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H^∞-Optimal Tracking Control Techniques for Nonlinear Underactuated Systems

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Abstract

This paper presents new techniques for controlling the motion of an underactuated vehicle when disturbances are present and only imperfect state measurements are available for feedback. A state feedback controller is developed and then it is converted to an imperfect state measurement feedback controller. The state feedback tracking control law uses an H^∞-optimal design and produces a locally exponentially stable closed-loop system. The imperfect state measurement feedback controller combines the state feedback control law with an H^∞-£lter to estimate the states and achieves a modi£ed form of disturbance attenuation. The state estimator exploits a unique structure in the nonlinear equations of motion to develop a direct solution. The MATLAB simulations illustrate both control algorithms for an underactuated ship model.

1 Introduction

We consider in this paper the problem of designing a control law to make a nonlinear underactuated vehicle track a trajectory when disturbances are present and only imperfect state measurements are available for feedback. The primary design techniques will rely on H^∞-optimal control methods. We start with a brief outline of the problem and a review of the associated literature to provide a framework for our contributions. After this introduction, the following sections will describe the underactuated vehicle model, the controller design process, and the simulation results.

The vehicles under consideration display four attributes that make the motion planning and control problems challenging. The attributes are that (i) the vehicles are underactuated, (ii) the equations of motion are nonlinear, (iii) there are unknown disturbances entering the equations of motion, and (iv) the state of the system is only available through noisy partial state measurements. Individually, these factors can be dif£cult to handle, so their combination in a single design problem presents a formidable task.

To formally state the problem, we assume we are given an accurate, time-invariant model of a nonlinear underactuated vehicle that we want to control. In addition, we are given a feasible trajectory for the vehicle to track. A feasible trajectory is one that the vehicle could track exactly if there were no initial condition errors and no disturbances. Our objective is to design a controller that causes the vehicle to track the feasible trajectory while also attenuating the effect of disturbances. The control design problem can be subdivided into two parts. The £rst part will be to design a state feedback controller for the nonlinear underactuated system assuming perfect state information is available. The second part will be to modify the state feedback controller to account for the imperfect state measurement case.

There are two main contributions described in this paper. First, we present the design of a state feedback controller for an underactuated vehicle that locally exponentially tracks a desired trajectory in the presence of disturbances. Second, we develop an imperfect state measurement controller for the same system by using a decomposition of the equations of motion to estimate the states and then prove a form of disturbance attenuation for the closed-loop system.

Our original contributions build directly on some of the recent results in the literature. We used a model of an underactuated ship and posed a tracking problem inspired by the work of Pettersen and Nijmeijer [5]. Their controller is based on a variation of the backstepping technique and allows the ship to recover from initial errors to track a reference trajectory. Godhavn developed a more traditional type of backstepping controller to make a ship track a desired trajectory [3]. Under certain initial conditions, Godhavn's approach would allow the ship to rotate 180 degrees and track the reference path backwards. Our approach corrects this de£ciency, and the simulations show the vehicle makes natural maneuvers to recover from poor initial conditions. In addition and in contrast to both backstepping techniques, our approach also offers the advantage of explicitly accounting for disturbances in the controller design.

Other literature directly related to our problem addresses the solution to the H^∞ -optimal control problem. We have adopted a game-theoretic approach to solve the control problem, and Başar and Bernhard provide a complete account of the solution for linear and nonlinear systems [1]. Başar and Olsder provide an additional reference on noncooperative dynamic games [2], which will also be instrumental in our design. The H^∞ controller design presented here will build on contributions from Walsh *et al.* [7]. Our approach extends their results to account for the disturbances in the system. Finally, we will use the results by Pan and Başar [4] to complete the disturbance attenuation proofs.

