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On the homogeneity of the affine connection model for mechanical control systems

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Abstract

This work presents a review of a number of control results for mechanical systems. The key technical advances derive from the homogeneity properties of affine connections models for a large class of mechanical systems. Recent results on nonlinear controllability and on series expansions are presented in a unified fashion.

1 Introduction

In this paper we provide a review of some recent work concerning the employment of an affine connection framework to study mechanical systems. The emphasis of the presentation here is on the homogeneity properties possessed by these systems, and how this arises in various results, especially those of the authors. It is this property of homogeneity which accounts for a great deal of the structure seen in so-called affine connection control systems. The structure of these systems makes them an ideal proving ground for many techniques in nonlinear control—the systems are simple enough that one may fruitfully approach difficult problems, but are nontrivial enough to require sophisticated machinery to have any degree of success. For example, typical linearization techniques are not useful in this category of control systems.

The classic structure of mechanical system exploited in stabilization problems is *passivity*. Indeed, numerous important control problems rely in their essence on the existence of a total energy function and its use as a candidate Lyapunov function, see for example the books [2, 20, 23]. This paper focuses on a different property of mechanical systems, i.e., their *homogeneity*. This property characterizes the Lie algebraic structure of mechanical systems, and accordingly, it plays a key rôle in nonlinear controllability [15], normal forms [12, 16], series expansions [5], algorithms for motion planning [7, 21] averaging via the average potential [3, 6], optimal control [13], and various other areas of control theory. It is a contention of this paper that this property has for long time been neglected and that instead its consequences should be investigated in greater detail.

This paper relies on the notion of affine connection control systems to model a large class of systems which are of current interest in the control community. Broadly speaking, Lagrangian mechanical systems with kinetic energy Lagrangians are effectively modeled in the affine connection framework, and this is the topic of the papers [15] and [7]. If one adds constraints linear in velocity to this class of systems, the resulting systems may still be modeled using affine connections, and the control setting here is described by Lewis [14].

2 Affine connections and mechanics

In this section we begin with a brief overview of affine connections, and how they come up in mechanics. We also mention how homogeneity enters the picture in terms of the basic problem data for the control problem.

2.1 Affine connections

We refer to [10] for a comprehensive treatment on affine connections and Riemannian geometry. An *affine connection* on a manifold Q is a smooth map that assigns to a pair of vector fields X, Y a vector field $\nabla_X Y$ such that for any function f and for any third vector field Z:

1. $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$;
2. $\nabla_X(fY + Z) = (\mathcal{L}_X f)Y + f\nabla_X Y + \nabla_X Z$.

We also say that $\nabla_X Y$ is the *covariant derivative* of Y with respect to X. Vector fields can also be covariantly differentiated along curves, and this concept will be instrumental in writing the Euler-Lagrange equations. Consider a smooth curve $\gamma\colon [0, 1] \to Q$ and a vector field along γ, i.e., a map $v\colon [0, 1] \to TQ$ such that $\pi(v(t)) = \gamma(t)$ for all $t \in [0, 1]$ (π is the tangent bundle projection). Let the vector field V satisfy

$V(\gamma(t)) = v(t)$. The *covariant derivative of the vector field v along γ* is defined by

$$\frac{Dv(t)}{dt} \triangleq \nabla_{\dot\gamma(t)} v(t) = \nabla_{\dot\gamma(t)} V(q)\big|_{q=\gamma(t)}.$$

In a system of local coordinates $(q^1, \dots, q^n)$, an affine connection is uniquely determined by its *Christoffel symbols* Γ^i_{ij}:

$$\nabla_{\frac{\partial}{\partial q^i}} \Big(\frac{\partial}{\partial q^j}\Big) = \Gamma^k_{ij} \frac{\partial}{\partial q^k},$$

and accordingly, the covariant derivative of a vector field is written as

$$\nabla_X Y = \Big(\frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k\Big) \frac{\partial}{\partial q^i}.$$

In settings where Q possesses a Riemannian metric g (such as is provided, for example, by kinetic energy), one derives a canonical affine connection associated with g. This connection is called the *Levi-Civita* affine connection, and is most directly characterized by its Christoffel symbols, which are given in terms of the metric components as follows:

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\ell} \Big(\frac{\partial g_{j\ell}}{\partial q^k} + \frac{\partial g_{k\ell}}{\partial q^j} - \frac{\partial g_{jk}}{\partial q^\ell}\Big).$$

Although this *is* the affine connection commonly seen in applications, our treatment here is not restricted to the Levi-Civita connection. Indeed, systems with nonholonomic constraints are directly modelled within the same framework, but not using Levi-Civita connections; see [14].

