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# On Nonlinear Controllability of Homogeneous Systems Linear in Control

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**Abstract**

This work considers small-time local controllability (STLC) of single and multiple-input systems, $\dot{x} = f_\circ(x) + \sum_{i=1}^m f_i u^i$ where $f_\circ(x)$ contains homogeneous polynomials and $f_1, \ldots, f_m$ are constant vector fields. For single-input systems, it is shown that even-degree homogeneity precludes STLC if the state dimension is larger than one. This, along with the obvious result that for odd-degree homogeneous systems STLC is equivalent to accessibility, provides a complete characterization of STLC for this class of systems. In the multiple-input case, transformations on the input space are applied to homogeneous systems of degree two, an example of this type of system being motion of a rigid-body in a plane. Such input transformations are related via consideration of a tensor on the tangent space to congruence transformation of a matrix to one with zeros on the diagonal. Conditions are given for successful neutralization of bad type (1,2) brackets via congruence transformations.

## 1 Introduction

Various concepts of controllability for nonlinear systems were initially explored in [1, 2, 3]. In particular, [2] is primarily concerned with the property of accessibility of the analytic control system $\dot{x} = F(x, u)$, namely that the set of points attainable from a given initial point via application of feasible input is full in the sense that the interior is nonempty. Sussmann and Jurdjević demonstrated in [2] that a necessary and sufficient condition for accessibility of these systems is that the Lie algebra generated by the system have full rank, the so-called Lie Algebra Rank Condition (LARC).

In [4], the property of small-time local controllability (STLC) was explored for affine analytic single-input systems $\dot{x} = f_\circ(x) + f_1(x)u$ with $|u| \leq 1$. A system is said to be STLC at a point $x_\circ$ if that initial point is in the interior of the set of points attainable from it in time $T$ for all $T > 0$. In this case, the Lie algebra generated by the system, denoted by $L(\{f_\circ, f_1\})$, is the distribution spanned by iterated Lie brackets of $f_\circ$ and $f_1$. Sussmann gave various necessary and sufficient conditions for STLC in [4]. For example, a necessary condition for STLC is that the bracket $[f_1, [f_1, f_\circ]]_{x_\circ}$ be in the space spanned by all brackets with only one occurrence of $f_1$, which is denoted $\mathcal{L}^1(\{f_\circ, f_1\})_{x_\circ}$. More importantly, the conditions conjectured by Hermes were proved to be sufficient conditions for STLC. These Hermes Local Controllability Conditions (HLCC) consist of (1) $x_\circ$ is a (regular) equilibrium point, (2) the LARC is satisfied, and (3) $\mathcal{L}^k(\{f_\circ, f_1\})_{x_\circ} \subset \mathcal{L}^{k-1}(\{f_\circ, f_1\})_{x_\circ}$ for all even $k > 1$ where $\mathcal{L}^k(\{f_\circ, f_1\})_{x_\circ}$ denotes the span of all brackets with $k$ or less occurrences of $f_1$. Using the additional notation $\mathcal{S}^k(\{f_\circ, f_1\})_{x_\circ} := \sum_{i=0}^k \mathcal{L}^i(\{f_\circ, f_1\})_{x_\circ}$, Stefani provided an extension of Sussmann's necessary condition by demonstrating that STLC implies $(\mathrm{ad}_{f_1}^{2m} f_\circ)_{x_\circ} \in \mathcal{S}_{x_\circ}^{2m-1}$ for all $m \in \{1, 2, \ldots\}$ [5]. For an excellent summary and tutorial of these as well as other results in the single-input case, the inquisitive reader is directed to Kawski [6].

STLC of multiple-input affine analytic control systems was addressed by Sussmann in [7], where a general sufficiency theorem was proven for analytic systems of the form $\dot{x} = f_\circ(x) + \sum_{i=1}^m f_i(x)u_i$ with the constraints $|u^i| \leq 1$ for all $i \in \{1, \ldots, m\}$. Several results were presented by Sussmann in [7], but in the context of this paper the most appropriate result is based on the $\delta_\theta$ degree of brackets of the Lie subalgebra generated by the system. For a given iterated bracket $B$ of the vector fields $\{f_i\}$, $|B|_i$ is used to denote the number of occurrences of $f_i$ in $B$.[1] For $\theta \in [0, 1]$, $\delta_\theta$ is defined by $\delta_\theta(B) := \theta|B|_\circ + \sum_{i=1}^m |B|_i$. The general theorem states that systems that satisfy the LARC at $x_\circ$ and have $f_\circ(x_\circ) = 0$ are STLC if there exists a $\theta \in [0, 1]$ such that every bracket $B$ with $|B|_\circ$ odd and $|B|_1, \ldots, |B|_m$ all even can be expressed as a linear combination of $N$ brackets $B_k$ such that $\delta_\theta(B_k) < \delta_\theta(B)$ for all $k \in \{1, \ldots, N\}$. It has become conventional to refer to the brackets with $|B|_\circ$ odd and $|B|_1, \ldots, |B|_m$ all even as *bad brackets* or *potential obstructions.* Both the single-input and multiple-input re-

[1] To be precise, $B$ represents an element of the free monoid (*i.e.*, the set of parenthesized words) in the indeterminates $\{0, 1, \ldots, m\}$.

sults of Sussmann neutralize bad brackets with brackets that are of lower degree in some sense.