2 Underactuated Vehicle Model

In this section we introduce an underactuated ship model which we will use for our analysis. This model is based on the one that was studied earlier by Pettersen and Nijmeijer [5]. We have made only minor changes to simplify the notation and to include disturbances. The relevant equations of motion for this system are

$$\begin{aligned}
\dot{u} &= m_u v r - d_u u + u_1 + w_1 && (1)\\
\dot{v} &= m_v u r - d_v v + w_2 && (2)\\
\dot{r} &= m_r u v - d_r r + u_2 + w_3 && (3)\\
\dot{x} &= u\cos(\psi) - v\sin(\psi) && (4)\\
\dot{y} &= u\sin(\psi) + v\cos(\psi) && (5)\\
\dot{\psi} &= r && (6)
\end{aligned}$$

where u, v and r are the body-frame velocities in the forward (surge), lateral (sway) and rotational (yaw) directions, respectively, x and y are the inertial positions and ψ is the inertial rotation angle. The coef£cients m_i and d_i represent combined mass terms, including added mass, and damping coef£cients. The two control inputs are u_1 and u_2. The disturbances w_i only appear in equations (1) through (3) because they represent forces or torques that in¤uence the acceleration of the vehicle.

One of our research objectives is to design the control inputs u_1 and u_2 such that the vehicle tracks a feasible trajectory. A feasible trajectory can be generated by selecting control inputs and simulating the ship model described above without the disturbances. The equations that describe the desired system, are the same as equations (1) through (6), without the disturbances. We will append a subscript d to the variables to indicate the desired values for the states and control inputs.

An equivalent way to state the tracking problem is to stabilize the error dynamics of the system. We can write the error differential equations by subtracting the desired equations from the system equations and de£ning the error variables as $a_e = a - a_d$, where a represents any state or control variable

To help ease notation, we will use the following vector expression to represent the system given in (1)–(6):

$$\dot{\mathbf{q}} = f(t,\mathbf{q}) + B(t)\mathbf{u} + D(t)\mathbf{w}, \quad \mathbf{q}(t_0) = \mathbf{q}_0, \qquad (7)$$

where $f(t,\mathbf{q})$ is a vector-valued nonlinear function and we have used the following de£nitions:

$$\begin{aligned}
\mathbf{q} &:= \begin{bmatrix} u & v & r & x & y & \psi \end{bmatrix}'\\
\mathbf{u} &:= \begin{bmatrix} u_1 & u_2 \end{bmatrix}'\\
\mathbf{w} &:= \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \end{bmatrix}'.
\end{aligned}$$

Using the vector form of the equations will show how the design techniques are relatively general and not tailored to a speci£c model.

3 Perfect State Feedback Tracking Control

We begin the design process with the perfect state feedback tracking controller. Our current objective is to design a control law to track a feasible trajectory and to reject the effect of disturbances. We will assume that the feasible trajectory has already been created and that we have full knowledge of its required control inputs. At this stage, we also assume we can use the full state vector in the feedback control law. The initial control law uses H^∞-optimal design techniques applied to a linearized version of the system model to follow the reference trajectory. We prove that the solution to the linearized version of the problem provides a locally exponentially stabilizing control law for the nonlinear system and establish conditions under which the solution is valid.

The equations of motion for the vehicle are given by (7) and the initial condition $\mathbf{q}_0$ does not necessarily agree with the desired initial con£guration for the vehicle. The desired system uses the control inputs $\mathbf{u}_d(t)$ to generate the feasible trajectory we want to track. The desired trajectory is represented by the differential equation

$$\dot{\mathbf{q}}_d(t) = f(t,\mathbf{q}_d) + B(t)\mathbf{u}_d(t), \quad \mathbf{q}_d(t_0) = \mathbf{q}_{d0} \qquad (8)$$

where we assume that the initial condition $\mathbf{q}_{d0}$ is known. We want the actual trajectory to approach the desired trajectory, so we consider the error dynamics. If we subtract (8) from (7) we arrive at the equations for the error system

$$\dot{\mathbf{q}}_e(t) = f_e(t,\mathbf{q}_e) + B(t)\mathbf{u}_e(t) + D(t)\mathbf{w}(t), \quad \mathbf{q}_e(t_0) = \mathbf{q}_{e0} \qquad (9)$$

where $\mathbf{q}_e = \mathbf{q} - \mathbf{q}_d$ and $\mathbf{u}_e = \mathbf{u} - \mathbf{u}_d$. Note that in (9) the $B(t)$ and $D(t)$ matrices are the same as in (7) because the control inputs and disturbances enter the equations linearly.