2.2 Control systems described by affine connections

We introduce a class of control systems that is a generalization of Lagrangian control systems. This approach to modeling of vehicles and robotic manipulators is common to a number of recent works; see [4, 15, 14, 5]. An *affine connection control system* is defined by the following objects:

1. an n-dimensional configuration manifold Q, with $q \in Q$ being the configuration of the system and $v_q \in T_qQ$ being the system's velocity,
2. an affine connection ∇ on Q, whose Christoffel symbols are $\{\Gamma^i_{jk} : i, j, k = 1, \dots, n\}$,
3. a family of input vector fields $\mathcal{Y} = \{Y_1, \dots, Y_m\}$ on Q.

The corresponding equations of motion are written as

$$\nabla_{\dot q(t)} \dot q(t) = u^a(t) Y_a(q(t)) \tag{1}$$

or equivalently in coordinates as

$$\ddot q^i + \Gamma^i_{jk}(q) \dot q^j \dot q^k = u^a(t) Y^i_a(q), \tag{2}$$

where the indices $i, j, k \in \{1, \dots, n\}$. These equations are a generalized form of the Euler-Lagrange equations. That is to say, if one takes for ∇ the Levi-Civita affine connection associated with a kinetic energy Riemannian metric g, then the equations (1) are the forced Euler Lagrange equations for the associated kinetic energy Lagrangian and with input forces modeled by the vector fields $Y_1, \dots, Y_m$. However, as we mentioned in the previous section, we do not wish to restrict our attention to Levi-Civita affine connections, and so the equations (1) in consequence give, for example, the forced equations of motion for a nonholonomic system with a kinetic energy Lagrangian, and constraints linear in velocity.

The second-order system in equation (2) can be written as a first-order differential equation on the tangent bundle TQ. Using $\{\frac{\partial}{\partial q^i}, \frac{\partial}{\partial v^i}\}$ as a basis for vector fields on the tangent bundle to TQ, we define vector fields Z and Y^{lift}_a, $a = 1, \dots, m$, on TQ by

$$Z(v_q) = v^i \frac{\partial}{\partial q^i} - \Gamma(q)^i_{jk} v^j v^k \frac{\partial}{\partial v^i},$$
$$Y^{\text{lift}}_a(v_q) = Y^i_a(q) \frac{\partial}{\partial v^i},$$

so that the control system is rewritten as

$$\dot v(t) = Z(v(t)) + u^a(t) Y^{\text{lift}}_a(v(t)), \tag{3}$$

where $t \mapsto v(t)$ is now a curve in TQ describing the evolution of a first-order control affine system. We refer to [15, 11] for coordinate independent definitions of the lifting operation $Y_a \to Y^{\text{lift}}_a$ and of the drift vector field Z. The latter vector field is called the *geodesic spray*.

2.3 Homogeneity and Lie algebraic structure

One fundamental structure of the control system in equation (1) is the polynomial dependence of the vector fields Z and Y^{lift} on the velocity variables v^i. This structure leads to some enormous simplifications when performing iterated Lie brackets between the vector fields in the set $\{Z, Y^{\text{lift}}_1, \dots, Y^{\text{lift}}_m\}$. Apart from the papers of the authors concerning the consequences of the structure of these Lie brackets, we refer to the work of Sontag and Sussmann [22] on time-optimal control for robotic manipulators. Other work on optimal control for affine connection systems is that of Lewis [13].

We focus here on the notion of *geometric homogeneity*[1] as described in [9]. Generally, given two vector fields X and X_E, we say that the vector field X is *homogeneous*

[1] Geometric homogeneity corresponds to the existence of an infinitesimal symmetry in the equations of motion. For control systems described by an affine connection the symmetry is invariance under affine time-scaling transformations.

with degree $m \in \mathbb{Z}$ with respect to X_E if

$$[X_E, X] = mX.$$

For affine connection control systems, we introduce the Liouville vector field L on TQ (see [17, page 64]), as

$$L = v^i \frac{\partial}{\partial v^i}.$$

Straightforward computations verify the following.

