in \{1, \dots, n\}\}$ of the inertia tensor M are defined by

$$\Gamma^i_{jk} = \frac{1}{2} M^{\ell i} \left(\frac{\partial M_{\ell j}}{\partial q^k} + \frac{\partial M_{\ell k}}{\partial q^j} - \frac{\partial M_{kj}}{\partial q^\ell} \right) , \qquad (1)$$

where $M^{\ell i}$ are the components of M^{-1} (the summation convention is assumed throughout the paper). All relevant quantities are assumed to be smooth. In coordinates the equations of motion are

$$\ddot{q}^k + \Gamma^k_{ij} \dot{q}^i \dot{q}^j = \sum_{a=1}^{m} M^{kj} (F_a)_j u_a \, , \qquad (2)$$

where $(F_a)_j$ is the jth component of F_a.

To formulate these equations in a coordinate-free setting, it is useful to introduce some geometric concepts; see [6]. Given two vector fields X and Y, the *covariant derivative* of Y with respect to X is the vector field $\nabla_X Y$ with coordinates

$$(\nabla_X Y)^i = \frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k \, ,$$

where X^i and Y^i are the ith and jth component of X and Y. The operator ∇ is called an *affine connection* and it is determined by the functions Γ^i_{jk}. When these functions are computed according to equation (1), the affine connection is called *Levi-Civita*.

Let $\mathscr{L}_X f$ be the Lie derivative of a scalar function f with respect to the vector field X. Given a scalar function f, its gradient $\operatorname{grad} f$ is the unique vector field defined implicitly by

$$\langle\langle \operatorname{grad} f \, , \, X \rangle\rangle = \mathscr{L}_X f \, .$$

Given a one-form F, the vector field $M^{-1}F$ is defined implicitly by $\langle F \, , \, X \rangle = \langle\langle M^{-1} F \, , \, X \rangle\rangle$. The equations of motion (2) can be written in a coordinate-free fashion as

$$\nabla_{\dot{q}} \dot{q} = \sum_{a=1}^{m} (M^{-1} F_a) u_a \, . \qquad (3)$$

3 Systems with constraints, external forces, and symmetries

Rolling without sliding is a constraint on the system's velocity which cannot be written as a constraint on the system's configuration. A non-integrable constraint of this sort is called *nonholonomic* and can be written as

$$\langle \omega \, , \, \dot{q} \rangle = 0 \, ,$$

where ω is a constraint one-form, and $\langle \cdot \, , \, \cdot \rangle$ is the natural pairing between tangent and cotangent vector fields on Q.

A *simple mechanical control system subject to nonholonomic constraints* is described by a manifold, an inertia tensor, m input forces, and a collection of constraint one-forms $\{\omega_1, \ldots, \omega_p\}$. The annihilator of $\operatorname{span}\{\omega_1, \ldots, \omega_p\}$ is the $(n-p)$-dimensional distribution of feasible velocities that we call the *constraint distribution* $\mathcal{D}$.

3.1 Coordinate-free expressions for the equations of motion

Let $P : TQ \to \mathcal{D}$ denote the orthogonal projection onto $\mathcal{D}$. Orthogonality is taken with respect to the inertia tensor M. Let $\mathcal{D}^\perp$ denote the orthogonal complement to $\mathcal{D}$ with respect to M and let $P^\perp = I - P$, where I is the identity map. The Lagrange-d'Alembert principle [14] leads to the equations of motion

$$\begin{aligned} \nabla_{\dot{q}} \dot{q} &= \lambda + \sum_{a=1}^{m} (M^{-1} F_a) u_a \\ P^\perp(\dot{q}) &= 0 \, , \end{aligned} \tag{4}$$

where $\lambda \in \mathcal{D}^\perp$ is the Lagrange multiplier enforcing the constraints. Define the covariant derivative of the tensor $P^\perp$ along the vector field X as

$$\left(\nabla_X P^\perp \right)(Y) = \nabla_X \left(P^\perp(Y) \right) - P^\perp(\nabla_X Y) \, .$$

Lemma 3.1 (Constrained affine connection [11]) *The equations of motion* (4) *can be written as*

$$\widetilde{\nabla}_{\dot{q}} \dot{q} = \sum_{a=1}^{m} (P M^{-1} F_a) u_a \, , \tag{5}$$

where $\widetilde{\nabla}$ is the affine connection given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \left(\nabla_X P^\perp \right)(Y) \, , \tag{6}$$

for all vector fields X and Y. Furthermore, for all $Y \in \mathcal{D}$:

$$\widetilde{\nabla}_X Y = P(\nabla_X Y) \, . \tag{7}$$

Equation (7) makes it possible to efficiently compute the constrained affine connection $\widetilde{\nabla}$ without covariantly differentiating the orthogonal projection $P^\perp$.