Both Sussmann in [7] and Kawski in [6] apply a generalized definition of homogeneity to STLC. This concept of homogeneity begins with definition of a dilation $\delta_\epsilon$ as a parameterized map of $I\!R^n$ to $I\!R^n$ of the form $\delta_\epsilon(x) = (\epsilon^{r_1}x_1, \epsilon^{r_2}x_2, \ldots \epsilon^{r_n}x_n)$ where $r_i$ are nonnegative integers. A polynomial $p : I\!R^n \to I\!R$ is then said to be homogeneous of degree $k$ with respect to the dilation, symbolically $p \in H_k$ if $p(\delta_\epsilon(x)) = \epsilon^k(x)$. Traditional homogeneity is recovered via the dilation with $r_1 = \cdots = r_n = 1$. The definition of homogeneity is then extended to vector fields in the following manner: a vector field $f$ is said to be homogeneous of degree $j$ if $fp \in H_{k-j}$ whenever $p \in H_k$ for all $k \geq 0$. A related area of research that capitalizes on this generalized concept of homogeneity is that of nilpotent and high-order approximation of control systems presented by Hermes for example in [8]. One pertinent outcome of this research is that a system is STLC if its Taylor approximation is STLC, where order is defined relative to a foliation provided by a dilation. However, the converse question of whether STLC can be determined from a finite number of differentiations is still open [9].

## 2 Problem Exposition

In this paper, we address STLC of systems of the form

$$\dot{x} = f_\circ(x) + \sum_{i=1}^{m} f_i u^i \tag{1}$$

where $|u^i| \leq 1$ and $x \in I\!R^n$. $f_i$ for $i \in \{1, \ldots, m\}$ are assumed to be constant vector fields, *i.e.*, $f_i(x) \equiv f_i$, and the components of $f_\circ(x)$ are homogeneous polynomials of degree $k \geq 1$. We use the traditional definition of homogeneous polynomial $p$, namely that $p(\epsilon x) = \epsilon^k p(x)$. The set of such homogeneous vector fields is denoted by $\mathcal{H}_k$. Our definition of degree-$k$ homogeneous vector fields is equivalent to that used in [6, 7, 8] in the following manner: take $\delta_\epsilon : x \mapsto \epsilon x$ and then $\mathcal{H}_k$ is in the general framework the set of vector fields homogeneous of degree $1-k$. With this traditional definition, we have the following elementary facts: (i) $f(0) = 0$ for $f \in \mathcal{H}^k$ with $k \geq 1$, and (ii) $[f, g] \in \mathcal{H}^{j+k-1}$ for all $f \in \mathcal{H}^j$ and $g \in \mathcal{H}^k$, where $\mathcal{H}^{-1}$ is interpreted as the singleton containing the zero vector field.

Systems of this form are theoretically interesting because their Lie algebra at $x_\circ = 0$ has a diagonal structure, as depicted in Figure 1. In particular, the only brackets $B$ that have a nonzero value at $x_\circ$ are those with $|B|_{!\circ} = (k-1)|B|_\circ + 1$, where $|B|_{!\circ} := \sum_{i=1}^{m} |B|_i$. Brackets above this line have homogeneity of degree greater than zero, hence have zero value at $x_\circ$. Brackets below this line are identically zero.

Furthermore, systems of the form (1) arise in mechanics. An example of such a system is the motion of a rigid body in a plane expressed in body-fixed coordinates, as depicted in Figure 2. The equations of motion for this system are

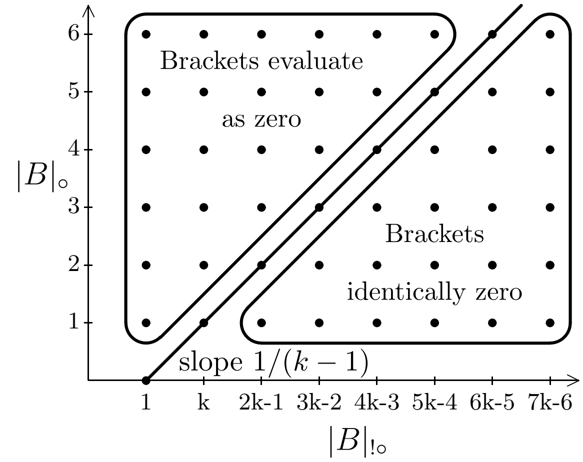


**Figure 1:** Graphical depiction of nonzero brackets of polynomial system (3) with homogeneity degree $k$.

$$\begin{aligned} \dot{\omega}_y &= u_2 + hu_1 \\ \dot{v}_x &= -\omega v_y \\ \dot{v}_y &= \omega v_x + u_1. \end{aligned} \tag{2}$$