Lemma 2.1 *Let ∇ be an affine connection on Q with geodesic spray Z, and let Y be a vector field on Q. The following statements hold:*

1. $[L, Z] = (+1)Z$;
2. $[L, Y^{lift}] = (-1)Y^{lift}$.

Hence, the vector field Z is homogeneous of degree $+1$, and the vector field Y^{lift} is homogeneous of degree -1 with respect to the Liouville vector field. In the following, a vector field X on TQ is simply *homogeneous of degree $m \in \mathbb{Z}$* if it is homogeneous of degree m with respect to L. Let $\mathcal{P}_j$ be the set of vector fields on TQ of homogeneous degree j, so that

$$Z \in \mathcal{P}_1, \quad \text{and} \quad Y^{\text{lift}} \in \mathcal{P}_{-1}.$$

Let us leave our general discussion of homogeneity at that for the moment, and in the next section we will investigate these properties, and illustrate how they may be used in an investigation of nonlinear controllability for affine connection control systems.

3 Controllability of affine connection control systems

The matter of controllability for affine connection control systems was first undertaken systematically by Lewis and Murray [15]. Here the precise character of the Lie bracket structure for affine connection control systems was undertaken, and in the presence of a potential energy term in the Lagrangian. In this section, we distill the essence of this structure without potential energy (with potential energy, the systems are not affine connection control systems as we have defined them in Section 2.2). As we shall see, the resulting structure allows us to quickly understand the character of the set of configurations one can reach starting from a state with zero velocity.

First we observe that the sets $\mathcal{P}_j$ enjoy various interesting properties: Figure 1 illustrates them, and their proof is via direct computation. Here are these properties illustrated in the table, but expressed via formulas:

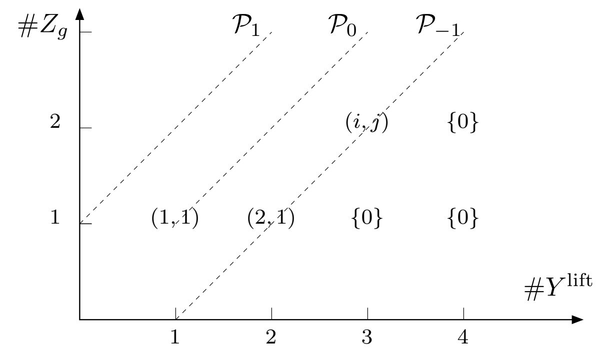

Figure 1: Table of Lie brackets between the drift vector field Z and the input vector fields Y_a^{lift}. The (i, j)th position contains brackets with i copies of Y^{lift} and j copies of Z. The corresponding homogeneous degree is $j - i$. All Lie brackets to the right of $\mathcal{P}_{-1}$ vanish. All Lie brackets to the left of $\mathcal{P}_{-1}$ vanish when evaluated at $v_q = 0_q$.

1. $[\mathcal{P}_i, \mathcal{P}_j] \subset \mathcal{P}_{i+j}$, i.e., the Lie bracket between a vector field in $\mathcal{P}_i$ and a vector field in $\mathcal{P}_j$ belongs to $\mathcal{P}_{i+j}$;
2. $\mathcal{P}_k = \{0\}$ for all $k \leq -2$;
3. for all $X(v_q) \in \mathcal{P}_k$ with $k \geq 1$, $X(0_q) = 0_q$.

The key observation here is that all brackets are homogeneous of *some* degree, and if one is evaluating brackets at points of zero velocity, the only contributions will come from those brackets which are homogeneous of degree -1 or 0. It turns out that one can characterize these brackets, and this is exactly what is undertaken by Lewis and Murray [15].

To understand what a vector field from $\mathcal{P}_i$ looks like, let us work in local coordinates. We write a vector field X on TQ as

$$X = X_h^i \frac{\partial}{\partial q^i} + X_v^i \frac{\partial}{\partial v^i}. \tag{4}$$

Here we think of the components X_h^i, $i = 1, \ldots, n$, as being "horizontal" and the components X_v^i, $i = 1, \ldots, n$, as being "vertical." Let $\mathcal{H}_i$ be the set of scalar functions in the local chart for TQ which are arbitrary functions of q and which are homogeneous polynomials in $\{v^1, \ldots, v^n\}$ of degree i. One verifies that a vector field X on TQ of the form (4) is in $\mathcal{P}_i$ exactly when the functions X_h^i, $i = 1, \ldots, n$, are in $\mathcal{H}_i$, and the functions X_v^i, $i = 1, \ldots, n$, are in $\mathcal{H}_{i+1}$.