3.2 Coordinate expressions for the equations of motion

Given a basis of vector fields $\{X_1, \ldots, X_n\}$ on Q, we introduce the *generalized Christoffel symbols* of ∇ as

$$\nabla_{X_i} X_j = \Gamma^k_{ij} X_k \, .$$

We are now ready to state the main result of the paper.

Theorem 3.2 *Let $\{X_1, \ldots, X_{n-p}\}$ be an orthogonal basis of vector fields for $\mathcal{D}$. The generalized Christoffel symbols of $\widetilde{\nabla}$ are*

$$\widetilde{\Gamma}^k_{ij} = \frac{1}{\|X_k\|^2} \langle\langle \nabla_{X_i} X_j \, , \, X_k \rangle\rangle \, ,$$

and the equations of motion (5) *read*

$$\dot{v}^k + \widetilde{\Gamma}^k_{ij} v^i v^j = \sum_{a=1}^{m} Y^k_a u_a \, ,$$

where v^i are the components of $\dot{q}$ along $\{X_1, \ldots, X_{n-p}\}$, i.e., $\dot{q} = v^i X_i$, and where the coefficients of the control forces are

$$Y^k_a = \frac{1}{\|X_k\|^2} \langle F_a \, , \, X_k \rangle \, .$$

Furthermore, if the control forces are differential of functions, that is, if $F_a = \mathrm{d}\varphi_a$ for some $a \in \{1, \ldots, m\}$, then $Y^k_a = \frac{1}{\|X_k\|^2} \mathscr{L}_{X_k} \varphi_a$.

PROOF. We compute

$$\begin{aligned} \widetilde{\nabla}_{\dot{q}} \dot{q} = \widetilde{\nabla}_{\dot{q}} (v^i X_i) &= \dot{v}^i X_i + v^i \left(\widetilde{\nabla}_{\dot{q}} X_i \right) \\ &= \dot{v}^i X_i + v^i v^j \widetilde{\nabla}_{X_j} X_i \, , \end{aligned}$$

and the inner product with X_k as

$$\begin{aligned}\langle\langle X_k\,,\,\widetilde{\nabla}_{\dot q}\dot q\rangle\rangle &= \dot v^i\langle\langle X_k\,,\,X_i\rangle\rangle + v^i v^j\langle\langle X_k\,,\,\widetilde{\nabla}_{X_j}X_i\rangle\rangle\\ &= \dot v^k\|X_k\|^2 + v^i v^j\langle\langle X_k\,,\,\widetilde{\nabla}_{X_j}X_i\rangle\rangle\,,\end{aligned}$$

where we used the equality $\langle\langle X_k\,,\,X_i\rangle\rangle = 0$ for all $k \neq i$. From equation (7) we further simplify:

$$\langle\langle\widetilde{\nabla}_{X_i}X_j\,,\,X_k\rangle\rangle = \langle\langle P\nabla_{X_i}X_j\,,\,X_k\rangle\rangle = \langle\langle\nabla_{X_i}X_j\,,\,X_k\rangle\rangle\,.$$

A similar simplification takes place when computing the effect of control forces:

$$\langle\langle X_k\,,\,P(\operatorname{grad}\varphi_a)\rangle\rangle = \langle\langle X_k\,,\,\operatorname{grad}\varphi_a\rangle\rangle = \mathscr{L}_{X_k}\varphi_a\,.$$

The usual definition of Christoffel symbols requires the basis $\{X_1,\dots,X_n\}$ to be of the form $X_i = \partial/\partial q^i$ for some coordinate system $\{q^1,\dots,q^n\}$. Only under this assumption the velocity variables v^i satisfy the usual relationship $v^i = \dot q^i$. In general, the equality $\dot q = v^i X_i(q)$ in Theorem 3.2 is a nontrivial kinematic equation, and the components v^i are sometimes referred to as pseudo-velocities, as they do not correspond to the time derivative of any configuration variable. For example, when no constraints are present and the configuration space is the group itself, the equations of motion in Theorem 3.2 coincide with the classic Euler-Poincarè equations; see [2,14].