Tracking the desired trajectory is equivalent to £nding a set of control inputs $\mathbf{u}_e(t)$ for the error system to drive the error state to the origin and keep it there. If we can £nd these control inputs, then we can calculate the tracking control inputs for the underactuated vehicle as $\mathbf{u} = \mathbf{u}_d + \mathbf{u}_e$.

The tracking controller uses a relatively simple linearization approach to achieve surprisingly good results. We start with the error differential equations in (9) and linearize the nonlinear system about the origin $\mathbf{q}_e = 0$ to get the new system matrices

$$A_e(t) = \left. \frac{\partial f_e(t, \mathbf{q}_e)}{\partial \mathbf{q}_e} \right|_{\mathbf{q}_e = 0}, \quad B_e(t) = B(t), \quad D_e(t) = D(t). \tag{10}$$

Note that linearizing the error equations about the origin is equivalent to linearizing the actual system equations about the desired trajectory. We can now write (9) as

$$\dot{\mathbf{q}}_e = A_e(t)\mathbf{q}_e + B_e(t)\mathbf{u}_e + D_e(t)\mathbf{w} + [f_e(t, \mathbf{q}_e) - A_e(t)\mathbf{q}_e] \tag{11}$$

with the same initial condition and where the term in square brackets is the remaining nonlinear portion and is $o(|\mathbf{q}_e|)$.

We will start the controller design process by formulating an optimal disturbance attenuation problem and then modify the results to prove local exponential stability. For the initial design, we will ignore the nonlinear term in square brackets in (11), but we will account for it in our stability analysis.

If we associate a quadratic performance index with the linear portion of the system given in (11), we can compute a solution to the H^∞ control problem and £nd the optimal disturbance attenuating controller for the *linear* problem. Accordingly, we select the performance index to be

$$\begin{aligned} L_\gamma(\mathbf{u}_e, \mathbf{w}) \quad = \quad & \int_{t_0}^{t_f} \left\{ |\mathbf{q}_e(t)|^2_{Q(t)} + |\mathbf{u}_e(t)|^2 - \gamma^2 |\mathbf{w}(t)|^2 \right\} dt \\ & + |\mathbf{q}_e(t_f)|^2_{Q_f} \end{aligned} \tag{12}$$

where $Q_f > 0$, $Q(t) \geq 0$, $t \in [t_0, t_f]$, and $\gamma > 0$. The solution to the optimal control problem depends on £nding a unique positive de£nite solution, $Z_\gamma(t)$, to the generalized Riccati differential equation (GRDE)

$$\begin{aligned} \dot{Z} + A_e' Z + Z A_e + Q - Z \left(B_e B_e' - \gamma^{-2} D_e D_e' \right) Z = 0, \\ Z(t_f) = Q_f. \end{aligned} \tag{13}$$

Following the development in [1], we de£ne the in£mum of values for γ that allow a solution to (13) as

$$\begin{aligned} \gamma^* := \inf\{\gamma > 0 \ : & \text{ The GRDE (13) does not have a} \\ & \text{conjugate point on } [t_0, t_f]\}. \end{aligned} \tag{14}$$

As described in [1], there always exists a γ^* such that for any $\gamma > \gamma^*$ we can £nd a positive de£nite solution to (13). The solution to the GRDE leads to a unique feedback saddle-point solution to the differential game, with

$$\mathbf{u}^*(t, \mathbf{q}_e(t)) \quad = \quad -B'(t) Z_\gamma(t) \mathbf{q}_e(t) \tag{15}$$

$$\mathbf{w}^*(t, \mathbf{q}_e(t)) \quad = \quad \gamma^{-2} D'(t) Z_\gamma(t) \mathbf{q}_e(t), \quad t \geq t_0 \tag{16}$$

where $\mathbf{u}^*$ and $\mathbf{w}^*$ are the optimal control and worst-case disturbance, respectively.