Let us focus for a moment on the Lie bracket $[Y_a^{\text{lift}}, [Z, Y_b^{\text{lift}}]]$ where $a, b \in \{1, \ldots, m\}$. Since this Lie bracket belongs to $\mathcal{P}_{-1}$, there must exist a vector field on Q, which we denote $\langle Y_a : Y_b \rangle$, such that

$$\langle Y_a : Y_b \rangle^{\text{lift}} = [Y_b^{\text{lift}}, [Z, Y_a^{\text{lift}}]].$$

This vector field we call the *symmetric product* between Y_a and Y_b and a direct computation shows that it satisfies

$$\langle Y_b : Y_a \rangle = \nabla_{Y_a} Y_b + \nabla_{Y_b} Y_a,$$

or equivalently in coordinates

$$\langle Y_b : Y_a \rangle^i = \frac{\partial Y_a^i}{\partial q^j} Y_b^j + \frac{\partial Y_b^i}{\partial q^j} Y_a^j + \Gamma^i_{jk}\big(Y_a^j Y_b^k + Y_a^k Y_b^j\big).$$

The adjective "symmetric" comes from the obvious equality $\langle Y_a : Y_b \rangle = \langle Y_b : Y_a \rangle$. It turns out, in fact, that *all* Lie brackets of vector fields from the set $\{Z, Y_1^{\text{lift}}, \dots, Y_m^{\text{lift}}\}$ which are also vector fields in $\mathcal{P}_{-1}$ are vertical lifts of iterated symmetric products of the vector fields $\{Y_1, \dots, Y_m\}$. We denote distribution spanned by all such iterated symmetric products by $\overline{\text{Sym}}(\mathcal{Y})$.

Now let us focus on another type of bracket, those of the form $[[Z, Y_a^{\text{lift}}], [Z, Y_b^{\text{lift}}]]$ for $a, b \in \{1, \dots, m\}$. This bracket, under our classification scheme, is in $\mathcal{P}_0$. Therefore, if $\tau_Q \colon TQ \to Q$ is the tangent bundle projection, there is a vector field X_{ab} on Q which satisfies

$$\tau_Q([[Z, Y_a^{\text{lift}}], [Z, Y_b^{\text{lift}}]](v_q)) = X_{ab}(q).$$

A routine computation shows that in fact $X_{ab} = -[Y_a, Y_b]$. Thus when we evaluate brackets in $\mathcal{P}_0$, we expect to get something involving Lie brackets of vector fields whose vertical lifts are brackets from $\mathcal{P}_{-1}$. Indeed, all Lie brackets of vector fields from the set $\{Z, Y_1^{\text{lift}}, \dots, Y_m^{\text{lift}}\}$ which are also vector fields in $\mathcal{P}_0$ project to a vector field on Q which is a Lie bracket of two iterated symmetric products. Somewhat more precisely, if $\overline{\text{Lie}}(D)$ denotes the smallest integrable distribution containing a distribution D, the distribution on Q generated by the projection to Q of brackets from $\mathcal{P}_0$ is given by $\overline{\text{Lie}}(\overline{\text{Sym}}(\mathcal{Y}))$.

To summarize, we have made believable the following result of Lewis and Murray [15].

Theorem 3.1 *For an analytic affine connection control system, the set of configurations reachable from $q \in Q$ starting at zero initial velocity forms an open subset of the integral manifold through q of the integrable distribution $\overline{\text{Lie}}(\overline{\text{Sym}}(\mathcal{Y}))$.*

To make this precise requires some effort, but we hope to have made it clear the important rôle homogeneity plays in studying affine connection control systems.

4 A series expansion for the forced evolution from rest

As in the previous section, the homogeneity and Lie algebraic structure of mechanical systems leads to a novel characterization of their flow. Assuming zero initial velocity, the evolution of the second order initial value problem in equation (3) can be described via a first order differential equation. Precise statements and proof are available in [5].

