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Kinematic Controllability and Decoupled Trajectory Planning for Underactuated Mechanical Systems

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Abstract

We introduce the notion of *kinematic controllability* for second-order underactuated mechanical systems. For systems satisfying this property, the problem of planning fast collision-free trajectories between zero velocity states can be decoupled into the computationally simpler problems of path planning for a kinematic system followed by time-optimal time scaling. While this approach is well known for fully actuated systems, until now there has been no way to apply it to underactuated dynamic systems. The results in this paper form the basis for efficient collision-free trajectory planning for a broad class of underactuated mechanical systems.

1 Introduction

The problem of finding the time-optimal trajectory for a fully actuated robot manipulator along a specified path is a classical one in robotics. This problem has been solved by algorithms proposed by Bobrow *et al.* [1] and Shin and McKay [2], and later enhancements due to Pfeiffer and Johanni [3], Slotine and Yang [4], and Shiller and Lu [5]. These algorithms find the minimum-time time scaling of the path which respects the actuator constraints.

With the time-scaling algorithms in hand, the problem of finding a fast collision-free trajectory for an n joint manipulator in its $2n$-dimensional state space can be decoupled into the computationally simpler problems of planning paths in the n-dimensional configuration space (considering joint limits and obstacles) followed by time-optimal time scaling according to the manipulator dynamics. Any complex geometric configuration constraints are dealt with in the first phase, irrespective of the robot dynamics. Shiller and Dubowsky [6] use this decoupling to find globally near-time-optimal trajectories for a manipulator by considering the time-optimal time scaling of a large set of candidate paths.

Unfortunately, the decoupled approach to trajectory planning does not extend in general to underactuated dynamic systems (second-order systems with fewer actuators than degrees-of-freedom). If the system has n degrees-of-freedom and m actuators ($m < n$), there are $n - m$ state-dependent equality constraints on the feasible accelerations of the system (second-order nonholonomic constraints). Examples of such systems include robot manipulators with passive joints, spacecraft, and underwater vehicles (ignoring drag). Since the acceleration constraints cannot be expressed as constraints on tangent vectors on the configuration space, meaningful path planning on the configuration space is precluded. In general, paths returned by a path planner will either be (1) infeasible for the robot, due to constraints arising from underactuation, or (2) feasibly followed at only a certain speed.

In this paper we define a class of *kinematically controllable* underactuated systems for which it is possible to decouple trajectory planning between zero velocity states. The path planner uses a set of *decoupling velocity vector fields* defined on the configuration space to find paths which can be time scaled without violating the second-order nonholonomic constraints. As a result, for this broad class of underactuated dynamic systems, we have the basis for efficient collision-free trajectory planning. This basic approach was first introduced in deriving a trajectory planner for a 3DOF robot with a passive third joint (Lynch *et al.* [7, 8]). In this paper we formalize the notions of "kinematically controllable" and "decoupling vector field" in the geometric framework of [9, 10, 11, 12], provide tests for them, and give several examples of underactuated dynamic systems which admit decoupling in trajectory planning.

1.1 Brief overview

Here we briefly summarize the major points of the paper. The dynamics of an underactuated second-order system can be written in the form

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \begin{pmatrix} \tau \\ 0 \end{pmatrix},$$

where $q \in Q = \Re^n$ is the configuration, $\tau \in \Re^m$ is the control, and there are $n-m$ second-order nonholonomic constraints due to underactuation. Consider a path of the system $q(s)$ parameterized by $s \in [0,1]$ and a time scaling $s(t)$ which assigns a point on the path for each $t \in [0,T]$. $s(t)$ is twice-differentiable and $\dot{s}(t) > 0$ for all $t \in (0,T)$. (Such a time scaling is the output of the minimum-time algorithms described above.) Then the trajectory of the system can be written $q(s(t))$, and each of the $n-m$ second-order nonholonomic constraints has the form

