Modeling and Controllability for a Class of Hybrid Mechanical Systems

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Abstract— **This work studies a class of hybrid mechanical systems that locomote by switching between constraints defining different dynamic regimes. We develop a geometric framework for modeling smooth phenomena such as inertial forces, holonomic, and nonholonomic constraints, as well as discrete features such as transitions between smooth dynamic regimes through plastic and elastic impacts. We focus on devices that are able to switch between constraints at an arbitrary point in the configuration space. This class of hybrid mechanical control systems can be described in terms of affine connections and jump transition maps that are linear in the velocity. We investigate two notions of local controllability, the equilibrium and kinematic controllability, and provide sufficient conditions for each of them. The tests rely on the assumption of zero velocity switches. We illustrate the modeling framework and the controllability tests on a planar sliding, clamped, and rolling device. In particular, we show how the analysis can be used for motion planning.**

Keywords— **mechanical control systems, nonlinear controllability, hybrid systems**

I. Introduction

A. Problem description and motivation

This work studies a class of mechanical systems that locomote by switching between constraints. Each constraint results in different dynamic equations. Such system therefore form a subclass of hybrid systems. Important representatives in this subclass are locomotion and grasping devices where changes in the dynamics stem from switches in the constraints describing the interaction between the device and the environment. While a locomotion device should be able to move between two arbitrary configurations, it is desirable to minimize the number of actuators that are necessary to move the device in order to simplify the mechanical design, reduce the cost, and increase the reliability. Nonholonomic constraints can often make up for the missing actuation, as was for example demonstrated in [1]. Typically, the price for the reduction in actuation is the increased complexity of the control algorithms.

In this work we explore a different class of locomotion devices, where the reduction in the number of actuated degrees of freedom is obtained by enabling the device to switch among several different constrained regimes. A motivation for studying such devices comes from legged locomotion. However, while in the case of legged locomotion a switch between constraints can only occur when a leg hits the ground, we consider the case where the device can switch between the constrained regimes at an arbitrary point in the configuration space. We describe one such device, a planar mechanism that can locomote by clamping one of its links to the ground. A device that could be studied in this context is the roller-walker described in [2], [3].

Hybrid systems that locomote by switching between constraints cannot be analyzed using methods derived for smooth mechanical systems. On the other hand, techniques developed for general hybrid systems are not able to account for the special geometric structure of the mechanical systems. In this paper we propose a new modeling framework that achieves both: it models the hybrid nature of the mechanical system and takes into account its special structure. The modeling framework allows us to study local controllability of hybrid locomotion devices. It is worth remarking that controllability analysis of a mechanism is useful at several levels. First, it is a necessary step in the analysis and design of mechanisms as it directly depends on allocation and availability of actuators. Second, it provides guidance for design of motion planning algorithms, And finally, it suggests which control strategies are most appropriate. For example, controllability analysis has lead to effective motion planning schemes for various classes of mechanical control systems, e.g., see the works on driftless systems on Lie groups [4], [5], on mechanical systems on Lie groups [6], [7], and on kinematically controllable systems [8], [9].

B. Previous work

Recent advances in control of smooth Lagrangian systems have led to a theoretical framework that encompasses numerous results on modeling and controllability. Coordinate-free models for nonholonomic constraints are discussed in Bloch and Crouch [10] and Lewis [11]. The modeling framework in both these works is that of affine connections; see the textbooks by Do Carmo [12] and Marsden and Ratiu [13] for background information and for alternative modeling paradigms. Impact models are discussed in Brogliato [14]. Controllability results for smooth mechanical systems are presented in Lewis and Murray [6]; these results exploit the work on small-time local controllability by Sussmann [15]. Kinematically controllable systems are characterized by Bullo and Lynch [9].

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One inherent difficulty in studying local controllability properties for hybrid systems is that Lie bracket computations are not well defined between vector fields belonging to different regimes. To overcome this obstacle, Goodwine and Burdick [16], [17] study the setting of kinematic mechanical systems defined over stratified manifolds. A careful construction allows them to compute Lie brackets between vector fields defined on different strata. The hybrid mechanical control systems studied in this work differ from those considered in [16], [17] in two respects: 1) the dynamic equations are second order, i.e., our systems are dynamic (and have drift) instead of being kinematic, and 2) in our case, different constrained regimes are defined over the same configuration space. This second assumption leads to a well defined computation of Lie brackets.

Several alternatives were proposed for modeling of hybrid systems. Alur et al. [18] and Nicollin et al. [19] define the notion of hybrid automaton, building their work on the automata theory. In Brockett [20], a model is proposed that augments a state-space model of a dynamical system with a map that describes the evolution of the discrete state. Further models and various applications are documented for example in a sequence of proceedings starting with [21] through [22]. The focus of most existing studies is on formulating a general model for a hybrid system.

Controllability of hybrid mechanical systems has also attracted considerable attention. Some of the contributions in this category are [23], [24], [25]. However, as in the case of modeling, the focus has mostly been on general hybrid systems and global results. Given that a satisfactory theory for global controllability is not available even in the smooth case, the scope of these studies is limited and the results are difficult to apply in practice. Our aim is instead to focus on local controllability properties, provide computable tests, and investigate the relationship with controllability notions for smooth systems.

C. Statement of contribution

The contribution of this paper is twofold. On one side we provide a modeling framework that encompasses geometric models for holonomic and nonholonomic constraints, plastic and elastic impact mechanics, and systems with changing dynamics. In particular, we present an intrinsic definition of hybrid mechanical control systems in terms of affine connections and linear jump transition maps. Furthermore, we present a novel instructive example consisting of a planar mechanism switching between sliding, clamped and rolling regimes.

Secondly, we present a local nonlinear controllability analysis for a subclass of hybrid mechanical systems which are allowed to switch between dynamic regimes at an arbitrary point in the configuration space. We review the notions of equilibrium and kinematic controllability and provide coordinate-free sufficient tests that extend the results for the smooth case to this subclass of hybrid mechanical systems. These algebraic tests are easily verified and therefore immediately applicable. We illustrate the tests on the planar mechanism and determine equilibrium and kinematic controllability of various configurations of the mechanism. We subsequently use the analysis in designing motion primitives and a complete local motion planning algorithm for the device.

The paper is organized as follows. Section II presents a unified treatment of mechanical systems with constraints and impacts and a notion of hybrid mechanical control system. Section III introduces a sliding, clamped, and rolling machine as an example of hybrid mechanical control system. Section IV characterizes the equilibrium and kinematic controllability properties of a class of hybrid mechanical control systems and shows its implications for the planar device. We present our conclusions in Section V.

II. Smooth and hybrid mechanical systems

We start by reviewing some modeling concepts for smooth mechanical systems. We assume the reader to be familiar with some geometric concepts employed in nonlinear control theory [26] and in geometric mechanics [13]. In this work we consider dynamical systems that are not subject to any potential forces such as gravity other than possibly the control inputs. This assumption is natural since we are primarily interested in locomotion devices where it is desirable that with no inputs the system can be in equilibrium at any point in the configuration space. We start with systems with total energy equal to kinetic energy, and later generalize this model to include constraints and impacts.