We can solve (13) in reverse time for $Z_\gamma(t)$ because A_e depends on only the desired trajectory, which we assumed can be computed in advance, and we can pick $\gamma > \gamma^*$ and $Q_f > 0$ in our design. The computational requirements for the solution depend on the £delity of the model (i.e., the size of the state vector) and the length of the time interval for the motion.

The control input $\mathbf{u}^*$ provides an optimal controller for the linearized error equations (11), so we will set $\mathbf{u}_e = \mathbf{u}^*$. We construct the locally optimal solution for (7) by summing $\mathbf{u}_e(t) + \mathbf{u}_d(t) =: \mathbf{u}(t)$, since we know $\mathbf{u}_d(t)$ in advance. We can then apply

$$\mathbf{u}(t) = \mathbf{u}_d(t) - B'(t) Z_\gamma(t) \mathbf{q}_e(t) \tag{17}$$

as the state feedback controller for the full nonlinear system.

Before we present the stability proof for this design, we make one slight adjustment to the control law that introduces the design parameter $\kappa(t)$. Set the controller for the error system to be

$$\mathbf{u}_e(t) = -\kappa(t) B' Z_\gamma(t) \mathbf{q}_e(t) \tag{18}$$

where $\kappa(t) > 0$ for all $t \in [t_0, t_f]$. This control law is just a scaled version of the optimal control law and offers some additional ¤exibility in the design. We can now prove that (18) leads to a locally exponentially stable closed-loop system under mild restrictions on $\kappa(t)$.

To prove that the controller provides reasonable tracking performance, we examine the stability of the closed-loop system without the presence of disturbances. We will work directly with the error equations for the system and we rewrite (11) without the disturbances to get

$$\dot{\mathbf{q}}_e = A_e \mathbf{q}_e + B_e \mathbf{u}_e + o(|\mathbf{q}_e|). \tag{19}$$

Assume we have picked $Q = qI$, $Q_f = q_f I$, where q and q_f are positive scalar constants, and $\gamma > \gamma^*$ to solve (13) for $Z_\gamma(t)$. If we apply control law (18) and suppress the γ subscript on Z_γ, we get the following closed-loop system

$$\dot{\mathbf{q}}_e = \left(A_e - \kappa B_e B_e' Z \right) \mathbf{q}_e + o(|\mathbf{q}_e|). \tag{20}$$

We choose as a candidate Lyapunov function

$$V(t, \mathbf{q}_e) = \mathbf{q}_e' Z(t) \mathbf{q}_e, \tag{21}$$

and note that for all $t \in [t_0, t_f]$ we have that $V(t, \mathbf{q}_e)$ is positive de£nite and radially unbounded because $Z(t) > 0$ for $t \in [t_0, t_f]$. After solving for $Z(t)$, we can also £nd real constants m and M such that

$$0 < mI \leq Z(t) \leq MI. \tag{22}$$

Since $A_e(t)$ is time-varying and depends on the desired trajectory, the constants m and M will have to be determined numerically. To show that the closed-loop system is locally exponentially stable, we want to show that the time derivative of the candidate Lyapunov function is negative de£nite in a neighborhood of the origin. We compute the time derivative as

$$\begin{aligned}\dot{V} &= -\mathbf{q}_e'\left[Q+(2\kappa-1)ZB_eB_e'Z+\gamma^{-2}ZD_eD_e'Z\right]\mathbf{q}_e \\ &\quad +2\mathbf{q}_e'Z\,o(|\mathbf{q}_e|). \end{aligned} \tag{23}$$

If we choose $\kappa(t) \geq \frac{1}{2}$ on $[t_0, t_f]$, then for all $t \in [t_0, t_f]$ we have

$$\left[Q+(2\kappa-1)ZB_eB_e'Z+\gamma^{-2}ZD_eD_e'Z\right] > 0.$$

The remaining term in (23) is sign inde£nite, but vanishes as $|\mathbf{q}_e|$ approaches the origin, so we can use the bounds on Z and our knowledge of Q to £nd a neighborhood of the origin where $\dot{V} < 0$. Following the approach presented in [7], we can £nd some $\varepsilon > 0$ such that

$$o(|\mathbf{q}_e|) \;\leq\; \frac{1}{4}\frac{q}{M}|\mathbf{q}_e|, \quad \forall\,\mathbf{q}_e \text{ such that } |\mathbf{q}_e| < \varepsilon. \tag{24}$$

Using (24) and standard arguments [6], we can show that in the ε neighborhood of the origin, we have exponential stability for the closed-loop system. Several remarks about the control law are in order.