$$a(s)\ddot{s} + b(s)\dot{s}^2 + c(s) = 0. \tag{1}$$

The path $q(s)$ is a *kinematic motion* if the constraints are satisfied for any time scaling, i.e., $\dot{s}, \ddot{s}$ arbitrary. A velocity vector field $V(q)$ is a *decoupling vector field* if all paths $q(s)$ satisfying

$$\frac{\partial q(s)}{\partial s} = V(q)$$

are kinematic motions. The system is *locally kinematically controllable* if there exist p decoupling vector fields such that the kinematic system

$$\begin{aligned} \dot{q} &= \sum_{c=1}^{p} V_c(q) w_c \qquad (2)\\ (w_1, \dots, w_p) &\in \{(1,0,\dots,0),(0,1,0,\dots,0),\dots,\\ &\qquad (0,\dots,0,1)\} \end{aligned}$$

is locally controllable.

It is clear that the constraints (1) can only be satisfied for all time scalings $s(t)$ if the potential is zero in the constrained directions ($c(s) = 0$). In this paper we focus on systems with no potential terms.

If the system is locally kinematically controllable, then there exists a path between any two configurations in the same open connected component of free space, and we can apply any collision-free path planner for the driftless kinematic system (2). Examples include path planners for car-like mobile robots [13, 14]. Because each segment of the resulting path follows one of the decoupling vector fields, the time scaling along the segment is limited only by actuator saturation limits, not by the second-order nonholonomic constraints. Switches between decoupling vector fields must occur at zero velocity to avoid discontinuous velocities, so it is appropriate for the path planner to minimize the number of switches. The decoupling vector fields define a p-dimensional subspace of the n-dimensional tangent space at any configuration, so it is not possible to plan a trajectory between arbitrary states. We focus on trajectory planning between zero velocity states.

1.2 Organization

Section 2 gives a coordinate-free description of underactuated second-order mechanical systems, and Section 3 provides tests for decoupling vector fields and kinematic controllability. Section 4 gives examples of kinematically controllable systems. Although the problem of finding decoupling vector fields is difficult in general, Section 5 describes a method for finding them for the case of underactuated vehicles.

2 Models of mechanical systems

We consider mechanical control systems with total energy equal to kinetic energy and control inputs bounded in magnitude.

Definition 2.1. *Let $q = (q^1, \dots, q^n) \in Q = \Re^n$ be the configuration of the mechanical system and consider the control system:*

$$\begin{aligned} \ddot{q}^i + \Gamma^i_{jk}(q)\dot{q}^j\dot{q}^k &= Y^i_1(q)u_1 + \dots + Y^i_m(q)u_m \qquad (3)\\ q(0) = q_0, &\quad \dot{q}(0) = 0, \end{aligned}$$

where the summation convention is in place for the indices j,k that run from 1 to n, and where:

(i) $M(q)$ is the inertia matrix defining an inner product between vector fields,

(ii) the $\{\Gamma^i_{jk} : i,j,k = 1,\dots,n\}$ are n^3 Christoffel symbols, derived from M according to

$$\Gamma^k_{ij} = \frac{1}{2} M^{mk} \left(\frac{\partial M_{mj}}{\partial q^i} + \frac{\partial M_{mi}}{\partial q^j} - \frac{\partial M_{ij}}{\partial q^m} \right)$$

where M^{mk} is the (m,k) component of M^{-1},

(iii) $\{F_a(q) : a = 1,\dots,m\}$ are the m input co-vector fields, and $\{Y_a(q) = M^{-1}(q)F_a(q) : a = 1,\dots,m\}$ are the m input vector fields.