The state consists of rotational velocity $\omega$ and the two body-fixed translation velocities $v_x$ and $v_y$. The input consists of the pure torque $u_2$ and the force $u_1$ applied at a moment arm of $h$. Hence $f_\circ(x) = (0, -x_1x_3,\ x_1x_2)$, $f_1 = (h, 0, 1)$ for some constant $h$, $f_2 = (1, 0, 0)$, and the system is of the form (1) with homogeneity degree two. This provides a simple example of a system for which Sussmann's sufficient condition is not invariant with respect to input transformations. In particular, if a pure force ($h = 0$) and a pure torque are used as inputs, then the system satisfies Sussmann's sufficient condition for the multiple-input case. However, if an offset force ($h \neq 0$) is used, then obstructions appear in the type (1,2) brackets, *i.e.*, brackets with $|B|_\circ = 1$ and $|B|_{!\circ} = 2$. In general, we employ the phrase *type (k,ℓ) brackets* to refer to all iterated brackets $B$ of the vector fields $\mathcal{F} := \{f_\circ, \ldots, f_m\}$ with $|B|_\circ = k$ and $|B|_{!\circ} = \ell$, and denote the distribution spanned by such brackets as $\mathcal{L}^{(k,\ell)}(\mathcal{F})$.[2] Using this system as a motivating example, we explore the neutralization via congruence transformation of bad brackets of type (1,2) with other brackets (perhaps also bad) of type (1,2).

## 3 Single Input Systems

In this section we consider the single-input system

$$\dot{x} = f_\circ(x) + f_1 u \tag{3}$$

where $x \in I\!R^n$, $u \in [-1, 1]$, $f_1 \in I\!R^n$ and $f_\circ \in \mathcal{H}_k$. In light of HLCC and Sussmann's general result, it is

[2] The notation $\mathcal{L}$ is used instead of $L$ to emphasize that $\mathcal{L}^{(k,\ell)}(\mathcal{F})$ is not necessarily a Lie subalgebra.

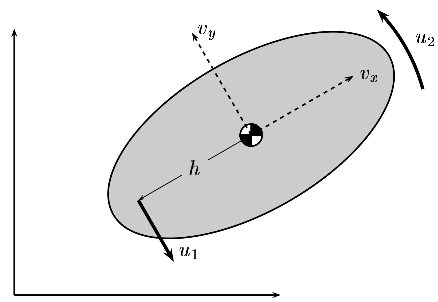


**Figure 2:** Motion of a rigid body in a plane expressed in body-fixed coordinates.

clear that for a system as in (3) with odd homogeneity degree, accessibility is equivalent to STLC, since there are no nonzero brackets with $|\cdot|_\circ$ odd and $|\cdot|_1$ even. In other words, the question of STLC reduces to the LARC. The following lemma asserts that if the brackets of type (1,$k$) do not add to the Lie algebra rank, then neither do the higher-degree brackets.

Lemma 1
For system (3) with homogeneity degree $k > 0$, if $(\mathrm{ad}^k_{f_1} f_\circ)_{x_\circ} \in \mathrm{span}\{f_1\}$, then $L(\{f_\circ, f_1\})_{x_\circ} = \mathrm{span}\{f_1\}$.

**Proof:** Since the Lie algebra structure is invariant to state transformations, without loss of generality we can take $f_1$ to be the basis vector $e_1$. Define $\mathcal{H}^{11}_k$ to be the vector fields of homogeneity degree $k$ with the form $(Cx_1^k + \eta_1(x), \eta_2(x), \ldots, \eta_n(x))$, with the power of $x_1$ in $\eta_i$ being less than $k$ for all $i \in \{1, \ldots, m\}$. Then $(\mathrm{ad}^k_{f_1} f_\circ)_{x_\circ} \in \mathrm{span}\{e_1\}$ implies that $f_\circ \in \mathcal{H}^{11}_k$, since $\mathrm{ad}^k_{f_1}$ corresponds to the partial derivative operator $\partial^k/\partial x_1^k$. Furthermore, if $g \in H^{11}_m$ for some $m \geq 1$, then $\mathrm{ad}_{f_1} g \in H^{11}_{m-1}$ and $\mathrm{ad}_{f_\circ} g \in H^{11}_{m+k-1}$. Since the Lie subalgebra of (3) is spanned by brackets of the form $[f_{i_1}, [f_{i_2}, \cdots [f_{i_{r-1}}, f_{i_r}] \cdots]]$ (apply for example Proposition 3.8 of [10]), all brackets in $L(\{f_\circ, f_1\})_{x_\circ}$ are a multiple of $e_1$. ∎

Turning our attention to systems with even homogeneity degree $k$, we see that $L(\{f_\circ, f_1\})_{x_\circ}$ does include bad brackets. In particular, type $(m, (k-1)m+1)$ brackets are (odd, even) for $m$ even. We make use of the necessary condition of Stefani restated here for convenience.