Theorem 3.2 leads to remarkable simplifications in computing the Christoffel symbols of a constrained affine connection. First of all, the formulas in the theorem do *not* require knowledge of the orthogonal projection P nor of the covariant derivative $\nabla P^{\perp}$. Since the tensor $\nabla P^{\perp}$ is a complex object to compute and simplify symbolically, this is a considerable simplification over the procedure in [11] that directly uses equation (6). Furthermore, our approach relies on computing the generalized Christoffel symbols of $\widetilde{\nabla}$ *only* over the constraint distribution $\mathcal{D}$ as opposed to the whole space TQ. Finally, the computation of Christoffel symbols and control coefficients can be further simplified by properly accounting for group actions and symmetries; we discuss this topic in the next section.

3.3 Invariance under group actions

We start by reviewing some basic definitions from [6,15]. Let G be a Lie group with identity element e. A map $\Phi : Q \times G \mapsto Q; \Phi(q,g) = \Phi_g(q)$ is a *(left) group action* on Q if it satisfies $\Phi_e(q) = q$ and $\Phi_{g_1}\Phi_{g_2}(q) = \Phi_{g_1 g_2}(q)$ for all $q \in Q$ and $g_1, g_2 \in G$. We let $T_q\Phi_g$ denote the tangent map to Φ_g.

Given a group action on Q, a scalar function f is *invariant* if $f(\Phi_g(q)) = f(q)$ for all $q \in Q$ and $g \in G$. For simplicity, we shall neglect the argument q, and write $f \circ \Phi_g = f$. A vector field X is invariant if $X \circ \Phi_g = T\Phi_g \circ X$ for all $g \in G$. A one-form ω is invariant if $\langle \omega\,,\,X\rangle$ is an invariant function for any invariant vector field X. An metric tensor M, or equivalently the inner product $\langle\langle\cdot\,,\,\cdot\rangle\rangle$, is invariant if $\langle\langle X\,,\,Y\rangle\rangle = \langle\langle T\Phi_g \circ X\,,\,T\Phi_g \circ Y\rangle\rangle \circ \Phi_g$, for all $g \in G$ and for all X and Y tangent vectors at q. Finally, a simple mechanical control systems subject to nonholonomic constraints is invariant if its inertia tensor, its input forces, and constraint one-forms are invariant.

Lemma 3.3 *Consider an simple mechanical control systems subject to nonholonomic constraints and invariant under a group action. Select a base of invariant vector fields* $\{X_1,\dots,X_m\}$ *for the constraint distribution. Then the corresponding generalized Christoffel* $\tilde\Gamma^k_{ij}$ *and control force coefficients* Y^k_a *are invariant functions.*

PROOF. The key notion is that invariance is preserved under natural operations on a manifold. For example, the inner product of invariant vector fields X and Y is an invariant function:

$$\langle\langle X\,,\,Y\rangle\rangle \circ \Phi_g = \langle\langle T\Phi_{g^{-1}} \circ X \circ \Phi_g\,,\,T\Phi_{g^{-1}} \circ Y \circ \Phi_g\rangle\rangle = \langle\langle X\,,\,Y\rangle\rangle\,.$$

Similarly, one can prove that Lie derivative of an invariant function along an invariant vector field is invariant, and that the Levi-Civita connection of an invariant system is invariant, that is $\nabla_{(T\Phi_g \circ X)}(T\Phi_g \circ Y) = T\Phi_g \circ \nabla_X Y$ for all vector fields X and Y and for all $g \in G$. In summary, the generalized Christoffel symbols and the control forces coefficients are invariant functions since their corresponding equalities in Theorem 3.2 involve only natural operations on a manifold (inner products, Lie and covariant derivatives of invariant quantities).