Systems endowed with fewer control actuators m than degrees of freedom n are called *underactuated*, i.e., $m < n$. Assuming the vector fields $Y_1, \dots, Y_m$ span the first m directions, the $n-m$ second-order constraints in system (3) read

$$\ddot{q}^i + \Gamma^i_{jk}(q)\dot{q}^j\dot{q}^k = 0, \quad i = m+1, \dots, n. \tag{4}$$

Coordinate-free modeling

In a coordinate-free language, vector fields are written in terms of the canonical base $\{\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\}$, and co-vector fields in terms of $\{dq^1, \ldots, dq^n\}$. We will write both vectors and co-vectors fields as as column vectors; given a co-vector field F and two vector fields X_1, X_2, two well defined operations are:

$$\langle F, X_1 \rangle = F^T X_1$$
$$\langle\langle X_1 \, , \, X_2 \rangle\rangle = X_1^T M X_2.$$

Given two vector fields X, Y, the *covariant derivative* of Y with respect to X is the third vector field $\nabla_X Y$ defined via

$$(\nabla_X Y)^i = \frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k. \tag{5}$$

The operator ∇ is called the *Levi-Civita* connection for the mechanical system in equation (3). Using these concepts, the equations of motion can be rewritten as

$$\nabla_{\dot{q}} \dot{q} = \sum_{a=1}^{m} Y_a(q) u_a(t), \tag{6}$$

Equation (6) is a coordinate-free version of equation (3).

Consider the $m < n$ dimensional co-distribution generated by the input co-vector fields $\text{span}\{F_1, \ldots, F_m\}$. Its annihilator is an $n-m$ dimensional distribution, which we shall call the *constraint distribution*, generated by some vector fields $\{X_1, \ldots, X_{n-m}\}$, such that

$$\langle F_a(q), X_b(q) \rangle = 0$$

for all $q \in \Re^n$, $1 \le a \le m$, and $1 \le b \le n-m$. Note that the vector fields X_b are easy to compute since they do not depend on the inertia tensor M. Given these constraint vector fields, we compute

$$0 = \sum_{a=1}^{m} u_a \langle F_a, X_b \rangle = \langle M \nabla_{\dot{q}} \dot{q}, X_b \rangle = \langle\langle \nabla_{\dot{q}} \dot{q} \, , \, X_b \rangle\rangle.$$

Therefore, a curve $q(t)$ is a solution to the underactuated system in equation (3) if it satisfies the $n-m$ constraints

$$\langle\langle \nabla_{\dot{q}} \dot{q} \, , \, X_b \rangle\rangle = 0. \tag{7}$$

Equation (7) is a coordinate-free version of equation (4).

Computational issues

Given an arbitrary robot manipulator, it is very demanding to compute its Christoffel symbols, the inverse of its inertia matrix, and the covariant derivative of various relevant vector fields. This is true even for low-dimensional systems, such as the three degree-of-freedom manipulator later in the paper. Accordingly, these computations are conveniently implemented in a Mathematica™ library. The library `MechSys` is available at `http://motion.csl.uiuc.edu/~bullo/math`.

3 Decoupling vector fields and kinematic controllability

The solution to the equations of motion for a mechanical system obeys the second-order differential equation (3) on the n-dimensional configuration space Q, or in other words a first-order differential equation on the $2n$-dimensional phase space TQ.

For mechanical control systems, we introduce the notion of *first-order* solutions described by vector fields on the configuration space Q. We furthermore require the solution to start and stop at rest.

Let $s : [0, T] \mapsto [0, 1]$ be a twice-differentiable function such that $s(0) = 0, s(T) = 1, \dot{s}(0) = \dot{s}(T) = 0$, and $\dot{s}(t) > 0$ for all $t \in (0, T)$. We call a curve s with these properties a *time scaling.*

Definition 3.1 (Decoupling vector field). *The vector field V is a* decoupling vector field *for the mechanical system* (3) *if, for any time scaling s and for any initial condition q_0, the curve $q(t)$ on Q solving*

$$\dot{q}(t) = \dot{s}(t) V(q(t)), \qquad q(0) = q_0, \tag{8}$$

satisfies the $n-m$ constraints in equation (7).

We call one such curve $q(t)$ a *kinematic motion.*

Necessary and sufficient conditions for a decoupling vector field are as follows.