Remark 1. We do not take advantage of the term $[(2\kappa-1)ZB_eB_e'Z+\gamma^{-2}ZD_eD_e'Z]$ in our stability analysis. This term is nonnegative de£nite for $\kappa \geq \frac{1}{2}$ and will improve the stability of the design. $\diamond$

Remark 2. Even though we ignore the disturbances in the stability analysis, we expect the state feedback control law to exhibit reasonable disturbance attenuation properties. The reason for this expectation is that the control law closely resembles the optimal disturbance attenuating control law for the linearized system and uses the same $Z(t)$ matrix. If we choose $\kappa(t) \equiv 1$, we recover the optimal disturbance attenuating controller. $\diamond$

Remark 3. We are using a control law based on the linearization of the system, so we cannot draw any global conclusions about stability or the disturbance attenuation properties of the closed-loop nonlinear system. $\diamond$

This section has described the state feedback control law for an underactuated vehicle and outlined a proof of local exponential stability for the closed-loop system. The control approach relies on linearizing the equations of motion about the candidate trajectory and applying H^∞ design tools to determine the control inputs. The next section will address the imperfect state measurement version of the tracking problem.

4 Imperfect State Measurement Tracking Control

The controller developed in the previous section relies on perfect state measurements for the feedback law. In a realistic situation, the entire state for the system usually cannot be measured. In our example of an underactuated ship, we can expect to directly measure the position and orientation of the vehicle, but not the velocities. In addition, the position and orientation measurements will likely be corrupted by disturbances, which we will have to account for in the analysis. We will now modify the controller designed in the previous section to handle the imperfect state measurement case. We still rely on the H^∞ design tools to develop the solution and will exploit a unique structure in the model for the underactuated vehicle to achieve the results. The approach will allow us to prove a modi£ed version of disturbance attenuation.

As before, we want to determine the control inputs to cause the underactuated vehicle to track a feasible trajectory and attenuate the effect of disturbances. The feasible trajectory will be constructed in advance and the controller will be able to use the inputs which generated the trajectory.

We will use the same equations for the actual, desired and error systems as presented in Section 3. We introduce the following measurement equation

$$\mathbf{y}(t) = C(t)\mathbf{q}(t) + E(t)\mathbf{w}(t). \tag{25}$$

Our approach will be to use the measurements $\mathbf{y}(t)$ to estimate the full state of the system $\mathbf{q}(t)$ and then substitute the estimate for the actual state in the feedback controller developed in Equation (17). We will denote the estimate for the state as $\hat{\mathbf{q}}$.

The £rst step in this design process is to use the structure of the model for the underactuated vehicle to write the equations of motion in a more convenient format. The nonlinear error equations for the vehicle can be decomposed into two sets of equations, where each equation is af£ne if the state of the other subsystem is known. For this analysis, we will suppress the e subscript on the error equations. The equations for the two subsystems are given as

$$\dot{\mathbf{q}}_1 = A_{11}(\mathbf{q}_2)\mathbf{q}_1 + \alpha_1(\mathbf{q}_2) + B_1\mathbf{u}_1 + D_1\mathbf{w}_1 \tag{26}$$

$$\dot{\mathbf{q}}_2 = A_{22}(\mathbf{q}_1)\mathbf{q}_2 + \alpha_2(\mathbf{q}_1) + B_2\mathbf{u}_2 + D_2\mathbf{w}_2 \tag{27}$$

where

$$\mathbf{q} = \begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix}\mathbf{u}_1\\ \mathbf{u}_2\end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix}\mathbf{w}_1\\ \mathbf{w}_2\end{bmatrix}. \tag{28}$$