In the computation of series expansions as well as in the general study of perturbation methods for differential equations one key tool is the variation of constants formula. We start by introducing some notation. A time-varying vector field $(q,t) \mapsto X(q,t)$ gives rise to the initial value problem on Q

$$\dot{q}(t) = X(q,t), \qquad q(0) = q_0.$$

We denote its solution at time T via $q(T) = \Phi^X_{0,T}(q_0)$. We also refer to it as the flow of X. Consider the initial value problem

$$\dot{q}(t) = X(q,t) + Y(q,t), \qquad q(0) = q_0, \tag{5}$$

where X and Y are analytic (in q) time-varying vector fields. If we regard X as a perturbation to the vector field Y, we can describe the flow of $X+Y$ in terms of a nominal and perturbed flow. The following relationship is referred to as the *variation of constants* formula and describes the perturbed flow:

$$\Phi^{X+Y}_{0,t} = \Phi^Y_{0,t} \circ \Phi^{(\Phi^Y_{0,t})^* X}_{0,t}, \tag{6}$$

where, given any vector field X and any diffeomorphism ϕ, the $\phi^* X$ is the pull-back of X along ϕ. The result is proven in [1, equation (3.15)], see also [5, Appendix A.1]. If X and Y are time-invariant, the classic infinitesimal Campbell-Backer-Hausdorff formula provides a mean of computing the pull-back:

$$(\Phi^Y_{0,t})^* X = \sum_{k=0}^{\infty} \text{ad}_Y^k X \, \frac{t^k}{k!}.$$

If instead X and Y are time-varying, a generalized expression is, see [1]:

$$(\Phi^Y_{0,t})^* X(q,t) = X(q,t) + \sum_{k=1}^{\infty} \int_0^t \cdots \int_0^{s_{k-1}} \big(\text{ad}_{Y(q,s_k)} \dots \text{ad}_{Y(q,s_1)} X(q,t)\big) \, ds_k \dots ds_1. \tag{7}$$

In general the convergence of this series expansion is a delicate matter. However, if the Lie brackets $\text{ad}_{Y(s_k)} \dots \text{ad}_{Y(s_1)} X$ vanish for all k greater than a given N, the series in equation (7) becomes a finite sum, and this is the key observation for us.

Now let us apply this result to the differential equation (3) on TQ, which we rewrite here for convenience:

$$\dot{v} = Z(v) + u^a(t) Y_a^{\text{lift}}(v) \triangleq Z(v) + Y(v,t)^{\text{lift}}.$$

The homogeneous structure described in Figure 1 simplifies the application of the variation of constant formula. Let the geodesic spray Z play the role of the perturbation to the vector field Y^{lift}. Then the infinite series in equation (7) collapses. We briefly illustrate this process in what follows.

Let 0_{q_0} denote the point on TQ, with configuration q_0 and zero velocity. The solution from rest to the previous equation is

$$v(T) = \Phi_{0,T}^{Z+Y^{\text{lift}}}(0_{q_0}).$$

Utilizing equation (6) we compute

$$v(T) = \Phi_{0,T}^{Y^{\text{lift}}}(w(T)) = w(T) + \int_0^t Y(q,s)ds$$

where some straightforward manipulations lead to

$$\begin{aligned}\dot{w}(t) &= \left(\Phi_{0,T}^{Y^{\text{lift}}}\right)^* Z(w(t)) \\ &= Z + \int_0^t [Y^{\text{lift}}(q,s)Z]ds \\ &\quad - \frac{1}{2}\int_0^t\int_0^t \langle Y(q,s_1) : Y(q,s_2)\rangle^{\text{lift}} ds_1 ds_2,\end{aligned}$$

with initial condition $w(0) = 0_{q_0}$. It is worth noting that the transformed initial value problem in w does now again enjoy the same homogeneity properties as the original one in equation (3). In other words, the resulting system satisfies a set of equations similar to the original one, except for some different forcing terms. One can therefore iterate this procedure for an infinite number of times and, under mild assumptions, obtain a locally convergent solution.

To summarize, we have made believable the following result of Bullo [5].