A. Mechanical control systems

Let Q be the configuration manifold of a system with coordinates $q = (q_1, \dots, q_n)$. At every $q \in Q$, the kinetic energy of the system defines a Riemannian metric M. We neglect any potential forces so that the total energy equals the kinetic energy. If $\dot{q} = \left[\dot{q}^1, \dots, \dot{q}^n\right]^T$ denotes the velocity variables, the total energy $\mathcal{E}$ for the system is thus

$$\mathcal{E}(q, \dot{q}) = \frac{1}{2} M_{ij}\, \dot{q}^i \dot{q}^j .$$

Note that the summation convention is assumed throughout the paper. The Christoffel symbols Γ^i_{jk} of the metric M are

$$\Gamma^i_{jk} = \frac{1}{2} M^{li} \left(\frac{\partial M_{lj}}{\partial q^k} + \frac{\partial M_{lk}}{\partial q^j} - \frac{\partial M_{kj}}{\partial q^l} \right), \tag{1}$$

where M^{li} are the components of M^{-1}. Let $F^1, \dots, F^m$ be the input forces. The input vector fields are then $Y_k = M^{-1} F^k$ and $\mathcal{Y} = \text{span}\{Y_1, \dots, Y_m\}$ is the input distribution. Then the forced Euler-Lagrange equations for the system are

$$\ddot{q}^i + \Gamma^i_{jk}(q) \dot{q}^j \dot{q}^k = \left(Y_k(q) u^k(t) \right)^i . \tag{2}$$

Throughout the paper we will assume that the functions u^k are piecewise smooth so that the existence and uniqueness of solutions to (2) are guaranteed.

The last equation suggests that we can formally define a *mechanical control system* as a quadruple $\Sigma = \{Q, M, \mathcal{F}, U\}$ where

(i) Q is an n-dimensional configuration manifold with local coordinates $q = \{q^1, \ldots, q^n\}$,
(ii) $M : TQ \times TQ \to \mathbb{R}$ is a metric on Q (the kinetic energy), that uniquely determines an inertia matrix M,
(iii) $\mathcal{F} = \mathrm{span}\{F^1, \ldots, F^m\}$ is an m-dimensional codistribution defining the input forces, and
(iv) $U \subset \mathbb{R}^m$ is the co-domain for the functions u^k.

B. Coordinate-free description

To formulate the equations of motion in a coordinate-free setting, it is useful to introduce some geometric concepts; see Do Carmo [12]. Given two vector fields X, Y, the *covariant derivative* of Y with respect to X is a new vector field $\nabla_X Y$ with coordinates

$$(\nabla_X Y)^i = \frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k. \tag{3}$$

The operator ∇ is called an *affine connection* and it is determined by the functions Γ^i_{jk}. When the functions Γ^i_{jk} are computed according to equation (1), the affine connection is called *Levi-Civita*. Using these concepts, the equations of motion (2) can be written in a coordinate-free fashion as

$$\nabla_{\dot q} \dot q = Y_k u^k. \tag{4}$$

C. Holonomic and nonholonomic constraints

We are interested in a class of locomotion devices that interact with the surrounding environment via holonomic or nonholonomic constraints. For both types of constraints, the Lagrange-d'Alembert principle provides a unified way for deriving the constrained equations of motion.

Holonomic Constraints Clamping a sliding body to a surface is an example of a holonomic constraint. Formally, a holonomic constraint is described by an equation of the form $\varphi(q) = 0$. We assume that the map $\varphi : Q \to \mathbb{R}^{n-p}$ is smooth and that 0 is a regular value of φ, so that $R = \varphi^{-1}(0)$ is a smooth submanifold of Q. The constraint on $q(t)$ induces a constraint on $\dot q(t)$ via

$$0 = \frac{d}{dt}\varphi^i(q(t)) = \mathrm{d}\varphi^i \cdot \dot q, \quad i = 1, \ldots, n-p. \tag{5}$$

This implies that at each point $q \in Q$, the feasible velocities $\mathcal{D}(q) \subset T_q Q$ correspond to the annihilator of the set of covector fields $\{\mathrm{d}\varphi^1(q), \ldots, \mathrm{d}\varphi^{n-p}(q)\}$. If the covector fields are represented in coordinates as (column) vectors $\alpha^1, \ldots, \alpha^{n-p}$, the set of feasible velocities $\mathcal{D}$ corresponds to the null-space of the matrix $\left[\alpha^1 \cdots \alpha^{n-p}\right]^T$. The characterizing feature of the systems we consider is that the constraint can become active at any configuration so that the constraint distribution is defined at least over an open subset of Q.

Nonholonomic Constraints Rolling without sliding is an example of a nonholonomic constraint. We describe a nonholonomic constraint by a p-dimensional *constraint distribution* $\mathcal{D}$. At each point $q \in Q$, $\mathcal{D}(q)$ describes the set of feasible velocities. In other words, $\mathcal{D}(q)$ is the set of directions in which the device can move instantaneously. Quite often, the constraint distribution $\mathcal{D}(q)$ for nonholonomic constraints is also described by a set of equations (5).

The advantage of using constraint distributions is that both holonomic and nonholonomic constraints can be represented in the same way. In both cases, we can write $\dot q \in \mathcal{D}(q)$ for an appropriate distribution $\mathcal{D}(q)$.

A mechanical control system together with a constrained distribution is said to be a *constrained mechanical control system* and can be represented by a tuple $\Sigma = \{Q, M, \mathcal{F}, \mathcal{D}, U\}$. A system subject to no constraint can be thought of as a constrained system by setting $\mathcal{D} = TQ$.

D. Constrained equations of motion

Let $P : TQ \to \mathcal{D}$ denote the orthogonal projection onto the distribution of feasible velocities $\mathcal{D}$. Let $\mathcal{D}^\perp$ denote the orthogonal complement to $\mathcal{D}$ with respect to the metric M and let $P^\perp = Id - P$, where Id is the identity operator. The Lagrange-d'Alembert constrained variational principle [13] leads to the equations of motion

$$\nabla_{\dot q} \dot q = \lambda(t) + Y_k u^k, \tag{6}$$

$$P^\perp(\dot q) = 0, \tag{7}$$

where $\lambda(t) \in \mathcal{D}^\perp$ is the Lagrange multiplier enforcing the constraint. Following the treatment in [11], equation (6) can be written as:

$$\widetilde{\nabla}_{\dot q} \dot q = P(Y_k) u^k, \tag{8}$$

where $\widetilde{\nabla}$ is the affine connection given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \left(\nabla_X P^\perp\right)(Y), \qquad \forall X, Y. \tag{9}$$

The term $\nabla_X P^\perp$ stands for the covariant derivative of the $(1,1)$ tensor $P^\perp$:

$$\left(\nabla_X P^\perp\right)(Y) = \nabla_X \left(P^\perp(Y)\right) - P^\perp(\nabla_X Y).$$

In this paper, the connection $\widetilde{\nabla}$ will be only applied to the vector fields that belong to $\mathcal{D}$. A short computation shows that for $Y \in \mathcal{D}$:

$$\widetilde{\nabla}_X Y = P(\nabla_X Y). \tag{10}$$

The last expression allows us to evaluate $\widetilde{\nabla}_X Y$ directly and thus avoid significant amount of computation needed to explicitly compute $\widetilde{\nabla}$ from equation (9); see [27].