For our model, we have

$$\mathbf{q}_1 = \begin{bmatrix} u & v & x & y \end{bmatrix}', \quad \mathbf{q}_2 = \begin{bmatrix} r & \psi \end{bmatrix}'$$
$$\mathbf{u}_1 = u_1, \quad \mathbf{u}_2 = u_2$$
$$\mathbf{w}_1 = \begin{bmatrix} w_1 & w_2 & w_4 & w_5 \end{bmatrix}', \quad \mathbf{w}_2 = \begin{bmatrix} w_3 & w_6 \end{bmatrix}'.$$

The $\alpha_i(\mathbf{q}_j)$ terms are nonlinear and we would like to make a £rst order approximation of them. To do so, we take the Jacobian of the $\alpha_i(\mathbf{q}_j)$ terms with respect to $\mathbf{q}_j$ to identify the linear portions

$$A_{12} = \left.\frac{\partial\alpha_1(\mathbf{q}_2)}{\partial\mathbf{q}_2}\right|_{\mathbf{q}_2=0} \qquad A_{21} = \left.\frac{\partial\alpha_2(\mathbf{q}_1)}{\partial\mathbf{q}_1}\right|_{\mathbf{q}_1=0}.$$

We now rewrite the subsystem equations as

$$\begin{aligned}\dot{\mathbf{q}}_1 &= A_{11}(\mathbf{q}_2)\mathbf{q}_1 + A_{12}\mathbf{q}_2 + [\alpha_1(\mathbf{q}_2) - A_{12}\mathbf{q}_2] \quad (29)\\ &\quad + B_1\mathbf{u}_1 + D_1\mathbf{w}_1\\ \dot{\mathbf{q}}_2 &= A_{22}(\mathbf{q}_1)\mathbf{q}_2 + A_{21}\mathbf{q}_1 + [\alpha_2(\mathbf{q}_1) - A_{21}\mathbf{q}_1] \quad (30)\\ &\quad + B_2\mathbf{u}_2 + D_2\mathbf{w}_2.\end{aligned}$$

Combine the subsystem equations to get

$$\begin{aligned}\dot{\mathbf{q}} &= \begin{bmatrix} A_{11}(\mathbf{q}_2) & A_{12} \\ A_{21} & A_{22}(\mathbf{q}_1) \end{bmatrix} \begin{bmatrix} \mathbf{q}_1 \\ \mathbf{q}_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} \alpha_1(\mathbf{q}_2) - A_{12}\mathbf{q}_2 \\ \alpha_2(\mathbf{q}_1) - A_{21}\mathbf{q}_1 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{bmatrix} \\ &\quad + \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} \begin{bmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \end{bmatrix}\end{aligned}$$

which we write compactly as

$$\dot{\mathbf{q}} = A(\mathbf{q})\mathbf{q} + \alpha(\mathbf{q}) + B\mathbf{u} + D\mathbf{w}. \quad (31)$$

We are going to estimate the state $\mathbf{q}$ using $\mathbf{y}$ so we rewrite (31) as

$$\begin{aligned}\dot{\mathbf{q}} &= A(\hat{\mathbf{q}})\mathbf{q} + \alpha(\hat{\mathbf{q}}) + B\mathbf{u} + D\mathbf{w} \quad (32)\\ &\quad + [A(\mathbf{q})\mathbf{q} - A(\hat{\mathbf{q}})\mathbf{q} + \alpha(\mathbf{q}) - \alpha(\hat{\mathbf{q}})]\end{aligned}$$

where the estimate $\hat{\mathbf{q}}$ depends on past measurements $y_{[t_0,t]} := \{y(\tau) \,:\, \tau \in [t_0,t]\}$.