4 The snakeboard

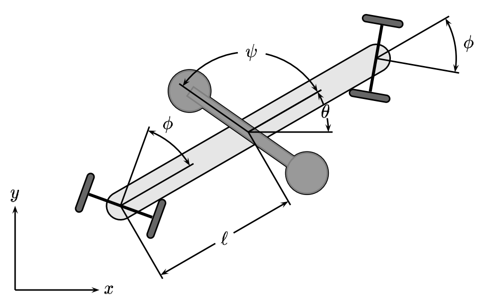

Fig. 1. The snakeboard is a modified skate-board where the angles of the front (top-right) and back (bottom-left) wheels are free to rotate. The absolute angle of the front wheels is $\theta - \phi$, the angle of the back wheels is $\theta + \phi$.

We study the snakeboard system presented in [13], see Figure 1. The configuration manifold $SE(2) \times \mathbb{S}^2$ is the Cartesian product of the group of planar displacements and a torus. In coordinates we write $q = \{x, y, \theta, \psi, \phi\}$, where (x, y) is the location of the system's center of mass, θ is the angle of the main body relative to the horizontal axis, ψ is the relative angle between the main body and the rotor, and ϕ is the relative angle between the main body and the back wheel. The inertia tensor is

$$M = \begin{pmatrix} m & 0 & 0 & 0 & 0 \\ 0 & m & 0 & 0 & 0 \\ 0 & 0 & \ell^2 m & J_r & 0 \\ 0 & 0 & J_r & J_r & 0 \\ 0 & 0 & 0 & 0 & J_w \end{pmatrix}$$

and therefore the Christoffel symbols of the Levi-Civita connection ∇ all vanish, $\Gamma^k_{ij} = 0$. The system is subject to two control inputs: a torque u_ψ that controls the angle ψ, and a torque u_ϕ controlling the angle ϕ. The location of front wheel is $(x_{\text{front}}, y_{\text{front}}) = (x + \ell\cos\theta, y + \ell\sin\theta)$, and a similar relationship holds for the back wheel. The non-slip constraints are

$$\dot{x}_{\text{front}} \sin(\theta - \phi) - \dot{y}_{\text{front}} \cos(\theta - \phi) = 0$$
$$\dot{x}_{\text{back}} \sin(\theta + \phi) - \dot{y}_{\text{back}} \cos(\theta + \phi) = 0\,,$$

which can be expressed via the one-forms

$$\omega_1 = \sin(\phi - \theta)\mathrm{d}x + \cos(\phi - \theta)\mathrm{d}y + \ell\cos\phi\mathrm{d}\theta$$
$$\omega_2 = -\sin(\phi + \theta)\mathrm{d}x + \cos(\phi + \theta)\mathrm{d}y - \ell\cos\phi\mathrm{d}\theta.$$

We refer to [15,11] for more details on the assumptions in this model.

4.1 *An orthogonal basis for the feasible velocities*

We start by introducing the convenient vector fields

$$\begin{aligned} V_x &= \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} \\ V_y &= -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}\,. \end{aligned} \tag{8}$$

They are motivated by the $SE(2)$ symmetry and describe body fixed translation along the x and y body fixed axis. The set of feasible velocities is generated by the three vector fields

$$X_1 = \ell(\cos\phi)V_x - (\sin\phi)\frac{\partial}{\partial\theta},$$
$$X_2' = \frac{\partial}{\partial\psi}, \quad X_3' = \frac{\partial}{\partial\phi}\,.$$

Note that X_3' is perpendicular to X_1 and X_2'. A direct way of computing an orthogonal basis $\{X_1, X_2, X_3\}$ from the basis $\{X_1, X_2', X_3'\}$ is to define

$$X_2 = X_2' - \frac{\langle\langle X_2'\,, X_1\rangle\rangle}{\langle\langle X_1\,, X_1\rangle\rangle} X_1\,, \tag{9}$$

and $X_3 = X_3'$. After a rescaling step, X_2 becomes

$$X_2 = \frac{J_r}{m\ell}(\cos\phi\sin\phi)V_x - \frac{J_r}{m\ell^2}(\sin\phi)^2\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\psi}\,.$$

These vector fields have the following physical interpretation: X_1 describes the instantaneous rotation when the internal angles $\{\psi, \phi\}$ are fixed, while $\{X_2', X_3'\}$ correspond to changes in the internal angles.