Lemma 3.2. *The vector field V is decoupling for the mechanical system* (3) *if and only if*

$$\langle\langle V \, , \, X_b \rangle\rangle = 0 \tag{9}$$
$$\langle\langle \nabla_V V \, , \, X_b \rangle\rangle = 0 \tag{10}$$

for all $1 \le b \le n-m$.

Proof. Consider a curve $q(t)$ such that $q(t) = \dot{s}(t) V(q(t))$. The property $\nabla_{fX} Y = f(\nabla_X Y)$ leads to:

$$\begin{aligned} \nabla_{\dot{q}} \dot{q} &= \nabla_{\dot{s}V} (\dot{s} V) \\ &= \ddot{s} V + \dot{s} \nabla_{\dot{s}V} V = \ddot{s} V + \dot{s}^2 \nabla_V V. \end{aligned}$$

The curve $q(t)$ is a kinematic motion and V is a decoupling vector field if the constraints (7) are satisfied:

$$0 = \langle\langle \nabla_{\dot{q}} \dot{q} \, , \, X_b \rangle\rangle$$

which can be written

$$0 = \langle\langle \left(\ddot{s} V + \dot{s}^2 \nabla_V V \right) \, , \, X_b \rangle\rangle$$

for all $1 \le b \le n-m$. Note that $\langle\langle V \, , \, X_b \rangle\rangle$ and $\langle\langle \nabla_V V \, , \, X_b \rangle\rangle$ are equivalent to $a(s)$ and $b(s)$ in equation (1), respectively. Since s is an arbitrary time scaling and q_0 is an arbitrary point, V and $\nabla_V V$ must separately have a vanishing inner product with X_b. The same argument also shows the other implication. $\square$

Remark 3.3. *Roughly speaking, equation* (10) *encodes the requirement that motion along V at constant speed be feasible. Equation* (9) *requires the system to be able to speed up and slow down the motion along V.*

As described in the introduction, decoupling vector fields reduce the complexity of motion planning problems by turning a dynamic problem into a driftless kinematic one. Accordingly, it is of interest to define the class of systems for which this approach applies.

Definition 3.4 (Kinematic controllability). *The mechanical system* (3) *is* kinematically controllable *if every point in the configuration space Q is reachable via a sequence of kinematic motions. The system* (3) *is* locally kinematically controllable *if for any $q \in Q$ and any neighborhood U_q of q, the set of reachable configurations from q by kinematic motions remaining in U_q contains q in its interior.*

Obviously, the main difficulty is that there simply might not be enough decoupling fields for controllability. A sufficient test for local kinematic controllability is given below.

Lemma 3.5. *The system* (3) *is locally kinematically controllable if there exist $p \leq m$ vector fields $\{V_1, \dots, V_p\}$ such that*

(i) $0 = \langle\langle X_b \,, V_c \rangle\rangle = \langle\langle X_b \,, \nabla_{V_c} V_c \rangle\rangle$ *for all* $1 \leq b \leq n-m$, $1 \leq c \leq p$ *, and*

(ii) $\mathrm{Lie}\{V_1, \dots, V_p\}$ *has full rank at all* $q \in \Re^n$.

Proof. Property *(i)* ensures that the vector fields V_c are decoupling. Property *(ii)* ensures the local controllability of the driftless kinematic system (2)

$$\begin{aligned} \dot{q} &= \sum_{c=1}^{p} V_c(q) w_c \\ (w_1, \dots, w_p) &\in \{(1,0,\dots,0), (0,1,0,\dots,0), \dots, \\ &\quad (0,\dots,0,1)\} \end{aligned}$$

and therefore every point in the configuration space is reachable. In the presence of obstacles, a collision-free path exists between any two points in an open connected set of the configuration space. □

Note that scalar multiples of decoupling vector fields are again decoupling, but linear combinations may not be decoupling. There are mechanical control systems for which no decoupling vector fields can be found. The maximum number of linearly independent decoupling vector fields is m.