We can consider (32) as an affine system with time-varying elements and a nonlinear perturbation term in square brackets. We would like to use results similar to those by Pan and Başar [4] to prove disturbance attenuation for this system. To mimic the problem presented in [4], we use a change of variables to rewrite (32) as

$$\begin{aligned}\dot{\mathbf{q}} &= A(\hat{\mathbf{q}})\mathbf{q} + \alpha(\hat{\mathbf{q}}) + B\mathbf{u} + D\mathbf{w} \quad (33)\\ &\quad + \varepsilon\left[a(t,\mathbf{q}) + b(t,\mathbf{q})\mathbf{u} + d(t,\mathbf{q})\mathbf{w}\right].\end{aligned}$$

Equation (33) is similar to Equation (1a) in [4], with the difference being that (33) contains the affine term $\alpha(\hat{\mathbf{q}})$. We can develop a disturbance attenuating controller with the affine term in the equations if we make minor modifications to the derivation in [4].

The candidate control law will substitute the estimate for the state, $\hat{\mathbf{q}}$, in the control law (17) from the linearized error equations to get

$$\mu(t, \mathbf{y}_{[t_0,t]}) = -B'(t)Z(t)\hat{\mathbf{q}}. \quad (34)$$

We use the following equation to estimate the state of the system

$$\begin{aligned}\dot{\hat{\mathbf{q}}} &= [A - (BB' - \gamma^{-2}DD')Z]\hat{\mathbf{q}} + \alpha(\hat{\mathbf{q}}) \quad (35)\\ &\quad + [I - \gamma^{-2}\Sigma Z]^{-1}\Sigma C' N^{-1}(\mathbf{y} - C\hat{\mathbf{q}}),\\ \hat{\mathbf{q}}(t_0) &= 0\end{aligned}$$

along with the following H^∞-filter error covariance equation

$$\begin{aligned}\dot{\Sigma} &= A\Sigma + \Sigma A' - \Sigma(C'N^{-1}C - \gamma^{-2}Q)\Sigma + DD', \quad (36)\\ \Sigma(t_0) &= [Q_0 - \eta I]^{-1}\end{aligned}$$

where $Q(t) \geq 0$, $Q_0 - \eta I > 0$, $\eta > 0$, and $N(t) := E(t)E'(t) > 0$. The performance index associated with the imperfect state measurement case is

$$\begin{aligned}L(\mathbf{u},\mathbf{w},\mathbf{q}_0;\varepsilon) &= |\mathbf{q}(t_f)|^2_{Q_f} + \int_{t_0}^{t_f}\left[|\mathbf{q}(t)|^2_Q + |\mathbf{u}(t)|^2\right. \quad (37)\\ &\quad \left. + \varepsilon\left(q(t,\mathbf{q}(t)) + \mathbf{u}'(t)r(t,\mathbf{q}(t))\mathbf{u}(t)\right)\right]dt.\end{aligned}$$

We will assume that the initial state is unknown and treat it as part of the disturbance. The associated zero-sum differential game has the following soft-constrained cost function

$$L_\gamma(\mathbf{u},\mathbf{w},\mathbf{q}_0;\varepsilon) = L(\mathbf{u},\mathbf{w},\mathbf{q}_0;\varepsilon) - \gamma^2\left(|\mathbf{q}_0|^2_{Q_0} + \|\mathbf{w}\|^2\right). \quad (38)$$

As presented in [6], we can find a solution to dynamic game and prove that for some $M > 0$, we have

$$\sup_{\mathbf{q}_0 \in \mathbb{R}^n} \sup_{\mathbf{w} \in \mathcal{H}_w} \left\{L^* - \gamma^2\left(|\mathbf{q}_0|^2_{Q_0} + \|\mathbf{w}\|^2\right)\right\} \leq M. \quad (39)$$

Inequality (39) implies that the difference between L^* and $\gamma^2\left(|\mathbf{q}_0|^2_{Q_0} + \|\mathbf{w}\|^2\right)$ is always bounded by a fixed value. This translates into a condition relating the rates of growth for L^* and the disturbance terms. This result is a modified form of disturbance attenuation for the nonlinear system. Since, in general, we have to account for the affine terms which are sign indefinite, we can only guarantee the modified form of disturbance attenuation.