In summary, it is indeed possible to write the equations of motion for all constrained mechanical control systems, be it holonomic or nonholonomic, in the form (4). In every case the systems evolve over the same manifold Q but they are described with different affine connections and in each case the set of input vector fields must be projected to the appropriate constrained distribution

$$\mathcal{Y} = \mathrm{span}\{P(Y_k) \mid \ Y_k = M^{-1}F^k, k = 1, \ldots, m\}.$$

E. Plastic and elastic impacts

Next, we provide a geometric interpretation of the classical treatment of impacts; see Brogliato [14]. Loosely speaking, an impact causes a switch in the equations of motions and a jump in the system's velocity. Let $(Q, M, \mathcal{F})$ be a mechanical control system, let $\mathcal{D}^-$ and $\mathcal{D}^+$ be two constraint distributions, and let $(\nabla^-, \mathcal{Y}^-)$ and $(\nabla^+, \mathcal{Y}^+)$ be the corresponding affine connections and input distributions. We say that the mechanical system undergoes an *impact* at time t if the following events occur

(i) the dynamic equations switch from $(\nabla^-, \mathcal{Y}^-)$ to $(\nabla^+, \mathcal{Y}^+)$,

(ii) the state $(q, \dot{q})$ undergoes a discontinuous change in velocity described by a linear map $J_q : T_qQ \to T_qQ$. Formally, the linear map is a tensor field $J : TQ \to TQ$, such that for all $q \in Q$, $J_q : T_qQ \to T_qQ$. Denoting $q(t^-)$ and $q(t^+)$ as the limiting processes $\lim_{s\to t^-} q(s)$ and $\lim_{s\to t^+} q(s)$, we write:

$$q(t^+) = q(t^-)$$
$$\dot{q}(t^+) = J_q \cdot \dot{q}(t^-).$$

This definition generalizes the classic notions of plastic and elastic impacts. For example, if a particle hits a surface at the configuration q_0 with nonzero velocity, then the linear operator J_{q_0} annihilates the normal component of the velocity in the plastic impact case and reverses it in the elastic impact case (a coefficient of restitution e is used to account for energy dissipation). Formally, we define:

Plastic impact: The two constraint distributions $\mathcal{D}^-$ and $\mathcal{D}^+$ are distinct (for example $\mathcal{D}^- = TQ$ and $\mathcal{D}^+ = TR$ is the tangent space of a submanifold $R \subset Q$). The operator $J_q = P_{\mathcal{D}^+}$ is the orthogonal projection from T_qQ onto $\mathcal{D}^+$.

Elastic impact: The equations of motion do not change, as the connections and the input distributions do not change. There exists a distribution $\mathcal{D}$ such that

$$J_q = P_{\mathcal{D}} + (-e)P_{\mathcal{D}}^{\perp},$$

where $P_{\mathcal{D}}$ is the orthogonal projection from T_qQ onto $\mathcal{D}^+(q)$ and $0 < e \leq 1$ is the coefficient of restitution.

Finally, we note that the above definition of impact applies to both holonomic and nonholonomic impacts, that is, to impacts that possibly involve either holonomic or nonholonomic or both types of constraints. This is an important advantage of the geometric framework we present.

F. Hybrid mechanical control systems

Finally, we introduce a special class of hybrid systems by merging the notion of "control systems on manifolds with an affine connection," see [6], [11], and that of "controlled general hybrid dynamical system," see [28].

The fundamental discrete phenomena we model are controlled switches between distinct sets of constraints, resulting in impacts. The underlying structure is a mechanical control system $(Q, M, \mathcal{F})$ together with a given set of constraint distributions $\mathcal{D}_i$, where i belongs to an index set I. Each constraint $\mathcal{D}_i$ gives rise to a different regime in which

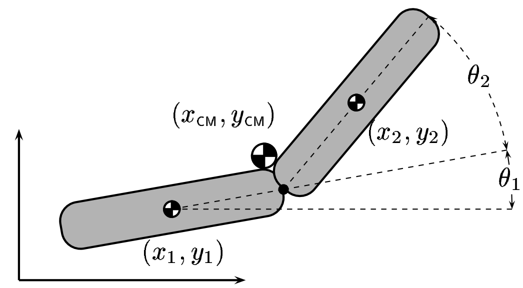

Fig. 1. A planar mechanism with two links. The mechanism is in one of the following three regimes: sliding, clamped (the position and orientation of the first link is clamped to the ground), and rolling (a wheel parallel to the first link and located at the center of mass of the link is placed on the ground).

the mechanism can operate. A regime will be also called a *discrete state* and is formally modeled by a constrained mechanical control system $\Sigma_i = [Q, M, \mathcal{F}, \mathcal{D}_i, U]$, with associated affine connection ∇_i and input distribution $\mathcal{Y}_i$. For simplicity we assume that the system can switch from a discrete state to any other discrete state, but additional switching structure could be added. Formally, we define the *hybrid mechanical control system* as

$$\text{HMCS} = [I, Q, \Sigma_Q, \Delta] \tag{11}$$

where:

(i) I is the *index set* of constraints,

(ii) Q is the n-dimensional *configuration manifold*,

(iii) $\Sigma_Q = \{\Sigma_i = [Q, M, \mathcal{F}, \mathcal{D}_i, U]\}_{i\in I}$ is the collection of constrained mechanical control systems on Q,

(iv) $\Delta = \{\delta_i | i \in I\}$ is the set of *jump transition maps*, where $\delta_{iq} : T_qQ \to D_i(q)$, and $\delta_i(q, \dot{q}) = (q, (J_q)_i \cdot \dot{q})$.

The evolution of a hybrid mechanical control system can be described as follows. The system starts in a state $(i, (q_0, \dot{q}_0)) \in I \times TQ$ and it evolves according to the dynamics given by ∇_i and the chosen set of controls. At any point, we can choose to switch to any other discrete state. In general, the switch results in an impact and a change in the velocity. If we require that we can only perform a finite number of impacts in a finite time interval, we can guarantee the existence and uniqueness of the solution of the equations of motion. Since impacts are under our control, this is not a very restrictive assumption.

III. A planar mechanisms subject to switching constraints

We present a simple example that illustrates the concepts in the previous section and motivates the following sections. We consider a planar mechanism moving in a horizontal plane (no gravity) consisting of two homogeneous links of equal length 2ℓ, of unit density, width and depth; see Figure 1. The configuration manifold of the two body system is $Q = \mathbb{R}^2 \times \mathbb{T}^2$, with a configuration $q = (x_{\mathsf{CM}}, y_{\mathsf{CM}}, \theta_1, \theta_2)$. The variables $(x_{\mathsf{CM}}, y_{\mathsf{CM}})$ are the coordinates of the center of mass, θ_1 denotes the angle of the first link with respect to the horizontal axis and θ_2 is the relative angle between the first and second link. Both angles are measured counterclockwise.

The inertia matrix M of the linkage is

$$\frac{1}{3}\begin{bmatrix} 12\ell & 0 & 0 & 0 \\ 0 & 12\ell & 0 & 0 \\ 0 & 0 & 2\ell^3(5+3\cos\theta_2) & \ell^3(5+3\cos\theta_2) \\ 0 & 0 & \ell^3(5+3\cos\theta_2) & 5\ell^3 \end{bmatrix}.$$

The control input is a torque applied to the internal joint, that is, the input codistribution is $\mathcal{F} = \operatorname{span}\{\mathrm{d}\theta_2\}$.

A. Three constrained mechanical controlled systems

We assume that at any location and at any time, the system can operate in and switch between any of the three following regimes:

Sliding regime $(i = 0)$ This regime corresponds to the mechanism that slides without friction on the horizontal plane; we denote this regime with the index 0. The constraint distribution is $\mathcal{D}_0(q) = T_qQ$.