4.2 *The equations of motion*

We use the results in Theorem 3.2 to obtain the equations of motion in coordinates. [2] The required computations are performed with a symbolic manipulation package. The only non-vanishing Christoffel symbols are

$$\widetilde{\Gamma}^1_{32} = \frac{J_r}{m\ell^2}\cos\phi\,, \quad \widetilde{\Gamma}^2_{31} = -\frac{m\ell^2\cos\phi}{m\ell^2 + J_r(\sin\phi)^2}\,,$$
$$\widetilde{\Gamma}^2_{32} = -\frac{J_r(\cos\phi\sin\phi)}{m\ell^2 + J_r(\sin\phi)^2}\,.$$

The next step is to compute how the two inputs come into the equations. All relevant Lie brackets vanish, except for $\mathscr{L}_{X_2}\psi = 1$, and $\mathscr{L}_{X_3}\phi = 1$. We also compute the two norms

$$\|X_2\|^2 = J_r + \frac{J_r^2(\sin\phi)^2}{m\ell^2}\,, \quad \|X_3\|^2 = J_w\,.$$

Next, we write the kinematic equations of motion $\dot q = X_1 v + X_2\dot\psi + X_3\dot\phi$. We write $\dot\psi$ and $\dot\phi$ for the velocity components along X_2 and X_3 since X_2 has a unit component along $\partial/\partial\psi$, and X_3 has a unit component along $\partial/\partial\phi$. In coordinates the kinematic equations are

$$\begin{pmatrix}\dot x\\ \dot y\\ \dot\theta\end{pmatrix} = \begin{pmatrix}\ell\cos\phi\cos\theta\\ \ell\cos\phi\sin\theta\\ -\sin\phi\end{pmatrix} v + \begin{pmatrix}\frac{J_r}{m\ell}\cos\phi\sin\phi\cos\theta\\ \frac{J_r}{m\ell}\cos\phi\sin\phi\sin\theta\\ -\frac{J_r}{m\ell^2}(\sin\phi)^2\end{pmatrix}\dot\psi$$

and the dynamic equations are

$$\begin{aligned}
&\dot v + \frac{J_r}{m\ell^2}(\cos\phi)\dot\phi\dot\psi = 0\\
&\ddot\psi - \frac{m\ell^2\cos\phi}{m\ell^2 + J_r(\sin\phi)^2}v\dot\phi - \frac{J_r\cos\phi\sin\phi}{m\ell^2 + J_r(\sin\phi)^2}v\dot\psi\\
&\qquad = \frac{m\ell^2}{m\ell^2 J_r + J_r^2(\sin\phi)^2}u_\psi\\
&\ddot\phi = \frac{1}{J_w}u_\phi\,.
\end{aligned}$$

5 The roller racer

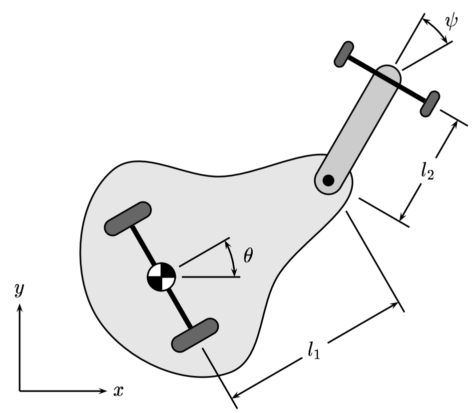

Fig. 2. The roller racer is a planar two-link device with wheels on both links and a control torque applied to the central joint.

We study the roller racer system presented in [8], see Figure 2. The configuration manifold is $SE(2)\times\mathbb{S}$. In coordinates we write $q = \{x, y, \theta, \psi\}$, where θ is the angle of the main body relative to the horizontal axis, and ψ is the relative angle between the main body and the front link. Neglecting the inertia of the front link, see [8], we get

$$M = \begin{pmatrix} m & 0 & 0 & 0\\ 0 & m & 0 & 0\\ 0 & 0 & I_1+I_2 & I_2\\ 0 & 0 & I_2 & I_2\end{pmatrix}.$$

Therefore, the Christoffel symbols of the Levi-Civita connection ∇ all vanish. The system is subject to a

[2] The code is available at `http://motion.csl.uiuc.edu/~bullo/math`.

single control input: a pure torque u_ψ that controls the angle ψ. The constraint one-forms are

$$\begin{aligned}\omega_1 &= \sin\theta \mathrm{d}x - \cos\theta \mathrm{d}y \\ \omega_2 &= \sin(\theta+\psi)\mathrm{d}x - \cos(\theta+\psi)\mathrm{d}y \\ &\quad - (\ell_2 + \ell_1\cos\psi)\mathrm{d}\theta - \ell_2 \mathrm{d}\psi\,.\end{aligned}$$

The system has a kinematic singularity at $\ell_2 + \ell_1\cos\psi = 0$. At that value, the system can only rotate about its center of mass, and (x, y, ψ) are constants.