Theorem 2 (Stefani [5])
If the system $\dot{x} = f_\circ(x) + f_1(x)u_1$ is STLC, then $(\mathrm{ad}^{2k}_{f_1} f_\circ)_{x_\circ} \in \mathcal{S}^{2k-1}(\{f_\circ, f_1\})_{x_\circ}$.

When applied to even systems, Theorem 2 states that $(\mathrm{ad}^{2k}_{f_1} f_\circ)_{x_\circ} \in \mathrm{span}\{f_1\}$ is necessary for STLC. But if $(\mathrm{ad}^{2k}_{f_1} f_\circ)_{x_\circ} \in \mathrm{span}\{f_1\}$, then Lemma 1 asserts $\dim L(\{f_\circ, f_1\}) = 1$, and systems with $n \geq 2$ cannot be STLC (for otherwise LARC is violated). This reasoning and the fact that the system $\dot{x} = x^{2k} + u$ with $x \in \mathbb{R}$ is STLC provides the following result.

Proposition 3
If the system in (3) has odd homogeneity degree, then it is STLC if and only if it satisfies the LARC. On the other hand, if the system has even homogeneity degree $2k > 0$, then it is STLC if and only if the state $x$ is scalar.

## 4 Multiple Input Systems

In this section, we return to consideration of the system in (1) where $f_\circ \in H^2(x)$. An extension of the above results to this multiple-input case is problematic. In particular, the necessary condition of Stefani runs into the problem of a possibility of balancing between bad brackets of the same degree. This consideration, along with the motivating example of planar rigid-body motion lead us to investigate neutralization of potential obstructions by brackets of the same type.

Of course, for the system in (1) the general sufficient condition of Sussmann can be applied to determine STLC. Since the Lie algebra has a diagonal structure, the choice of $\theta \in [0, 1]$ in the theorem is immaterial. Using Sussmann's concepts of good and bad brackets, the sufficient condition allows us to neutralize bad brackets with good brackets of lower degree. Our goal with this section is to address the case where there are bad brackets that cannot be neutralized in this manner, and to neutralize these bad brackets with other brackets of the *same* degree via appropriate choice of linear transformation on the input space. In this endeavor, the diagonal structure of the Lie algebra is particularly useful.

Returning to the motivating example of planar motion in equation (2), it is clear that this system is STLC. (For example, use the feedback transformation $u_1 = \omega v_x + \bar{u}_1$ and $u_2 = \bar{u}_2 - hu_1$ to obtain the system described by $\dot{\omega} = \bar{u}_2$, $\dot{v}_x = -\omega v_y$, and $\dot{v}_y = \bar{u}_1$.) However, in attempting to apply Sussmann's general sufficiency result directly on the unchanged equations of motion, we have the apparent obstruction $[f_1, [f_1, f_\circ]] = (0, -2h, 0) \notin \mathrm{span}\{f_1, f_2\}$. This potential obstruction prevents the use of Sussmann's sufficient condition for any $h \neq 0$. On the other hand, if $h = 0$ (corresponding to a force input $u_1$ through the center of mass) then we can apply the sufficient condition, for we have the good bracket $[f_1, [f_2, f_\circ]] = (0, -1, 0)$ bringing the Lie algebra to full rank, and STLC is demonstrated. Furthermore, the system for $h \neq 0$ can be transformed into the pure force system ($h = 0$) via the input transformation

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -h & 1 \end{pmatrix} \begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix}.$$

Since STLC is clearly invariant to full rank input transformations, we see that a particular choice of input transformation may provide a means of removing a potential obstruction, thus extending the applicability of Sussmann's general theorem for this class of systems.

*Remark.* It is worth noting that Kawski in [6] has considered techniques for neutralizing and balancing bad brackets. However, this technique is not clearly related to ours. Kawski's technique applies in the single-input setting, and neutralizes brackets possibly with brackets of different degree via parameterized families of controls carefully tailored to the system and the brackets in question. In some cases these families of controls involve switching between control limits, with the parameter affecting the switching times. Our technique utilizes the freedom of multiple independent inputs to enforce a linear relation between the inputs in order to neutralize brackets of the same type.