Clamped regime $(i = 1)$ The system's first link is clamped to the ground at some point and orientation $(x_{10}, y_{10}, \theta_{10})$. The coordinates of the center of mass of the first link are given by the kinematic relationships

$$\begin{aligned} x_1 &= x_{\mathsf{CM}} - \frac{\ell}{2}(\cos(\theta_1) + \cos(\theta_1+\theta_2)) \\ y_1 &= y_{\mathsf{CM}} - \frac{\ell}{2}(\sin(\theta_1) + \sin(\theta_1+\theta_2)). \end{aligned} \tag{12}$$

The constraint map is therefore $\varphi(x_{\mathsf{CM}}, y_{\mathsf{CM}}, \theta_1, \theta_2) = (x_1, y_1, \theta_1)$. This holonomic constraint induces a one dimensional constraint distribution $\mathcal{D}_1(q)$. Feasible velocities will be aligned with

$$\ell\sin(\theta_1+\theta_2)\frac{\partial}{\partial x_{\mathsf{CM}}} - \ell\cos(\theta_1+\theta_2)\frac{\partial}{\partial y_{\mathsf{CM}}} - 2\frac{\partial}{\partial\theta_2}.$$

Rolling regime $(i = 2)$ The system has two wheels in contact with the ground; the wheels are located at the center of mass of the first and second link. The wheels prevent the links from sliding sideways:

$$\begin{aligned} \dot{x}_1\sin\theta_1 - \dot{y}_1\cos\theta_1 &= 0 \\ \dot{x}_2\sin(\theta_1+\theta_2) - \dot{y}_2\cos(\theta_1+\theta_2) &= 0 \end{aligned}$$

where x_1, y_1 are given by (12), and x_2, y_2 satisfy similar relationships. In the coordinates $q = (x_{\mathsf{CM}}, y_{\mathsf{CM}}, \theta_1, \theta_2)$, the constraints read

$$\begin{aligned} \begin{bmatrix} 2\sin\theta_1 & -2\cos\theta_1 & \ell(1+\cos\theta_2) & \ell\cos\theta_2 \end{bmatrix} \cdot \dot{q} &= 0 \\ \begin{bmatrix} -2\sin(\theta_1+\theta_2) & 2\cos(\theta_1+\theta_2) & \ell(1+\cos\theta_2) & \ell \end{bmatrix} \cdot \dot{q} &= 0. \end{aligned}$$

It can be verified that the distribution of admissible velocities $\mathcal{D}_2(q)$ is generated by the two vector fields

$$\ell(1+c_2)(c_1+c_{12})\frac{\partial}{\partial x_{\mathsf{CM}}} + \ell(1+c_2)(s_1+s_{12})\frac{\partial}{\partial y_{\mathsf{CM}}} + 2s_2\frac{\partial}{\partial\theta_1}$$
$$\ell(c_1+c_2c_{12})\frac{\partial}{\partial x_{\mathsf{CM}}} + \ell(s_1+c_2s_{12})\frac{\partial}{\partial y_{\mathsf{CM}}} + 2s_2\frac{\partial}{\partial\theta_2},$$

where we introduce the shorthands

$$c_i = \cos\theta_i, \quad s_i = \sin\theta_i, \quad c_{ij} = \cos(\theta_i+\theta_j), \quad \text{etc.}$$

This concludes the definition of three distinct constrained mechanical control systems $\Sigma_i = [Q, M, \mathcal{F}, \mathcal{D}_i]$, for $i \in \{0, 1, 2\}$. The transitions from any regime to any other regime are assumed to be ideal plastic impacts.

B. The dynamics of the hybrid mechanical control system

In this section we describe the dynamics in the three regimes, i.e., we compute (i) the affine connection ∇_0 in the unconstrained regime, (ii) the transition maps δ_1 and δ_2 corresponding to the projection maps $P_{\mathcal{D}_1}$ and $P_{\mathcal{D}_2}$ onto the constraint distributions $\mathcal{D}_1$ and $\mathcal{D}_2$, respectively, and (iii) the input vector field in the three regimes[1]. These quantities completely determine the equations of motion in all three regimes since the constrained connections ∇_1 and ∇_2 can be evaluated through equation (10).

The affine connection and the input distribution $(\nabla_0, \mathcal{Y}_0)$ are computed via a direct application of equations (1). The only non-vanishing Christoffel symbols for the un-clamped regime are

$$\begin{aligned} \Gamma^3_{33}(q) = \Gamma^3_{34}(q) = \Gamma^3_{43}(q) &= \frac{30\sin\theta_2}{-41+9\cos(2\theta_2)}, \\ \Gamma^3_{44}(q) = \frac{3\sin\theta_2}{-5+3\cos\theta_2} \quad \Gamma^4_{33}(q) &= \frac{6\sin\theta_2}{5-3\cos\theta_2}, \\ \Gamma^4_{44}(q) = \Gamma^4_{34}(q) = \Gamma^4_{43}(q) &= \frac{3\sin\theta_2}{5-3\cos\theta_2}. \end{aligned}$$

Using Gram-Schmidt decomposition, the orthogonal projection $P_{\mathcal{D}_1}$ onto $\mathcal{D}_1$ is

$$\frac{1}{16\ell}\begin{bmatrix} 6\ell s_{12}^2 & -6\ell c_{12}s_{12} & -5\ell^2 s_{12} - 3\ell^2 c_2 s_{12} & -5\ell^2 s_{12} \\ -6\ell c_{12}s_{12} & 6\ell c_{12}^2 & 5\ell^2 c_{12} + 3\ell^2 c_2 c_{12} & 5\ell^2 c_{12} \\ 0 & 0 & 0 & 0 \\ -12s_{12} & 12c_{12} & 10\ell + 6\ell c_2 & 10\ell \end{bmatrix},$$

and the orthogonal projection $P_{\mathcal{D}_2}$ onto $\mathcal{D}_2$ is

$$\frac{1}{16\ell d_1 d_2}\left[\begin{matrix} 6\ell d_1(8(9c_2 - c_{222})c_{112}) & 6\ell d_1(c_{222} - 9c_2)s_{112} \\ 6\ell d_1(c_{222} - 9c_2)s_{112} & 6\ell d_1((9c_2 - c_{222})c_{112} - 8) \\ 48d_1s_2(2s_1s_2 - c_1c_2) & -48d_1s_2(s_{12} + c_1s_2) \\ 24d_2s_2(c_{12} - c_1) & 24d_2s_2(s_{12} - s_1) \end{matrix}\right.$$
$$\left.\begin{matrix} 4\ell^2d_1^2(5+3c_2)(c_1+c_{12})s_2 & \ell^2d_1(3(c_1c_{22} + s_1s_{22}) + c_{122} - 36c_1)s_2 \\ 4\ell^2d_1^2(5+3c_2)s_2(s_1+s_{12}) & \ell^2d_1s_2(3\sin(\theta_1 - 2\theta_2) + s_{112} - 36s_1) \\ 16\ell d_1^2(5+3c_2)\sin(\theta_2/2)^2 & 6\ell d_1(9c_2 - c_{222}) \\ 0 & 2\ell d_2(3c_{22} - 7 - 4c_2) \end{matrix}\right],$$

where we let $d_1 = c_2 - 2$ and $d_2 = c_{22} - 7$.

Finally, we compute the input distributions in the three constrained regimes. The input distribution $\mathcal{Y}_0$ is generated by the vector field

$$Y_1^0 = M^{-1}\mathrm{d}\theta_2 = \frac{3}{\ell^3(3\cos\theta_2 - 5)}\left(\frac{\partial}{\partial\theta_1} - 2\frac{\partial}{\partial\theta_2}\right).$$

[1]All the computations were performed in Mathematica. The code can be obtained at `http://motion.csl.uiuc.edu/~bullo/math`.