5.1 An orthogonal basis for the feasible velocities

The set of feasible velocities is generated by the vector fields

$$\begin{aligned}X_1 &= V_x + \left(\frac{\sin\psi}{\ell_2+\ell_1\cos\psi}\right)\frac{\partial}{\partial\theta} \\ X_2' &= -\left(\frac{\ell_2}{\ell_2+\ell_1\cos\psi}\right)\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\psi}\,,\end{aligned}$$

where V_x is defined as in equation (8) above. Equation (9) can then be used to obtain an orthogonal basis $\{X_1, X_2\}$. We define the shorthands:

$$\begin{aligned}f_1(\psi) &= m(\ell_2+\ell_1\cos\psi)^2 + (I_1+I_2)(\sin\psi)^2 \\ f_2(\psi) &= m\ell_2^2 I_1 + \ell_1^2 I_2 m(\cos\psi)^2 + I_1 I_2(\sin\psi)^2\,.\end{aligned}$$

In coordinates we have

$$\begin{aligned}X_2 = {}& \frac{(\ell_2 I_1 - \ell_1 I_2\cos\psi)\sin\psi}{f_1(\psi)}V_x \\ & - \frac{m\ell_2(\ell_2+\ell_1\cos\psi) + I_2(\sin\psi)^2}{f_1(\psi)}\frac{\partial}{\partial\theta} + \frac{\partial}{\partial\psi}\,.\end{aligned}$$

These vector fields have the following physical interpretation: X_1 encodes the instantaneous rotation when the internal angle ψ is fixed, and X_2 encodes a change in ψ and other variables.

5.2 The equations of motion

From Theorem 3.2 we compute the non-vanishing Christoffel symbols as

$$\begin{aligned}\widetilde{\Gamma}^1_{21} &= \left(\frac{\ell_1+\ell_2\cos\psi}{\ell_2+\ell_1\cos\psi}\right)\frac{(I_1+I_2)\sin\psi}{f_1(\psi)} \\ \widetilde{\Gamma}^1_{22} &= \frac{-m(\ell_1+\ell_2\cos\psi)(\ell_2+\ell_1\cos\psi)(\ell_1 I_2\cos\psi - \ell_2 I_1)}{f_1(\psi)^2} \\ \widetilde{\Gamma}^2_{21} &= \left(\frac{\ell_1+\ell_2\cos\psi}{\ell_2+\ell_1\cos\psi}\right)\frac{m(\ell_1 I_2\cos\psi - \ell_2 I_1)}{f_2(\psi)} \\ \widetilde{\Gamma}^2_{22} &= \frac{-m(\ell_1 I_2\cos\psi - \ell_2 I_1)(\sin\psi)f_3(\psi)}{f_1(\psi)f_2(\psi)}\,,\end{aligned}$$

where $f_3(\psi) = (\ell_1 I_2 - \ell_2 I_1\cos\psi) + m\ell_1\ell_2(\ell_2 + \ell_1\cos\psi)$. Note that all the Christoffel symbols are well-defined away from the kinematic singularity. To establish how the input torque comes into the equations we compute

$$\mathscr{L}_{X_1}\psi = 0, \quad \frac{1}{\|X_2\|^2}\mathscr{L}_{X_2}\psi = \frac{f_1(\psi)}{f_2(\psi)}\,.$$