Our analysis has designed a control law for the imperfect state measurement case and proven a form of local disturbance attenuation for the closed-loop system. In Section 5 we will simulate a system using this controller, as well as the state feedback controller designed in Section 3, to examine their tracking and disturbance attenuation performance.

5 Simulation Results

This section presents the results of simulations that implement the the state feedback controller and the imperfect state measurement controller algorithms. For the simulations, we used the following values for the coefficients in equations (1) through (3):

$$\begin{gathered} m_u = 0.5, \quad m_v = -2.0, \quad m_r = 0.5 \\ d_u = 1.0, \quad d_v = 2.0, \quad d_r = 1.0. \end{gathered}$$

The simulations fixed the disturbance attenuation parameter at $\gamma = 5$ and use weighting matrices $Q = I$ and $Q_f = I$.

The state feedback controller simulations start the vehicle with an initial condition that did not match the desired initial condition and monitored the tracking and disturbance attenuation performance of the system. Figure 1 displays the results for one sample desired trajectory where the system disturbances were sinusoidal signals with $w_1 = w_2 = w_3 = 0.1\cos[2\pi(0.2)t]$. The disturbances have amplitudes which are 10% of the magnitude of the input signal used to create the desired trajectory. The controller successfully corrected for the poor initial condition and approximately tracked the desired trajectory. We note that the vehicle makes a natural maneuver to recover from the poor initial condition and capture the desired trajectory.

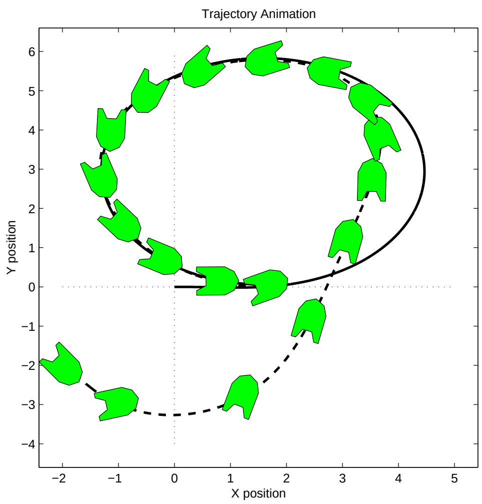

Figure 1: State feedback tracking control results. The desired (solid) and actual (dashed) *x*-*y* trajectories are shown.

We then applied the imperfect state measurement tracking controller to the same desired trajectory and system disturbances used in the £rst simulation. The only change in the problem was that we also estimated the state of the system. The measurement noise was sampled white noise with $\sigma = 0.5$. Figure 2 shows the results. The simulation indicates that the imperfect state measurement controller performs very well in terms of tracking and attenuating the disturbances.

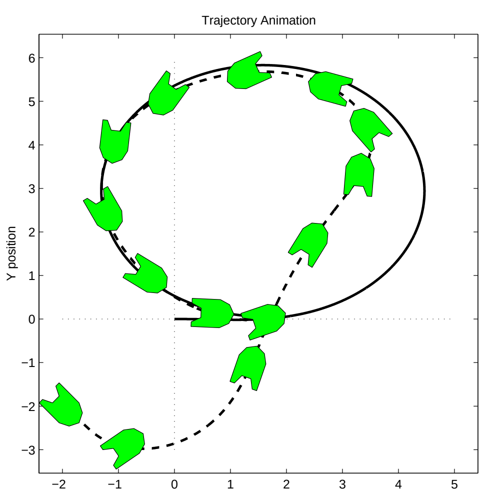

Figure 2: Imperfect state measurement tracking control results.

6 Conclusion

We have developed in this paper a controller that allows a nonlinear underactuated vehicle to track a feasible trajectory in the presence of disturbances and when only imperfect state measurements are available. The design process starts with a perfect state feedback controller that uses linearization about the desired trajectory to £nd an optimal control law. We proved local exponential stability for the closed loop state feedback system. The approach then incorporates an H^∞-optimal estimate for the states to handle the imperfect state measurement case. We then proved a form of local disturbance attenuation for the resulting system, and presented the results of two simulations to illustrate the performance of the approach.

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