The input distribution $\mathcal{Y}_1$ is generated by the vector field

$$Y_1^1 = P_{\mathcal{D}_1} Y_1^0 = \frac{1}{16\ell^3}\left(-3\ell\sin(\theta_1+\theta_2)\frac{\partial}{\partial x_{\mathsf{CM}}} +3\ell\cos(\theta_1+\theta_2)\frac{\partial}{\partial y_{\mathsf{CM}}}+6\frac{\partial}{\partial\theta_2}\right).$$

The input distribution $\mathcal{Y}_2$ is generated by

$$Y_1^2 = P_{\mathcal{D}_2} Y_1^0 = \frac{3}{8\ell^3(c_2-2)}\left(\ell s_2(c_{12}-c_1)\frac{\partial}{\partial x_{\mathsf{CM}}} + \ell s_2(s_{12}-s_1)\frac{\partial}{\partial y_{\mathsf{CM}}}+2(1+c_2)\frac{\partial}{\partial\theta_1}-4(1+c_2)\frac{\partial}{\partial\theta_2}\right).$$

IV. Equilibrium and kinematic controllability

In this section we investigate the controllability properties of hybrid mechanical control systems defined as in Section II-F, that is, systems that can switch between the constrained regimes at an arbitrary point in the configuration space. We provide sufficient controllability conditions that rely on the assumption of impact at zero velocity. (In the terminology of [28], we assume the system undergoes controlled switches rather than controlled jumps).

For mechanical devices, it is important to distinguish between the system's evolution on the full configuration manifold TQ (in coordinates, the space $(q,\dot q)$) and its evolution when just observed on Q. For example, when studying locomotion systems subject to velocity constraints we are primarily interested in how the configuration changed, not with what velocity the system moved. This is the motivation behind the notions of configuration, equilibrium, and kinematic controllability [6], [9] which we now review.

A. Preliminary definitions

We review some basic definitions and notations. Given a pair of smooth vector fields X, Y on Q, we shall perform two operations on them, Lie bracket and symmetric product:

$$[X,Y] = \nabla_X Y - \nabla_Y X, \quad \langle X : Y\rangle = \nabla_X Y + \nabla_Y X.$$

Corresponding to these operations between pairs of vector fields, we introduce two operations on a family of vector fields $\mathcal{X} = \{X_1,\dots,X_m\}$. We let $\overline{\mathrm{Lie}}(\mathcal{X})$ be the closure of $\mathcal{X}$ under the Lie bracket operation (the involutive closure), and we let $\overline{\mathrm{Sym}}(\mathcal{X})$ be the closure of $\mathcal{X}$ under the symmetric product operation. Within the set $\overline{\mathrm{Sym}}(\mathcal{X})$, we define the *order* of a symmetric product to be the number of vector fields X_j present in it. We say that a symmetric product is *bad* if it contains an even number of each X_i. Otherwise the product is said to be *good*. It is worth noting that these definitions can be stated more accurately via the notion of free Lie algebra; see [15], [6].

B. Equilibrium and kinematic controllability for smooth mechanical control systems

We can now present equilibrium controllability definitions and tests from [6] and kinematic controllability definitions and tests from [9]; see also [8], [29], [30] for extensions and related works. In what follows, we consider a smooth constrained mechanical control system $\Sigma = [Q, M, \mathcal{F}, \mathcal{D}, U]$ with the associated connection and input distribution $(\nabla, \mathcal{Y})$. The equations of motion are

$$\nabla_{\dot q}\dot q = Y_k u^k(t), \tag{13}$$

where $\{Y_1,\dots,Y_m\}$ is a base for $\mathcal{Y}$. Let q_0 be a point in Q and let W be a neighborhood of q_0.

B.1 Equilibrium controllability

The set of configurations reachable from $q_0 \in W \subset Q$ starting at zero initial velocity is defined as

$$\mathcal{R}_Q^W(q_0,\le T) = \cup_{t\le T}\{x\in Q \mid \exists \text{ a solution to (13) s.t. } \dot q(0)=0,\ q(t)\in W \text{ for } t\in[0,T], \text{ and } q(T)=x\}.$$

The system (13) is *small-time locally configuration controllable* at q_0 if there exists a time T such that the reachable set $\mathcal{R}_Q^W(q_0,\le T)$ contains a non-empty open subset of Q containing q_0. The system (13) is *equilibrium controllable* on $W\subset Q$, if, for $q_{\text{initial}}, q_{\text{final}} \in W$, there exist an input $\{u^k(t), t\in[0,T]\}$ and a solution $\{q(t), t\in[0,T]\}$ such that $q(0)=q_{\text{initial}}$, $q(T)=q_{\text{final}}$, $q(t)\in W$ for all $t\in[0,T]$, and $\dot q(0)=0$, $\dot q(T)=0$.

The following results from [6] characterize $\mathcal{R}_Q^W(q_0,\le T)$ and provide sufficient tests for both controllability notions.

Lemma IV.1: The reachable set $\mathcal{R}_Q^W(q_0,\le T)$ forms an open subset of the integral manifold through q_0 of the distribution $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathcal{Y}))$.

Lemma IV.2: If the distribution $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathcal{Y}))$ is full rank at q_0, and if every bad symmetric product at q_0 is a linear combination of lower order good symmetric products, then the system (13) is small-time locally configuration controllable at q_0.

Lemma IV.3: If both assumptions in Lemma IV.2 are verified at every $q\in W$, then the system (13) is equilibrium controllable on W.

Roughly speaking, the symmetric closure of the input vector fields describes what velocities are reachable, while the involutive closure describes what configurations are reachable.

B.2 Kinematic controllability

A vector field V on Q is a *decoupling vector field* if its integral curves may be followed, with an arbitrary reparameterization, by controlled trajectories for the system (13). The system (13) is *kinematically controllable* on $W\subset Q$ if any two configurations in W can be connected via a sequence of integral curves of decoupling vector fields. Roughly speaking, a kinematically controllable system has the following property: it is possible to design feasible motion plans using concatenations of integral curves of the decoupling vector fields. The resulting concatenated curve, when reparameterized so that each segment begins and ends with zero velocity, is guaranteed to be a controlled trajectory for the mechanical system.

The following results from [9] characterize decoupling vector fields and kinematically controllable systems.

Lemma IV.4: The vector field V is a decoupling vector field if and only if both V and $\nabla_V V$ lie in the input distribution $\mathcal{Y}$.

Lemma IV.5: The system (13) is kinematically controllable on W if it possesses decoupling vector fields $\{V_1, \ldots, V_k\}$ whose involutive closure has maximal rank at all $q \in W$.

Remark IV.6: One can see that a kinematically controllable system is also equilibrium controllable. Furthermore, it turns out that various examples of equilibrium controllable systems are also kinematically controllable, e.g., the systems discussed in [9], [30]. The following system is an example of an equilibrium controllable mechanical system which is not kinematically controllable:

$$\ddot{x}_1 = u_1, \;\; \ddot{x}_2 = u_2, \;\; \ddot{x}_3 = \dot{x}_1 \dot{x}_2.$$

□

C. Equilibrium and kinematic controllability for hybrid mechanical control systems

We now analyse the controllability properties for the hybrid mechanical control system HMCS $= [I, Q, \Sigma_Q, \Delta]$ defined in Section II-F. Recall that $(\nabla_i, \mathcal{Y}_i)$ are the connection and the input distribution for the ith regime. We let $\mathcal{Y}_i = \operatorname{span}\{Y_1^i, \ldots, Y_m^i\}$ be the input distribution, $\langle \cdot : \cdot \rangle_i$ be the symmetric product, and $\overline{\mathrm{Sym}}_i(\cdot)$ be the symmetric closure for the ith regime.

We start by noting that the definition of equilibrium controllability for a smooth system relies only on the properties of the solutions to the equations of motion. Since these solutions are well-defined for hybrid mechanical control systems, the definition of equilibrium controllability is also applicable to the hybrid setting. Next, we define decoupling vector fields for a hybrid system. A vector field V on Q is decoupling for the hybrid mechanical control system $[I, Q, \Sigma_Q, \Delta]$ if it is decoupling for some smooth regime i, that is, if for some i its integral curves may be followed, with an arbitrary reparametrization, by controlled trajectories for the system $(\nabla_i, \mathcal{Y}_i)$. As in the smooth case, the hybrid mechanical control system is kinematically controllable on $W \subset Q$ if any two configurations in W can be connected via a sequence of integral curves of decoupling vector fields. As in the smooth case, a kinematically controllable system is also equilibrium controllable.