We are now ready to write the kinematic equations of motion as $\dot q = X_1 v + X_2\dot\psi$, where we write $\dot\psi$ for the velocity component along X_2 since X_2 has unit component along $\partial/\partial\psi$. In coordinates the kinematic equations are

$$\begin{pmatrix}\dot x\\ \dot y\\ \dot\theta\end{pmatrix} = \begin{pmatrix}\cos\theta \\ \sin\theta \\ \frac{\sin\psi}{\ell_2+\ell_1\cos\psi}\end{pmatrix} v + \begin{pmatrix}\frac{(\ell_2 I_1-\ell_1 I_2\cos\psi)\sin\psi}{f_1(\psi)}\cos\theta \\ \frac{(\ell_2 I_1-\ell_1 I_2\cos\psi)\sin\psi}{f_1(\psi)}\sin\theta \\ \frac{m\ell_2(\ell_2+\ell_1\cos\psi)+I_2(\sin\psi)^2}{-f_1(\psi)}\end{pmatrix}\dot\psi$$

and the dynamic equations are

$$\begin{aligned}&\dot v + \widetilde{\Gamma}^1_{21}(\psi)\dot\psi v + \widetilde{\Gamma}^1_{22}(\psi)\dot\psi^2 = 0 \\ &\ddot\psi + \widetilde{\Gamma}^2_{21}(\psi)\dot\psi v + \widetilde{\Gamma}^2_{22}(\psi)\dot\psi^2 = \frac{f_1(\psi)}{f_2(\psi)}u_\psi\,.\end{aligned}$$

6 Controllability analysis

In this section we show how the expressions in examples' Christoffel symbols and control input coefficients can be combined with the approach in [12] to perform an effective controllability analysis. We start with some definitions. A system is *locally configuration accessible at a configuration* q_0 if the set of all configurations that are reachable from q_0 starting with an initial velocity equal to zero is a non-empty open subset of Q. It is *locally configuration controllable at* q_0 if q_0 belongs to the interior of this set. The reference [12] presents sufficient conditions in the form of simple algebraic tests for local configuration accessibility and controllability. An algebraic operation that plays a prominent role in these tests is the *symmetric product* between two vector fields X and Y, which is defined as:

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X \,.$$

6.1 Controllability analysis for the snakeboard

With the notation in Section 4, the snakeboard has input vector fields X_2 and X_3. Compute the symmetric products

$$\begin{aligned} \langle X_2 : X_2 \rangle &= 0\,, \quad \langle X_3 : X_3 \rangle = 0\,, \\ \langle X_2 : X_3 \rangle &= \frac{J_r}{m\ell^2}(\cos\phi)X_1 - \frac{J_r(\cos\phi\sin\phi)}{m\ell^2 + J_r(\sin\phi)^2}X_2\,. \end{aligned}$$

Therefore, $\operatorname{span}\{X_2, X_3, \langle X_2 : X_3\rangle\}$ equals the constraint distribution $\mathcal{D}$ everywhere where $\cos\phi \neq 0$. The involutive closure of $\mathcal{D}$ is full rank because

$$\begin{aligned} [X_1, X_3] &= \ell(\sin\phi)V_x + (\cos\phi)\frac{\partial}{\partial\theta} \\ [X_1, [X_1, X_3]] &= -\ell(\sin\phi)V_y\,, \end{aligned}$$

and because the determinant of the matrix with columns $\{X_1, X_2, X_3, [X_1, X_3], [X_1, [X_1, X_3]]\}$ equals ℓ^2. According to the treatment in [12], the system is locally configuration controllable.

6.2 Controllability analysis for the roller racer

With the notation in Section 5, the roller racer has a single input vector field X_2. The only possible symmetric product is

$$\langle X_2 : X_2 \rangle = 2\widetilde{\Gamma}^1_{22}(\psi)X_1 + 2\widetilde{\Gamma}^2_{22}(\psi)X_2\,.$$

Provided $\widetilde{\Gamma}^1_{22}(\psi) \neq 0$, that is, $\ell_2 I_1 \cos\psi \neq \ell_1 I_2$, the distribution generated by $\operatorname{span}\{X_2, \langle X_2 : X_2\rangle\}$ equals the constraint distribution $\mathcal{D}$. Furthermore, the involutive closure of $\mathcal{D}$ is full rank because

$$\begin{aligned} [X_1, X_2] &= \frac{\ell_2}{\ell_2 + \ell_1\cos\psi}V_y - \frac{\ell_1 + \ell_2\cos\psi}{(\ell_2 + \ell_1\cos\psi)^2}\frac{\partial}{\partial\theta} \\ [X_1, [X_1, X_2]] &= \frac{-\ell_2\sin\psi}{(\ell_2 + \ell_1\cos\psi)^2}V_x + \frac{\ell_1 + \ell_2\cos\psi}{(\ell_2 + \ell_1\cos\psi)^2}V_y\,, \end{aligned}$$

and because the determinant of the matrix with columns $\{X_1, X_2, [X_1, X_2], [X_1, [X_1, X_2]]\}$ equals

$$\frac{\ell_1^2 + \ell_2^2 + 2\ell_1\ell_2\cos\psi}{(\ell_2 + \ell_1\cos\psi)^4}\,.$$