4 Examples

Numerous examples fit the requirements of Lemma 3.5, allowing the decoupling of trajectory planning. The most obvious example is a planar body with a force through the center of mass and a pure torque. We present this model in detail below for the purpose of establishing the required notation. Other locally kinematically controllable systems include:

(i) All controllable kinematic mechanical systems as described in [12]. Examples include the upright rolling penny (described in detail in [12]) and a 3R planar robot arm with a passive first joint (actuator configuration $(0,1,1)$).

(ii) A 3R planar robot arm with a passive third joint (actuator configuration $(1,1,0)$). This was the original motivating example in [8], and it is worked out below in full detail. For the actuator configuration $(1,0,1)$, we provide one decoupling vector field, but we do not answer the kinematic controllability question.

(iii) Numerous vehicle models including the idealized planar hovercraft (planar body with two forces away from center of mass) and a rigid body in $SE(3)$ with three thrusters away from the center of mass.

4.1 Planar body with pure force and torque

The configuration of the body is $(x, y, \theta) \in \Re^2 \times S^1$, the kinetic energy is $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2$, and all of the Christoffel symbols Γ^k_{ij} are zero. The input co-vectors are $F_1 = d\theta$ and $F_2 = \cos\theta \, dx + \sin\theta \, dy$. The annihilator vector field is

$$X = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y}.$$

As decoupling vector fields we propose pure rotation and translation along the line of force:

$$\begin{aligned} V_1 = Y_1 = M^{-1} F_1 &= \frac{1}{I} \frac{\partial}{\partial \theta} \\ V_2 = Y_2 = M^{-1} F_2 &= \frac{1}{m} \cos\theta \frac{\partial}{\partial x} + \frac{1}{m} \sin\theta \frac{\partial}{\partial y}. \end{aligned}$$

By construction we have $\langle\langle V_1 \,, X \rangle\rangle = \langle\langle V_2 \,, X \rangle\rangle = 0$. Furthermore, an application of equation (5) provides:

$$\begin{aligned} \nabla_{V_1} V_1 &= \frac{1}{I^2} \nabla_{\frac{\partial}{\partial \theta}} \frac{\partial}{\partial \theta} = 0 \\ \nabla_{V_2} V_2 &= \frac{1}{m^2} \nabla_{\left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}\right)} \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \right) = 0. \end{aligned}$$

In summary, V_1, V_2 are decoupling vector fields, and since $\{V_1, V_2, [V_1, V_2]\}$ is full rank, the system is locally kinematically controllable.

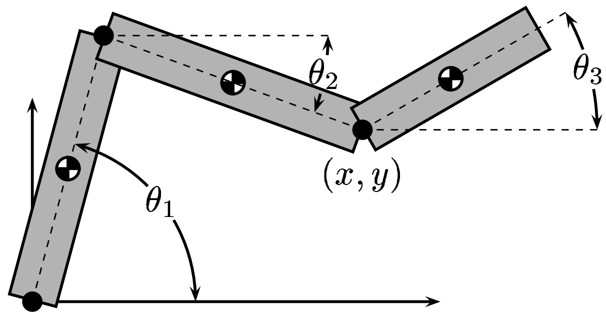

Figure 1: Three-link planar robot manipulator. This configuration is known as SCARA. The angles $\{\theta_1, \theta_2, \theta_3\}$ are measured counterclockwise. $\{x, y\}$ is the location of the third joint.

4.2 Three link planar robot manipulator with a passive joint

We consider a three joint robot manipulator moving in a horizontal plane (Figure 1). Different coordinates will be suited to different tasks: the set $\{\theta_1, \theta_2, \theta_3\}$ consists of the absolute angles (measured counterclockwise) of the three links with respect to the horizontal axis, the set $\{\theta_1, r_2, r_3\}$ measures the relative angles at the second and third joint, and $\{x, y, \theta_3\}$ measures the absolute location of the third joint and the absolute angle of the third link.