### 4.1 Neutralization via congruence transform

Moving to the generic multiple-input homogeneous system of degree two, suppose that there is some bracket of the form $[f_i, [f_i, f_\circ]]$ that is not in the span of $\{f_i\}_{i=1}^m$. We would like to find a transformation $T$ on the input space such that the transformed system

$$\dot{x} = f_\circ(x) + \sum_{j=1}^{m} \overbrace{(\sum_{i=1}^{m} f_i T_{ij})}^{\bar{f}_j} \; \bar{u}_j$$

has the corresponding potential obstruction removed. Of course, the input transformation will not affect $\text{span}\{f_i\}_{i=1}^m$. We make the restriction $|\bar{u}_j| \le 1/\bar{\lambda}$ where $\bar{\lambda}$ is the spectral radius of $T$ in order to have the resulting $u_i$ satisfy the bounds $|u_i| \le 1$. [3]

Suppose the subspace $B := \mathcal{L}^{(0,1)}(\mathcal{F})_{x_\circ}^{\perp} \cap \mathcal{L}^{(1,2)}(\mathcal{F})_{x_\circ}$ has dimension one and there is at least one bad bracket of type (1,2) with a nonzero projection onto this subspace. Let $b \in T_{x_\circ}I\!\!R^n$ be such that $\text{span}\{b\} = B$. Hence $b$ represents the direction of the potential obstruction, and if we could annihilate the bad brackets along this direction, the potential obstruction will have been removed. It is possible to choose a differential one-form $\beta$ such that $(\beta b)(x_\circ) \neq 0$ but $(\beta f_i)(x_\circ) = 0$ for all $i \in \{1, \ldots, m\}$. Roughly speaking, $\beta$ represents the projection of a vector field onto $B$. With $\beta$ in hand, we define the map $\psi_\beta$ from $T_{x_\circ}I\!\!R^n \times T_{x_\circ}I\!\!R^n$ to $I\!\!R$ by

$$\psi_\beta : (f, g) \mapsto (\beta[f, [g, f_\circ]])_{x_\circ} \tag{4}$$

This map inherits bilinearity from the Lie bracket, and hence is a tensor of covariant order two at $x_\circ$. Next we derive a matrix $\Psi_\beta \in I\!\!R^{m \times m}$ from $\psi_\beta$ via

$$(\Psi_\beta)_{ij} := \psi_\beta(f_i, f_j) \tag{5}$$

where $f_i, f_j$ are the input vector fields of the system in (1). Then $\Psi_\beta$ is the matrix whose $ij^{th}$ element is the projection of the tangent vector $[f_i, [f_j, f_\circ]]_{x_\circ}$ onto the direction $b$. By employing the Jacobi identity and the fact that the input vector fields commute, it is clear that $\Psi_\beta$ is also symmetric. Denoting by $\hat{\Psi}_\beta$ the corresponding matrix for the transformed system, it is easy to see that $\hat{\Psi}_\beta = T^T \Psi_\beta T$. In this manner, the question of whether the obstructing brackets can be neutralized is reduced to the linear algebra question:

> Given a symmetric matrix $\Psi_\beta \neq 0$, is there a full rank, square matrix $T$ such that the congruence transformation of $\Psi_\beta$, $\hat{\Psi}_\beta = T^T \Psi_\beta T$, has all zeros along the diagonal?

Supposing for a moment that such a congruence transform exists, since it is full rank it must be true that there is some nonzero, off-diagonal element $(\hat{\Psi}_\beta)_{ij}$. In other words, if such an input transformation exists, it not only neutralizes the bad bracket(s) in the $b$ direction, but also replaces them with good brackets that have nonzero projections in that direction. Furthermore, the input transformation will not create type (1,2) bad brackets in directions orthogonal to $\mathcal{L}^{(1,2)}(\mathcal{F})_{x_\circ}$. (This would be tantamount to $T^T 0 T \neq 0$.) Of course, the input transformation will also rotate the higher-degree brackets, possibly creating bad brackets from good.

Recalling that a symmetric matrix is called indefinite if it has at least one positive eigenvalue and at least one negative eigenvalue, we have the following answer to the posed question.

Lemma 4

*Given a matrix $\Psi_\beta = \Psi_\beta^T \neq 0$, there exists a full rank matrix $T$ such that $\hat{\Psi}_\beta = T^T \Psi_\beta T$ has all zeros on the diagonal if and only if $\Psi_\beta$ is indefinite.*

**Proof:** First recall that by virtue of its symmetry, the matrix $\Psi_\beta$ has orthonormal eigenvectors $V := (v_1, \ldots, v_m)$ such that $V^T \Psi_\beta V = \text{diag}(\lambda_1, \ldots, \lambda_m)$ where $\lambda_i$ are the real eigenvalues of $\Psi_\beta$. Expressing the columns $t_i$ of $T$ in terms of the orthonormal eigenvectors, we have $(\hat{\Psi}_\beta)_{ii} = \sum_{j=1}^m \lambda_j (t_i^T v_j)^2$. If $\Psi_\beta \neq 0$ is either positive or negative semidefinite, then for any full rank $T$ there is some column $t_i$ that has a nonzero projection onto an eigenvector that corresponds to some $\lambda_j \neq 0$, and the corresponding diagonal element $(\hat{\Psi}_\beta)_{ii}$ will be nonzero.