We can now state the main results in this section.

Theorem IV.7: The hybrid mechanical control system (11) is equilibrium controllable on an open set W if
(i) in each discrete state i, every bad symmetric product is a linear combination of lower order good symmetric products, and
(ii) the distribution $\overline{\mathrm{Lie}}(\sum_{i\in I} \overline{\mathrm{Sym}}_i(\mathcal{Y}_i))(q)$ is full rank at every $q \in W$.

Theorem IV.8: The hybrid mechanical control system (11) is kinematically controllable on an open set W if it possesses decoupling vector fields $\{V_1, \ldots, V_k\}$ whose involutive closure has maximal rank at all $q \in W$.

Proof of Theorem IV.7: We start by examining the set of configurations reachable at zero velocity for the ith regime. For any point $q_0 \in W$, consider the distribution $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}_i(\mathcal{Y}_i))$, and let $N_i \subset Q$ be its maximal integral manifold through the point q_0. Because of Lemma IV.1 the trajectories of the ith mechanical system starting from point q_0 at zero velocity are constrained to remain on N_i. Additionally, because each bad symmetric product is compensated by a lower order, good symmetric product, each configuration on $N_i \cap W$ is reachable at zero velocity (see Lemma IV.2). In other words, the set of configurations that can be reached starting and finishing at zero velocity for the control system

$$(\nabla_i)_{\dot{q}} \dot{q} = Y_1^i u_1^i + \ldots + Y_m^i u_m^i, \tag{14}$$

is equal to the set of configurations that can be reached for the control system

$$\dot{q} = X_1^i v_1^i + \ldots + X_{p_i}^i v_{p_i}^i, \tag{15}$$

where the family of vector fields $\{X_1^i, \ldots, X_{p_i}^i\}$ generate the distribution $\overline{\mathrm{Sym}}_i(\mathcal{Y}_i) = \overline{\mathrm{Sym}}_i(Y_1^i, \ldots, Y_m^i)$.

Next, we consider the first order control system

$$\dot{q} = \sum_{i \in I} (X_1^i v_1^i + \ldots + X_{p_i}^i v_{p_i}^i), \tag{16}$$

where vector fields from all regimes $i \in I$ are present. By assumption (ii), the system in equation (16) satisfies the Lie algebra rank condition and it is therefore locally controllable at every point $q \in W$. According to Proposition 2.3 in [15], for any pair $q_{\text{initial}}, q_{\text{final}} \in W$, there exist a sequence of vector fields $\{X_k, k \in \{1, \ldots, N\}\}$ belonging to $\cup_{i\in I}\{X_1^i, \ldots, X_{p_i}^i, -X_1^i, \ldots, -X_{p_i}^i\}$ and positive numbers $\{\epsilon_k, k \in \{1, \ldots, N\}\}$ such that

$$q_{\text{final}} = \Phi_{\epsilon_N}^{X_N} \circ \cdots \circ \Phi_{\epsilon_1}^{X_1}(q_{\text{initial}}),$$

where $\Phi_\epsilon^X(q)$ denotes the flow along the vector field X for time ϵ starting from point q.

Finally, we show how to construct a sequence of inputs to steer the second order control system in equation (14) from the configuration q_{initial} at zero velocity to the configuration q_{final} at zero velocity. We start by defining $q_k = \Phi_{\epsilon_k}^{X_k} \circ \cdots \circ \Phi_{\epsilon_1}^{X_1}(q_{\text{initial}})$. Observe that $q_k = \Phi_{\epsilon_k}^{X_k}(q_{k-1})$, and that there exist indices $i_k \in I$ and $j \in \{1, \ldots, p_{i_k}\}$ such that $X_k = X_j^{i_k}$ or $X_k = -X_j^{i_k}$. Therefore, the configuration $q_k \in W$ is reachable from the configuration $q_{k-1} \in W$ for the control system in equation (15) with $i = i_k$. Following our earlier argument leading to equation (15), the configuration $q_k \in W$ is reachable from the configuration $q_{k-1} \in W$ for the control system in equation (14) with $i = i_k$ starting and finishing at zero velocity. Since the system can switch between regimes at arbitrary points, we steer the second order control system in equation (14) from q_{initial} at zero velocity to q_{final} at zero velocity by switching between regimes i_k and i_{k+1} at configuration q_k and zero velocity. ■

Proof of Theorem IV.8: The argument follows the same three steps as the previous proof. We start by considering the set of configurations reachable for the ith regime via a sequence of integral curves of decoupling vector fields. This set is the maximal integral manifold N_i of the family of decoupling vector fields $\{V_1^i, \dots, V_{k_i}^i\}$ for regime i. In other words, the set of configurations that can be reached along integral curves of decoupling vector fields for the control system (14) is equal to the set of configurations that can be reached for the control system

$$\dot{q} = V_1^i v_1^i + \cdots + V_{k_i}^i v_{k_i}^i, \qquad (17)$$

where the input magnitudes $\{v_1^i, \dots, v_{k_i}^i\}$ take value in the discrete set $\{\{1, 0, \dots, 0\}, \dots, \{0, \dots, 0, 1\}\}$. The rest of the proof is identical to the previous proof of Theorem IV.7. Equation (17) plays the same role as equation (15), the decoupling vector fields V_j^i play the same role as the vector fields X_j^i above, invoking Proposition 2.3 in [15] we can design a sequence of integral curves connecting the required configurations, and the final argument is unchanged. ■

Remark IV.9: The two conditions for equilibrium controllability have the following interpretation. Condition (i) is the functional equivalent of the bad versus good Lie bracket condition in Lemma IV.2 and guarantees that the system is controllable (as opposed to accessible) when restricted to the maximal integral manifold of $\overline{\text{Lie}}(\overline{\text{Sym}}_i(\mathcal{Y}_i))$. Condition (ii) guarantees that by combining subsequent motions feasible in different regimes, an open neighborhood of the initial point is accessible. □

Remark IV.10: Both theorems provide sufficient tests for controllability assuming the velocity at impact is zero. Roughly speaking, this assumption leads to a decoupling of the regimes: the system evolves in one regime, stops, switches to another regime and starts again. By exploring the equilibrium controllability property (i.e., the fact that any configuration can be reached at zero velocity), we reduce the computation of the reachable configurations for a second order control system (with drift) to those for a first order kinematic control system (without drift).

Requiring zero velocity impacts is a restrictive assumption. Nonetheless, this assumption is closely related to the notions of equilibrium and kinematic controllability, and leads to tests that are powerful enough for the planar hybrid mechanism. How to provide general conditions involving impacts at non-zero velocity remains an open problem at this time; see some initial contributions in [31]. □

D. Controllability results for the hybrid planar mechanism

The controllability results in the previous subsections are directly applicable to the planar sliding, clamped, and rolling mechanism. In this section we discuss how each individual regime of the planar mechanism is neither kinematically nor equilibrium controllable, and how most mechanisms that can switch between two or more regimes are kinematically and equilibrium controllable.