Actuator configuration (0,1,1)

We rely on the coordinate system $\{\theta_1, r_2, r_3\}$. Accordingly, the input co-vector fields F_1, F_2, and the annihilator vector field X are

$$F_1 = dr_2, \quad F_2 = dr_3, \quad X = \frac{\partial}{\partial \theta_1}.$$

It is a straightforward computation to see that the inertia matrix M has the following structure:

$$\begin{bmatrix} M_{11}(r_2, r_3) & M_{12}(r_2, r_3) & M_{13}(r_2, r_3) \\ M_{12}(r_2, r_3) & M_{22}(r_3) & M_{23}(r_3) \\ M_{13}(r_2, r_3) & M_{32}(r_3) & M_{33} \end{bmatrix}. \tag{11}$$

In other words the components of the inertia matrix are independent of θ_1 and depend on $\{r_2, r_3\}$ in a specific manner.

Lemma 4.1. *Consider the manipulator with a passive first joint. The system is locally kinematically controllable and two decoupling vector fields are*

$$V_1 = Y_1 = M^{-1} F_1 = M^{-1} dr_2$$
$$V_2 = Y_2 = M^{-1} F_2 = M^{-1} dr_3.$$

Proof. By construction, we have

$$0 = \langle\langle X \,,\, V_1 \rangle\rangle = \langle\langle X \,,\, V_2 \rangle\rangle,$$

and it is easy to see that

$$[V_1, V_2] \notin \text{span}\{V_1, V_2\}.$$

Next, we show that $0 = \langle\langle X \,,\, \nabla_{V_i} V_i \rangle\rangle$ via Mathematica$^{\text{TM}}$. The following code illustrates the computations required to prove that the conditions in Lemma 3.2 are satisfied, so that $\{V_1, V_2\}$ are indeed decoupling.

```
(** Mathematica code for SCARA 011.         **
 ** Configuration, inertia, and connection **)
Needs["MechSys`"];
q = {th1,r2,r3};
M = {{M11[r2,r3],  M12[r2,r3],  M13[r2,r3]},
     {M12[r2,r3],  M22[r3],     M23[r3]   },
     {M13[r2,r3],  M23[r3],     M33       }};
InvM  = Inverse[M];
nabla = LeviCivita[M, q];

(** input one-forms and annihilator **)
F1={0,1,0}; F2={0,0,1}; X={1,0,0};

(** decoupling vector fields and **
 ** their covariant derivatives  **)
V1 = InvM . F1;  V2 = InvM . F2;
V11 = CovariantDerivative[V1, V1, nabla, q];
V22 = CovariantDerivative[V2, V2, nabla, q];

(** these quantities vanish **)
Simplify[{V1.M.X, V11.M.X, V2.M.X, V22.M.X}]
```

□

A second proof of the vanishing of $\langle\langle \nabla_{V_i} V_i \,,\, X \rangle\rangle$ is obtained through a detailed study of the problem's structure. The key observation is that the system has a *symmetry* [9, 15] and therefore a conserved quantity, in this case angular momentum about the first joint. Using these more advanced tools, one can prove not only that the system is locally kinematically controllable, but also that it indeed is kinematic, as defined in [12].[1]

Actuator configuration (1,1,0)

We rely on the coordinate system $\{x, y, \theta_3\}$. The kinetic energy of the first two links can be written as

$$\frac{1}{2} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}^T \begin{bmatrix} M_{11}(x, y) & M_{12}(x, y) \\ M_{12}(x, y) & M_{22}(x, y) \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}.$$

Let m_3, I_3 denote the mass and moment of inertia of the third link; let l_3 be the distance from the third joint to

[1] The key fact is that span$\{V_1, V_2\}$ is the horizontal subspace of the bundle induced by the symmetry.