Suppose $\Psi_\beta$ is indefinite, and group the eigenvalues into those which are positive $\{\lambda_i^+\}_{i=1}^{m_+}$, those which are negative $\{\lambda_j^-\}_{j=1}^{m_-}$, and those which are zero $\{\lambda_k^\circ\}_{k=1}^{m_\circ}$. The eigenvectors are similarly grouped into $\{v_i^+\}_{i=1}^{m_+}$, $\{v_j^-\}_{j=1}^{m_-}$, and $\{v_k^\circ\}_{k=1}^{m_\circ}$. We proceed by constructing the matrix $T$. The first $m_\circ$ columns $t_i$ of $T$ are chosen so that $t_i = v_i^\circ$, achieving $t_i \Psi_\beta t_i = 0$ for $i \in \{1, \ldots, m_\circ\}$. The next $m_+$ columns $t_j$ are chosen according to

$$t_{j+m_\circ} = \frac{v_j^+}{(\lambda_j^+)^{1/2}} - \frac{v_1^-}{(\lambda_1^-)^{1/2}} \tag{6}$$

[3] While modification of the control bound can result in difficulties in the balancing of brackets in [6], this is not a concern in our case, since the neutralization that we achieve is independent of the relative magnitudes of $\bar{u}_i$.

for all $j \in \{1,\ldots,m_+\}$. For this choice, $t_i \perp t_j$ for all $i \in \{1,\ldots,m_\circ\}$ and all $j \in \{1,\ldots,m_+\}$, and $t_{j+m_\circ}^T \Psi_\beta t_{j+m_\circ} = 0$ for all $j \in \{1,\ldots,m_+\}$. The final $m_-$ columns $t_k$ are chosen to be

$$t_{k+m_\circ+m_+} = \frac{v_1^+}{(\lambda_1^+)^{1/2}} + \frac{v_k^-}{(\lambda_k^-)^{1/2}} \tag{7}$$

for all $k \in \{1,\ldots,m_-\}$. Similarly, this final group of columns is orthogonal to $\{t_i\}_{i=1}^{m_\circ}$ and has the property $t_{k+m_\circ+m_+}^T \Psi_\beta t_{k+m_\circ+m_+} = 0$ for all $k \in \{1,\ldots,m_-\}$. Furthermore, $\{t_\ell\}_{\ell=m_\circ+1}^{m}$ is linearly independent. This completes the construction of $T$. ■

### 4.2 The planar vehicle example revisited

Applying this line of reasoning to the planar vehicle example presented above, the direction of the apparent obstruction projected onto $\mathcal{L}^{(0,1)}(\mathcal{F})_{x_\circ}$ is $b = (0,-2h,0)$, and the tensor $\psi_\beta$ in coordinates is $(0,0,2h;0,0,0;2h,0,0)$. The associated matrix $\Psi_\beta = (4h^2,2h;2h,0)$ has eigenvalues $2h^2 \pm 2\sqrt{h^4+h^2}$. Of course $h \neq 0$ is assumed, for otherwise there is no obstruction. It is easy to see that $\Psi_\beta$ is sign indefinite, and hence the construction in the proof of Lemma 4 provides the transformation

$$T_\beta = \begin{pmatrix} \sigma_1(h)h - \sigma_2(h)\sqrt{1+h^2} & \sigma_2(h)h - \sigma_1(h)\sqrt{1+h^2} \\ \sigma_1(h) & \sigma_2(h) \end{pmatrix}$$

where $\sigma_1$ and $\sigma_2$ are continuous functions of $h > 0$ with $\sigma_i > 0$ for all finite $h > 0$. This transformation yields $\hat{\Psi}_\beta = (0,-2;-2,0)$, and the resulting type (1,2) brackets of the transformed system are $[\bar{f}_1,[\bar{f}_1,f_\circ]] = (0,0,0)$, $[\bar{f}_1,[\bar{f}_2,f_\circ]] = (0,1/h,0)$, and $[\bar{f}_2,[\bar{f}_2,f_\circ]] = (0,0,0)$. Thus the apparent obstruction to STLC is removed, and the Lie subalgebra generated by the system at $x_\circ$ is spanned by the good brackets $\{\bar{f}_1,\bar{f}_2,[\bar{f}_1,[\bar{f}_2,f_\circ]]\}$. Hence application of Sussmann's general result demonstrates STLC. It is interesting to note that while $T_\beta$ is not equal to $T$ determined above, it does transform the system into one with pure force and pure torque input for any $h > 0$.