Therefore the system is locally configuration accessible everywhere $\ell_2 I_1 \cos\psi \neq \ell_1 I_2$. It is not locally controllable or configuration controllable as proven in [11].

References

[1] A. M. Bloch and P. E. Crouch. Nonholonomic control systems on Riemannian manifolds. *SIAM Journal on Control and Optimization*, 33(1):126–148, 1995.

[2] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and R. M. Murray. Nonholonomic mechanical systems with symmetry. *Archive for Rational Mechanics and Analysis*, 136(1):21–99, 1996.

[3] A. M. Bloch, M. Reyhanoglu, and N. H. McClamroch. Control and stabilization of nonholonomic dynamic systems. *IEEE Transactions on Automatic Control*, 37(11):1746–1757, 1992.

[4] F. Bullo. Series expansions for the evolution of mechanical control systems. *SIAM Journal on Control and Optimization*, 40(1):166–190, 2001.

[5] F. Bullo, N. E. Leonard, and A. D. Lewis. Controllability and motion algorithms for underactuated Lagrangian systems on Lie groups. *IEEE Transactions on Automatic Control*, 45(8):1437–1454, 2000.

[6] S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry. Vol. I*, volume 15 of *Interscience Tracts in Pure and Applied Mathematics*. Interscience Publishers, New York, NY, 1963.

[7] I. Kolmanovsky and N. H. McClamroch. Developments in nonholonomic control problems. *IEEE Control Systems Magazine*, 15(6):20–36, 1995.

[8] P. S. Krishnaprasad and D. P. Tsakiris. Oscillations, SE(2)-snakes and motion control. In *IEEE Conf. on Decision and Control*, pages 2806–11, New Orleans, LA, December 1995.

[9] P. S. Krishnaprasad and D. P. Tsakiris. Oscillations, $SE(2)$-snakes and motion control: a study of the roller racer. *Dynamics and Stability of Systems*, 16(4):347–397, 2001.

[10] A. D. Lewis. Affine connections and distributions with applications to nonholonomic mechanics. *Reports on Mathematical Physics*, 42(1/2):135–164, 1998.

[11] A. D. Lewis. Simple mechanical control systems with constraints. *IEEE Transactions on Automatic Control*, 45(8):1420–1436, 2000.

[12] A. D. Lewis and R. M. Murray. Configuration controllability of simple mechanical control systems. *SIAM Journal on Control and Optimization*, 35(3):766–790, 1997.

[13] A. D. Lewis, J. P. Ostrowski, R. M. Murray, and J. W. Burdick. Nonholonomic mechanics and locomotion: the snakeboard example. In *IEEE Int. Conf. on Robotics and Automation*, pages 2391–2400, San Diego, CA, May 1994.

[14] J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry*. Springer Verlag, New York, NY, second edition, 1999.

[15] J. P. Ostrowski and J. W. Burdick. The geometric mechanics of undulatory robotic locomotion. *International Journal of Robotics Research*, 17(7):683–701, 1998.

[16] N. Sarkar, X. Yun, and V. Kumar. Control of mechanical systems with rolling constraints: Application to dynamic control of mobile robots. *International Journal of Robotics Research*, 13(1):55–69, 1994.

[17] J. L. Synge. Geodesics in nonholonomic geometry. *Mathematische Annalen*, 99:738–751, 1928.

[18] A. J. van der Schaft and B. M. Maschke. On the Hamiltonian formulation of nonholonomic mechanical systems. *Reports in Mathematical Physics*, 34(2):225–233, 1994.

[19] A. M. Vershik. Classical and non-classical dynamics with constraints. In Y. G. Borisovich and Yu. E. Gliklikh, editors, *Global analysis: studies and applications*, volume 1108 of *Lecture Notes in Mathematics*, pages 278–301. Springer Verlag, New York, NY, 1984.