the center of mass of the third link. The kinetic energy of the third link is

$$\frac{1}{2}(I_3 + m_3 l_3^2)\dot{\theta}_3^2 + \frac{1}{2}m_3(\dot{x}^2 + \dot{y}^2) + m_3 l_3 \dot{\theta}_3(\dot{y}\cos\theta_3 - \dot{x}\sin\theta_3).$$

The input co-vector fields F_1, F_2, and the annihilator vector field X can be written as

$$F_1 = dx, \ \ F_2 = dy, \ \ X = \frac{\partial}{\partial \theta_3}.$$

Lemma 4.2. *Consider the manipulator with a passive third joint. The system is locally kinematically controllable, and two decoupling vector fields are*

$$V_1 = \cos\theta_3 \frac{\partial}{\partial x} + \sin\theta_3 \frac{\partial}{\partial y}$$
$$V_2 = \sin\theta_3 \frac{\partial}{\partial x} - \cos\theta_3 \frac{\partial}{\partial y} + \frac{1}{\lambda}\frac{\partial}{\partial \theta_3}.$$

These motions are translation along the third link and rotation of the third link about its center of percussion, respectively.

Proof. It is easy to compute that

$$0 = \langle\!\langle X \, , \, V_1 \rangle\!\rangle = \langle\!\langle X \, , \, V_2 \rangle\!\rangle$$
$$[V_1, V_2] \notin \operatorname{span}\{V_1, V_2\}.$$

Next, we show that $0 = \langle\!\langle X \, , \, \nabla_{V_i} V_i \rangle\!\rangle$ via Mathematica™. To streamline the computations, we let $\lambda = (I_3 + m_3 l_3^2)/m_3 l_3$, we redefine the terms $\{M_{11}, M_{12}, M_{22}\}$ to account for the term $\frac{1}{2}m_3(\dot{x}^2 + \dot{y}^2)$, and we scale them by a factor $m_3 l_3$.

```
(** Mathematica code for SCARA 110.        **
 ** Configuration, inertia, and connection **)
Needs["MechSys`"];
q = {x,y,th3};
M = {{M11[x,y],    M12[x,y],  -Sin[th3]},
     {M12[x,y],    M22[x,y],  Cos[th3] },
     {-Sin[th3],   Cos[th3],  lambda   }};
nabla = Simplify[ LeviCivita[M, q]];

(** input one-forms and annihilator **)
F1={1,0,0}; F2={0,1,0}; X={0,0,1};

(** decoupling vector fields and **
 ** their covariant derivatives  **)
V1 = {Cos[th3], Sin[th3], 0};
V2 = {Sin[th3], -Cos[th3], 1/lambda };
V11 = CovariantDerivative[V1, V1, nabla, q];
V22 = CovariantDerivative[V2, V2, nabla, q];

(** these quantities vanish **)
Simplify[{V1.M.X, V11.M.X, V2.M.X, V22.M.X}]
```

□

Actuator configuration (1,0,1)

This configuration resembles the $(0,1,1)$ setting—we rely on the same coordinate system and on the expression for inertia matrix M given in equation (11). Here $F_1 = d\theta_1, F_2 = dr_3, X = \frac{\partial}{\partial r_2}$, and $Y_1 = M^{-1}F_1, Y_2 = M^{-1}F_2$.

Lemma 4.3. *Consider the manipulator with a passive second joint. One decoupling vector field is*

$$V_1 = Y_2 - Y_1 \frac{Y_2^1}{Y_1^1} = \begin{bmatrix} 0 \\ \frac{M_{23}}{M_{23}^2 - M_{33}M_{22}} \\ \frac{-M_{22}}{M_{23}^2 - M_{33}M_{22}} \end{bmatrix},$$

where Y_2^1 is the first component of Y_2.