### 4.3 Example of balancing two bad brackets

Not only can this technique neutralize a bad type (1,2) bracket with a good type (1,2) bracket, but it may also balance two bad brackets. Consider the two input example with $f_\circ(x) = (x_2x_3,\, x_1x_3,\, x_1^2 - x_2^2)$, $f_1 = (1,0,0)$, and $f_2 = (0,1,0)$. This system has two apparent obstructions of type (1,2), namely $[f_1,[f_1,f_\circ]] = (0,0,2)$ and $[f_2,[f_2,f_\circ]] = (0,0,-2)$. Furthermore, the good bracket $[f_1,[f_2,f_\circ]]$ is $(0,0,0)$. For this example, $\Psi_\beta = (4,0;0,-4)$ is clearly indefinite, and hence we have the desired transformation $T = (0.5,-0.5;0.5,0.5)$. The resulting transformed type (1,2) brackets are $[\bar{f}_1,[\bar{f}_1,f_\circ]] = (0,0,0)$, $[\bar{f}_1,[\bar{f}_2,f_\circ]] = (0,0,1)$, and $[\bar{f}_2,[\bar{f}_2,f_\circ]] = (0,0,0)$, and again STLC is achieved. An interesting variation on this example is obtained if we replace $f_\circ(x)$ above with $(x_2x_3,\, x_1x_3,\, x_1^2 + x_2^2 + \alpha x_1x_2)$ where $\alpha \in \mathbb{R}$. This system has $\Psi_\beta = (4,2\alpha;2\alpha,4)$ indefinite for $|\alpha| > 2$. This condition has the interpretation that the two bad brackets which project onto $b$ *with the same sign* can be neutralized with the good bracket if its projection onto $b$ is large enough.

### 4.4 Effect of neutralization on other directions

Next we consider the effect of neutralization of brackets along one direction on the projection of the brackets in another direction. We consider the system that evolves on $x \in \mathbb{R}^4$ described by $f_\circ(x) = (x_2x_4,\ 0,\ x_1^2 + x_1x_2,\ x_1x_2)$, $f_1 = (1,0,0,0)$, and $f_2(0,1,0,0)$. The type (1,2) brackets for this system are $[f_1,[f_1,f_\circ]] = (0,0,2,0)$, $[f_2,[f_2,f_\circ]] = (0,0,0,0)$, and $[f_1,[f_2,f_\circ]] = (0,0,1,1)$. If we concentrate on neutralizing the bad bracket in the direction $b = (0,0,2,0)$, then we have $\Psi_\beta = (4,2;2,0)$ with eigenvalues $\lambda = 2 \pm 2\sqrt{2}$. The constructed transformation matrix $T = (-2^{-1/4}, 2^{-1/4}; 0, 2^{1/4})$ neutralizes the bad bracket along $b$. However, the transformation produces an obstruction along $(0,0,0,1)$, as evidenced by the resulting brackets $[\bar{f}_1,[\bar{f}_1,f_\circ]] = (0,0,0,-\sqrt{2})$, $[\bar{f}_1,[\bar{f}_2,f_\circ]] = (0,0,-1,-1)$, and $[\bar{f}_2,[\bar{f}_2,f_\circ]] = (0,0,0,0)$.

### 4.5 Interpretation and impact

We have developed a test for neutralizing bad brackets of type (1,2) for homogeneous degree-two systems that requires indefiniteness of the matrix $\Psi_\beta$ defined in (5). To interpret this requirement, we recall that a matrix is positive definite if and only if all of its principal minors are positive definite. On the other hand, a matrix is negative definite if and only if all of its principal minors are negative definite when of odd dimension and positive definite when of even dimension. Hence $\Psi_\beta$ will be indefinite if any principal minor is itself indefinite. Recalling that the $ij^{th}$ entry of $\Psi_\beta$ is the projection of the bracket $[f_i,[f_j,f_\circ]]$ onto the direction $b$, the implications of the indefiniteness test become intuitively clear. Restricting our attention for the moment to cases where $\mathcal{L}^{(1,2)}(\mathcal{F})_{x_\circ} \cap \mathcal{L}^{(0,1)}(\mathcal{F})_{x_\circ}$ is spanned by the single vector $b$, if there is a single obstructing bad bracket and a good bracket with nonzero projection onto $b$, then the obstruction can always be removed since the principal minor corresponding to these two brackets is always indefinite (*i.e.*, the matrix $(2a,b;b,0)$ has eigenvalues $a \pm \sqrt{a^2+4b^2}$.) If two or more bad brackets project along $b$, then they can all be simultaneously neutralized so long as a pair of the bad brackets project with opposite sign along $b$. On the other hand, if one or more bad brackets project onto $b$ with the same sign with all good brackets being orthogonal to $b$, then the technique fails. Notice that while the examples all had just two inputs, the technique applies without modification to homogeneous degree-two systems with $m \geq 2$.