Theorem 4.1 *Define recursively the time-varying vector fields V_k:*

$$\begin{aligned}V_1(q,t) &= \int_0^t u^a(s)Y_a(q)ds \\ V_k(q,t) &= -\frac{1}{2}\sum_{j=1}^{k-1}\int_0^t \Big\langle V_j(q,s) :\ V_{k-j}(q,s)\Big\rangle ds.\end{aligned}$$

The solution $t \to q(t)$ to equation (1) satisfies the formal series expansion

$$\dot{q}(t) = \sum_{k=1}^{+\infty} V_k(q(t),t).$$

5 Simplifications in example systems

While the treatment present up to here is always applicable, there are two situations in which further structure in the affine connection ∇ and in the input forces Y_a simplifies the computation of symmetric products.

5.1 Simple systems with integrable forces

Here we consider systems with Lagrangian equal to "kinetic minus potential" and with integrable forces; such systems are referred to as "simple." In the interest of brevity, we refer to the textbooks [8, 18] for a detailed presentation and review here only the necessary notation. The affine connection of a simple system is the Levi-Civita connection associated with the kinetic energy metric $\langle\langle \cdot\,,\,\cdot\rangle\rangle$. If the control forces are integrable, then for all input vector fields Y_a there exists a scalar function φ_a such that

$$Y_a(q) = \operatorname{grad}\varphi_a(q). \tag{8}$$

One remarkable simplification takes place under these assumptions: the set of gradient vector fields is closed under the operation of symmetric product. Let φ_1, φ_2 be scalar functions on $\mathbb{R}^n$ and define a symmetric product between functions according to

$$\langle\varphi_1 : \varphi_2\rangle \triangleq \langle\langle \operatorname{grad}\varphi_1\,,\,\operatorname{grad}\varphi_2\rangle\rangle.$$

Then the symmetric product of the corresponding gradient vector fields equals the gradient of the symmetric product of the functions. In equations:

$$\langle\operatorname{grad}\varphi_1 : \operatorname{grad}\varphi_2\rangle = \operatorname{grad}\langle\varphi_1 : \varphi_2\rangle\,.$$

Accordingly, Theorem 4.1 can be restated as follows.

Lemma 5.1 *Define recursively the time-varying functions:*

$$\begin{aligned}\phi_1(q,t) &= \int_0^t u^a(s)\varphi_a(q)ds \\ \phi_k(q,t) &= -\frac{1}{2}\sum_{j=1}^{k-1}\int_0^t \Big\langle \phi_j(q,s) :\ \phi_{k-j}(q,s)\Big\rangle ds, \quad k \geq 2.\end{aligned}$$

Then the solution $q : [0,T] \to Q$ satisfies

$$\dot{q}(t) = \operatorname{grad}\sum_{k=1}^{+\infty}\phi_k(q(t),t).$$

In other words, the flow of a simple system forced from rest is written as a time-varying gradient flow.

5.2 Invariant systems on Lie groups

Next we investigate systems with kinetic energy and input forces invariant under a certain group action. These system have a configuration space G with the structure of an n-dimensional matrix Lie group. Examples include satellites, hovercraft, and underwater vehicles.

The equations of motion (1) decouple into a kinematic and dynamic equation in the configuration variable $g \in G$ and the body velocity $v \in \mathbb{R}^n$.[2] The kinematic

[2]More precisely, the body velocity v lives in the Lie algebra of the group G.

equation can be written as a matrix differential equation using matrix group notation $\dot{g} = g\widehat{v}$; we refer to [19] for the details. The dynamic equation, sometimes referred to as Euler-Poincarè, is

$$\dot{v}^i + \gamma^i_{jk} v^j v^k = u^a(t) y^i_a, \tag{9}$$

where the coefficients γ^i_{jk} are uniquely determined from the knowledge of the inertia metric and of the group G. The input vectors y_a are constant.

Within this setting, the result in Theorem 4.1 is summarized as follows. The solution to the equation (9) with initial condition $v(0) = 0$ is $v(t) = \sum_{k=1}^{\infty} v_k(t)$, where

$$\begin{aligned} v_1(t) &= \int_0^t u^a(a) y_a ds \\ v_k(t) &= -\frac{1}{2} \sum_{j=1}^{k-1} \int_0^t \Big\langle v_j(s) : \ v_{k-j}(s) \Big\rangle ds, \qquad k \geq 2, \end{aligned}$$

and where the symmetric product between velocity vectors is $\langle x : y \rangle^i = -2\gamma^i_{jk} x^j y^k$. Local convergence for the series expansion can be easily established in this setting. This result supersedes the treatment in [7].

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