Proof. The proof uses Mathematica™.

```
(** Mathematica code for SCARA 101.        **
 ** Configuration, inertia, and connection **)
Needs["MechSys`"];
q = {th1,r2,r3};
M = {{M11[r2,r3],  M12[r2,r3],  M13[r2,r3]},
     {M12[r2,r3],  M22[r3],     M23[r3]   },
     {M13[r2,r3],  M23[r3],     M33       }};
InvM  = Inverse[M];
nabla = LeviCivita[M, q];

(** input one-forms and annihilator **)
F1={1,0,0}; F2={0,0,1}; X={0,1,0};
Y1 = InvM . F1;  Y2 = InvM . F2;

(** decoupling vector field   **
 ** its covariant derivative  **)
V1 = Simplify[ Y2 - Y1 (Y2[[1]]/Y1[[1]])];
V11 = CovariantDerivative[V1, V1, nabla, q];

(** these quantities vanish **)
Simplify[{V1.M.X, V11.M.X}]
```

□

5 Finding decoupling fields

A sufficient test for local kinematic controllability is given in Lemma 3.5. The primary difficulty is identifying decoupling vector fields. Intuition about the behavior of the system is helpful, but a direct algorithm would simplify the task.

Before proceeding we recall the problem data. The Y_a are the input vector fields, the X_c are the constraint vector fields, and ∇ is the connection.

Assume we look for a decoupling vector field V. According to Lemma 3.2, two conditions need to be satisfied. To automatically satisfy the first one while losing no generality, we write

$$V(q) = \sum_{a=1}^{m} f^a(q) Y_a(q),$$

where $f(q) = (f^1(q), \ldots, f^m(q))^T$ are m arbitrary functions on Q. Recalling the identity $\nabla_X(fY) = f(\nabla_X Y) + (\mathcal{L}_X f)Y$, we compute

$$\nabla_V V = \sum_{a=1}^{m}\sum_{b=1}^{m}\left(f^a f^b \nabla_{Y_a} Y_b + f^a \left(\mathcal{L}_{Y_a} f^b\right) Y_b\right),$$

so that the vector field V is decoupling if

$$0 = \sum_{a=1}^{m}\sum_{b=1}^{m} f^a(q) f^b(q) \langle\langle X_c \, , \, \nabla_{Y_a} Y_b \rangle\rangle(q) \qquad (12)$$

for all $1 \le c \le n - m$. Equation (12) is nonlinear in the unknown functions f^a and configuration dependent; no general solution methodology appears to be available.

In the case of vehicles modeled as rigid bodies with body-fixed forces, however, equation (12) may be simplified considerably. In this case, the input vector fields, their covariant derivatives, and the constraint vector fields are configuration independent (more precisely, left invariant), so the inner product $\langle\langle X_c \, , \, \nabla_{Y_a} Y_b \rangle\rangle$ is independent of q. Accordingly, define the $(n - m)$ matrices $B^c \in \Re^{m \times m}, c = 1, \ldots, n - m$ according to

$$B^c_{\alpha\beta} = \langle\langle X_b \, , \, \nabla_{Y_\alpha} Y_\beta \rangle\rangle,$$

and introduce the constant vector $f = (f^1, \ldots, f^m)^T \in \Re^m$. The condition (12) for a decoupling vector field simplifies to

$$f^T B^c f = 0 \qquad (13)$$

for all $1 \le c \le n - m$. If f is a solution to the $n - m$ quadratic constraints (13) then so is kf for all $k \in \Re$ (particularly $k = 0$), so we impose the additional quadratic constraint $f^T I f = 1$, where I is the identity matrix. Any solution f to the $n - m + 1$ quadratic equations defines a decoupling vector field $V = \sum_a f^a Y_a$.

6 Conclusion

The notions of decoupling vector fields and kinematic controllability allow us to use the structure of underactuated mechanical systems to construct computationally efficient collision-free trajectory planners. While kinematic path planners already exist for some kinematically controllable systems (see, e.g., [7, 8]), future work will be toward constructing a path planner for general locally kinematically controllable systems. For robust execution of the planned trajectories, we plan to study feedback stabilization of kinematic motions.

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