When other directions are involved, the neutralization may encounter difficulties. Supposing that

**Table 1:** Applicability of neutralization via congruence transformation. $B := \mathcal{L}^{(0,1)}(\mathcal{F})^{\perp}_{x_\circ} \cap \mathcal{L}^{(1,2)}(\mathcal{F})_{x_\circ}$.

| $\dim(B)$ | # of good brackets | # of bad brackets | outcome |
|---|---|---|---|
| 0 | n.a. | n.a. | no obstr. |
| 1 | 1 | 0 | no obstr. |
| 1 | 1 | 1 | neutralized |
| 1 | $\geq 0$ | 2 | possible neut. |
| $\geq 2$ | open question | | |

$\mathcal{L}^{(1,2)}(\mathcal{F})_{x_\circ} \cap \mathcal{L}^{(0,1)}(\mathcal{F})_{x_\circ}$ is spanned by $\{b_i\}_{i=1}^{k}$ with $k \geq 2$, the question of neutralization of bad brackets becomes one of simultaneously transforming the matrices $\Psi_{\beta_i}$ so that they all have zeros on the diagonals. If the ranges of the matrices $\Psi_{\beta_i}$ are orthogonal, then the problem could be solved with a block diagonal transformation $T$, where each block appropriately transforms each $\Psi_{\beta_i}$. This would require a straightforward modification of the construction of $T$ that would allow freedom in the choice of $t_i$ on the null space spanned by $\{v_i^\circ\}$. These interpretations are summarized in Table 1.

Finally, notice that homogeneity is not essential to the development of neutralization via congruence transform, the construction of the matrix $\Psi_\beta$ being sufficiently general that it applies to any nonlinear system linear in control. For example, neutralization of bad type (1,2) brackets for the system $f_\circ(x) = (x_2x_3,\, x_1x_3,\, \sin^2 x_1 - \sin^2 x_2)$, $f_1 = (1,0,0)$, and $f_2 = (0,1,0)$ proceeds identically to that of the previous example with $f_\circ(x) = (x_2x_3,\, x_1x_3,\, x_1 - x_2)$. Clearly the proposed technique provides for neutralization of bad brackets of type (1,2) for these more general systems. A generalization of neutralization via congruence transformation to nonhomogeneous nonlinear systems would involve incorporation of the rich differential geometry of nilpotent and higher-order approximations and foliations described for example by Hermes in [8].

## 5 Conclusion

We have presented a complete characterization of STLC for the class of single-input, homogeneous polynomial systems linear in control, where homogeneous is used in the traditional sense. Specifically for odd-degree systems, STLC is equivalent to the Lie Algebra Rank Condition, while even-degree systems are never STLC except for the degenerate case of a scalar state. For multiple-input homogeneous systems linear in control, we have investigated neutralization of bad brackets with brackets of the same type. The methodology presented in this paper provides a means of neutralizing bad brackets of type (1,2). By consideration of the tensor generated from the bracket structure $[\cdot,[\cdot,f_\circ]]$ applied to the direction containing an apparent obstruction, we have reduced the question of neutralizing an obstruction to that of finding a congruence transform that results in a matrix with all zeros along its diagonal. It is shown that such a transformation exists if and only if the matrix in question is indefinite. When this test is translated back to type (1,2) brackets, it has very intuitive implications, which are illustrated with several simple examples. The methodology presented is limited in its effectiveness by the fact that it removes an apparent obstruction *only along a particular direction in the tangent space*, although an extension to multiple directions appears attainable. Although the result has been presented in the context of homogeneous systems, its development does not rely on homogeneity, and hence applies to neutralization of type (1,2) brackets for any nonlinear system that is linear in control.

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## References

[1] H. Hermes, "On the structure of attainable sets for generalized differential equations and control systems," *J. of Diff. Eq.*, vol. 9, pp. 141–154, Jan. 1971.

[2] H. Sussmann and V. Jurdjevic, "Controllability of nonlinear systems," *J. of Diff. Eq.*, vol. 12, pp. 95–116, July 1972.

[3] R. Hermann and A. Krener, "Nonlinear controllability and observability," *IEEE Tr. Aut. Cntr.*, vol. AC–22, pp. 728–740, Oct. 1977.

[4] H. Sussmann, "Lie brackets and local controllability: A sufficient condition for scalar-input systems," *SIAM J. Cntr. & Opt.*, vol. 21, pp. 686–713, Sept. 1983.

[5] G. Stefani, "On the local controllability of a scalar input control system," in *Thry and Appl. of Nonl. Cntr. Sys.* (C. Byrnes and A. Lindquist, eds.), New York: Elsevier, 1986.

[6] M. Kawski, "Higher-order small-time local controllability," in *Nonl. Cntrlblty & Opt. Cntrl.* (H. Sussmann, ed.), vol. 133, ch. 14, Monticello, NY: Dekker, 1990.

[7] H. Sussmann, "General theorem on local controllability," *SIAM J. Cntr. & Opt.*, vol. 25, pp. 158–194, Jan. 1987.

[8] H. Hermes, "Nilpotent and high-order approximations of vector field systems," *SIAM Rvw.*, vol. 33, pp. 238–264, June 1991.

[9] A. A. Agrachev, "Is it possible to recognize controllability in a finite number of differentiations?," in *Opn Prblms in Math. Sys. and Cntr. Thry* (V. Blondel, ed.), ch. 4, New York, NY: Springer, 1999.

[10] H. Nijmeijer and A. van der Schaft, *Nonlinear Dynamical Control Systems.* New York: Springer, 1990.