Let us start by examining the smooth mechanical control systems corresponding to the three individual regimes for the hybrid planar mechanism. We compute the symmetric closure of the input distribution in the three regimes. Let $Y^i = Y_1^i$ denote the input vector field in the ith regime, and compute

$$\begin{aligned} \langle Y^0 : Y^0 \rangle_0 &= \frac{-9 \sin\theta_2}{\ell^3(-5 + 3\cos\theta_2)^2} Y^0, \\ \langle Y^1 : Y^1 \rangle_1 &= 0, \\ \langle Y^2 : Y^2 \rangle_2 &= \frac{9 \sin\theta_2}{4\ell^3(\cos\theta_2 - 2)^2} Y^2. \end{aligned} \qquad (18)$$

From these computations, we draw the following conclusions for all three individual regimes (i.e., for all $i \in \{0, 1, 2\}$):
(i) $\overline{\text{Sym}}_i(\mathcal{Y}_i) = \mathcal{Y}_i$, so that all (good and bad) symmetric products are linear combination of first order symmetric products,
(ii) because of the obvious equalities $\langle Y^i : Y^i \rangle_i = 2(\nabla_i)_{Y^i} Y^i$, the control vector fields Y^i are decoupling in their respective regimes,
(iii) let $N_i(q_0)$ be the integral manifold through $q_0 \in Q$ of the distribution $\mathcal{Y}_i$, i.e., the image of the integral curve of the control vector field Y^i through q_0. According to Lemma IV.1, the evolution of the system in regime i starting from rest at q_0 is confined to the 1-dimensional submanifold N_i, and
(iv) the mechanical system defined as the restriction of the ith regime to the N_i submanifold is kinematically controllable and therefore equilibrium controllable.

In summary, the three individual regimes evolve along 1-dimensional submanifolds, over which they are kinematically controllable. For example, the clamped regime is kinematically controllable over a configuration space consisting of the only variable $\theta_2 \in \mathbb{T}$. However, the three individual regimes are *neither* equilibrium *nor* kinematically controllable over the full configuration domain $Q = \mathbb{R}^2 \times \mathbb{T}^2$.

Remark IV.11: A smooth mechanical systems with a single input for which $\overline{\text{Sym}}_i(\mathcal{Y}_i) \neq \mathcal{Y}_i$ is the roller racer; see [32], [11], [27]. Similarly to the planar mechanism in the rolling regime, the roller racer is a two-link planar mechanism endowed with two wheels, the difference being that the two links are assumed to have different length and mass. It is known [32] that the set of reachable configurations for the roller racer is an open subset of $Q = \mathbb{R}^2 \times \mathbb{T}^2$. □

Since the planar mechanism is not equilibrium or kinematically controllable in any of his smooth regimes (over the full configuration domain $Q = \mathbb{R}^2 \times \mathbb{T}^2$), we set out to investigate controllability for the hybrid systems that can switch between any of these regimes. By combining the three regimes in all possible manners $\{\{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$, we examine four hybrid mechanical control systems. According to Theorem IV.7, we need to look at the following Lie brackets computations.

- For the hybrid mechanical control system $\{\Sigma_0, \Sigma_1\}$,

$$\begin{aligned} &\text{rank}\left\{\overline{\text{Lie}}(Y^0, Y^1)(q)\right\} \\ &\qquad = \text{rank}\left\{Y^0, Y^1, [Y^0, Y^1], [Y^0, [Y^0, Y^1]]\right\}(q) = 4, \end{aligned}$$

for all $(x_{\mathsf{CM}}, y_{\mathsf{CM}}, \theta_1, \theta_2)$.

- For the hybrid mechanical control system $\{\Sigma_1, \Sigma_2\}$

$$\begin{aligned}\operatorname{rank}\left\{\overline{\mathrm{Lie}}(Y^1, Y^2)(q)\right\} \\ = \operatorname{rank}\left\{Y^1, Y^2, [Y^1, Y^2], [Y^2, [Y^1, Y^2]]\right\}(q) = 4,\end{aligned}$$

for all $(x_{\mathsf{CM}}, y_{\mathsf{CM}}, \theta_1, \theta_2)$ such that $\theta_2 \neq \pm\pi$.

- For the hybrid mechanical control system $\{\Sigma_0, \Sigma_2\}$

$$\operatorname{rank}\left\{\overline{\mathrm{Lie}}(Y^0, Y^2)(q)\right\} = 2.$$

- No further computations are necessary to analyze the hybrid mechanical control system $\{\Sigma_0, \Sigma_1, \Sigma_2\}$.

Theorem IV.7 leads to the following statements. First, the hybrid mechanical control systems $\Sigma = \{\Sigma_0, \Sigma_1\}$, $\Sigma = \{\Sigma_1, \Sigma_2\}$ and $\Sigma = \{\Sigma_0, \Sigma_1, \Sigma_2\}$ are kinematic and equilibrium controllable. These three systems correspond to the hybrid devices: “slide and clamp,” “clamp and roll,” “slide and clamp and roll.” Second, the system $\Sigma = \{\Sigma_0, \Sigma_2\}$, corresponding to “slide and roll,” does not satisfy conditions of either theorem. Since the theorems only provide sufficient conditions, we are unable to provide a conclusive answer as to whether the system is kinematic or equilibrium controllable.

E. *Application example to motion planning*

This section illustrates an application of the modeling and controllability results above: for the hybrid planar device we devise a motion planning algorithm based on the notion of kinematic controllability. We consider for simplicity the hybrid system consisting of regimes $\{0, 1\}$, i.e., the planar device that can slide or clamp. The algorithm development is parallel to that for the smooth setting; see [9].

E.1 Integral curves of decoupling vector fields

First, we investigate the decoupling vector fields for regime 0 and regime 1, and obtain closed form expressions for their integral curves. We start by rescaling the two input vector fields Y^0 and Y^1, and redefining:

$$Y^0 = \frac{\partial}{\partial\theta_1} - 2\frac{\partial}{\partial\theta_2},$$
$$Y^1 = \frac{\ell}{2}\left(-\sin(\theta_1+\theta_2)\frac{\partial}{\partial x_{\mathsf{CM}}} + \cos(\theta_1+\theta_2)\frac{\partial}{\partial y_{\mathsf{CM}}}\right) + \frac{\partial}{\partial\theta_2}.$$

The flow along Y^0 for time Δt is

$$\Phi^{Y^0}_{\Delta t}(x_{\mathsf{CM}0}, y_{\mathsf{CM}0}, \theta_{10}, \theta_{20}) = (x_{\mathsf{CM}0}, y_{\mathsf{CM}0}, \theta_{10}+\Delta t, \theta_{20}-2\Delta t),$$

and along Y^1 for time Δt is

$$\begin{aligned}&\Phi^{Y^1}_{\Delta t}(x_{\mathsf{CM}0}, y_{\mathsf{CM}0}, \theta_{10}, \theta_{20}) = \\ &\quad\Big(x_{\mathsf{CM}0} + \frac{\ell}{2}\big(\cos(\Delta t+\theta_{10}+\theta_{20}) - \cos(\theta_{10}+\theta_{20})\big), \\ &\quad y_{\mathsf{CM}0} + \frac{\ell}{2}\big(\sin(\Delta t+\theta_{10}+\theta_{20}) - \sin(\theta_{10}+\theta_{20})\big), \theta_{10}, \theta_{20}+\Delta t\Big).\end{aligned}$$

E.2 Motion primitives for x_{CM}, y_{CM}, or θ_1 displacements

Next, we concatenate integral curves of the vector fields Y^0 and Y^1 to design the useful combinations which we refer to as *motion primitives*. We assume $\theta_2 = 0$ at beginning and end of each primitive, and design motion plans to steer the variables $(x_{\mathsf{CM}}, y_{\mathsf{CM}}, \theta_1)$ from $(0,0,0)$ to desired values.

`Primitive 1:` To rotate the device an angle $\Delta\theta_1$, we perform the sequence of motions
1: `Flow along` $\operatorname{sign}(\Delta\theta_1)Y^0$ `for time` $|\Delta\theta_1|$
2: `Flow along` $\operatorname{sign}(\Delta\theta_1)Y^1$ `for time` $|2\Delta\theta_1|$
Accordingly, we write the equations

$$P^1_{\Delta\theta_1}(q_0) = \Phi^{Y^1}_{2\Delta\theta_1}\left(\Phi^{Y^0}_{\Delta\theta_1}(q_0)\right).$$

While obtaining the required rotation, this primitive also results in a center of mass translation in the direction $(-\sin\theta_1, \cos\theta_1)$ for the amount $\ell\sin(\Delta\theta_1)$. □

`Primitive 2:` To translate the device's center of mass along the direction $(\cos\theta_1, \sin\theta_1)$, i.e., along the body-fixed x-direction, for a positive quantity $\Delta x_{\mathsf{CM}} \in [0, 2\ell]$ we perform the sequence of motions
1: `Flow along` $-\operatorname{sign}(\alpha_x)Y^0$ `for time` $|\alpha_x|$
2: `Flow along` $-\operatorname{sign}(\alpha_x)Y^1$ `for time` $|\alpha_x|$
3: `Flow along` $\operatorname{sign}(\alpha_x)Y^0$ `for time` $|\alpha_x|$
4: `Flow along` $\operatorname{sign}(\alpha_x)Y^1$ `for time` $|\alpha_x|$
where $\alpha_x = \arccos(1 - \Delta x_{\mathsf{CM}}/\ell)$. Accordingly, we write the equations

$$P^2_{\Delta x_{\mathsf{CM}}>0}(q_0) = \Phi^{Y^1}_{\alpha_x}\left(\Phi^{Y^0}_{\alpha_x}\left(\Phi^{Y^1}_{-\alpha_x}\left(\Phi^{Y^0}_{-\alpha_x}(q_0)\right)\right)\right).$$

To obtain a final negative displacement in the amount $\Delta x_{\mathsf{CM}} \in [-2\ell, 0]$, we employ the opposite order

$$P^2_{\Delta x_{\mathsf{CM}}<0}(q_0) = \Phi^{Y^0}_{\alpha_x}\left(\Phi^{Y^1}_{\alpha_x}\left(\Phi^{Y^0}_{-\alpha_x}\left(\Phi^{Y^1}_{-\alpha_x}(q_0)\right)\right)\right),$$

where $\alpha_x = \arccos(1 + \Delta x_{\mathsf{CM}}/\ell)$. To obtain translations of amount $|\Delta x_{\mathsf{CM}}| > 2\ell$, we iterate the primitive a number of times equal to the smallest integer greater than or equal to $|\Delta x_{\mathsf{CM}}|/2\ell$. □

`Primitive 3:` To translate the device's center of mass along the direction $(-\sin\theta_1, \cos\theta_1)$, i.e., along the body-fixed y-direction, for a quantity $\Delta y_{\mathsf{CM}} \in [-3\sqrt{3}/2\ell, 3\sqrt{3}/2\ell]$, we perform the sequence of motions:
1: `Primitive 1 for amount` α_y
2: `Primitive 2 for amount` α_y
3: `Primitive 1 for amount` $-\alpha_y$
where α_y is the unique solution in the range $[-2\pi/3, 2\pi/3]$ to the equation $\Delta y_{\mathsf{CM}}/\ell = 2\sin(\alpha_y)(1-\cos\alpha_y)$. Accordingly, we write the equations

$$P^3_{\alpha_y}(q_0) = P^1_{-\alpha_y}\left(P^2_{\alpha_y}\left(P^1_{\alpha_y}(q_0)\right)\right).$$

While obtaining the required translation, this primitive also results in a center of mass translation in the direction $(\cos\theta_1, \sin\theta_1)$ for an amount $\ell(\cos\alpha_y - \cos 2\alpha_y)$. □

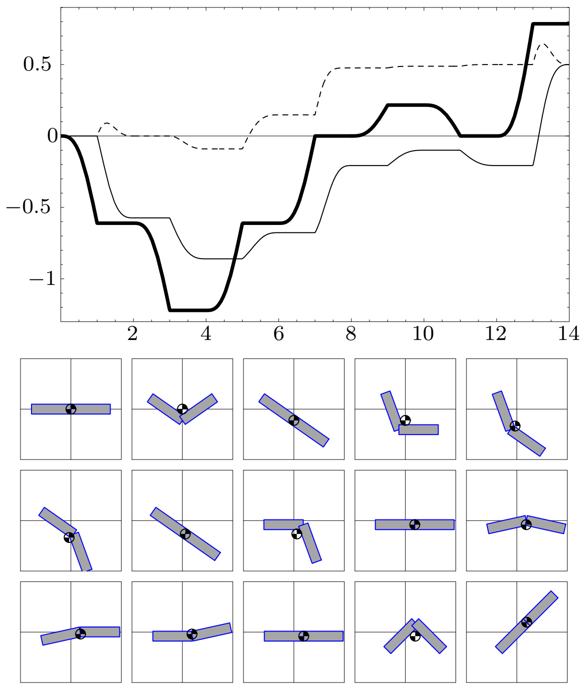

Fig. 2. Steering the sliding and clamping machine from the origin to $(x_{\mathsf{CM}}, y_{\mathsf{CM}}, \theta_1, \theta_2) = (1/2, 1/2, \pi/4, 0)$. The top figure displays the profile of θ_1 (thick line), x_{CM} (dashed line), and y_{CM} (solid line), and the bottom figure displays the machine's position at the moment of switches between different regimes. We let $\ell = 1$.

E.3 Motion planning via inversion

Finally, we combine motion primitives to reach a desired endpoint $(x_\mathrm{d}, y_\mathrm{d}, \theta_\mathrm{d}, 0)$ starting from $(x_{\mathsf{CM}}, y_{\mathsf{CM}}, \theta_1, \theta_2) = (0, 0, 0, 0)$. The key observation is that

$$(x_\mathrm{d}, y_\mathrm{d}, \theta_\mathrm{d}, 0) = P^1_{\theta_\mathrm{d}}\left(P^2_{x_\mathrm{d}-\ell(\cos(\alpha_y)-\cos(2\alpha_y))}\left(P^3_{y_\mathrm{d}-\ell\sin\theta_\mathrm{d}}(0,0,0,0)\right)\right),$$

provided $|y_\mathrm{d}/\ell - \sin\theta_\mathrm{d}| \leq 3\sqrt{3}/2$. In summary, we combine the primitives in the order and amount:

1: `Primitive 3 for amount` $y_\mathrm{d} - \ell\sin\theta_\mathrm{d}$
2: `Primitive 2 for amount` $x_\mathrm{d} - \ell(\cos(\alpha_y) - \cos(2\alpha_y))$
3: `Primitive 1 for amount` θ_d

Figure 2 illustrates a motion plan to the final configuration $(x_\mathrm{d}, y_\mathrm{d}, \theta_\mathrm{d}, 0) = (1/2, 1/2, \pi/4, 0)$. A total of 14 switches between integral curves of Y^0 and Y^1 are required, and the figure illustrates the total 15 initial and final moments of each regime.

V. Conclusions

We have presented some geometric tools for the study of hybrid mechanical control system. This class of hybrid systems has interesting features such as a very structured smooth dynamics (described by a set of affine connections) and jump transition maps linear in the velocity. We have presented a controllability test that characterizes the reachable set via zero velocity impacts. The essential technical step involves a "reduction" procedure from a dynamic to a kinematic analysis. Further research will focus on motion planning and on controllability problems with jumps at non-zero velocity; see some initial contributions in [31]. Finally, although our results are specific to Lagrangian systems, we hope to exploit the insight gained from these structured examples in more general problems.

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