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On Trajectory Optimization for Polynomial Systems via Series Expansions

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Abstract

In this paper, we present algorithms for the design of feasible and optimal trajectories of nonlinear control systems. We focus on stable polynomial control systems linear in the controls. We prove existence of local solutions near the minimum energy control for the linearized system and we investigate provably convergent iterative schemes. Finally, we formulate the trajectory optimization problem as a low dimensional nonlinear program.

1 Introduction

Versatile robots and autonomous vehicles will pervade scientific and commercial applications in the future and will impact a variety of areas such as factory automation, search and rescue operations, oceanographic and aerospace missions, and medical robotics. Technological advances in computing and manufacturing will enable these future devices to operate in diverse environments with increasing levels of autonomy and dexterity. These mechanical systems will interact with diverse environments via rolling constraints, impacts and viscous forces, and the primary tasks requested of them will be fast, reliable motion in Euclidean space.

Current research finds agile and efficient autonomous vehicles, trajectory generation for high dimensional multibody systems, and design and control of biomimetic locomotion devices as particularly relevant applications. The key problem common to these examples is how to quickly compute trajectories that satisfy the nonlinear dynamic equations of the system, as well as the constraints on the control authority.

Various current numerical and analytical techniques tackle this problem. In **numerical optimal control** open loop control and trajectories are obtained through a numerical optimization, see the classic Bryson and Ho [2]. Because the problem is infinite dimensional, various forms of *transcription* (discretization/parametrization) are used to cast the variational problem into a nonlinear program. Nonetheless, nonlinear programming algorithms are limited in speed and reliability. The high dimension and complexity of the motion to be designed render the numerical methods too slow, and the lack of convergence guarantees hinder their relevance in real time applications.

A great deal of mathematical work has been directed in the past towards the analytical understanding of optimal control and locomotion problems. One very successful method involves the notion of **feedback linearization and differential flatness**. Flat systems are presented in Fliess et al. [9], and a catalog of flat systems is described by Murray et al. [14]. However, there are a few fundamental limitations to this approach. Most systems do not fit the restrictive requirements of flatness, nor is there any established notion of approximately flat systems. Lastly, in trajectory optimization problems it is not clear how to deal with control magnitude and rate limits. One can set up a nonlinear program to deal with these limitations, but then the advantages of flatness disappear.

An important method to analyze locomotion and design trajectories revolves around the notion of Lie bracket and of controllability. Systems in chained form and the use of sinusoidal inputs were discussed in Murray and Sastry [15], driftless vehicles were considered in Leonard and Krishnaprasad [12], a very general approach to driftless systems was presented in [11]. We shall refer to these algorithms as **Lie brackets based planners**. These planners have been used only on systems that are driftless and accordingly neither linearly controllable nor linearly stable. The typical planner relies on oscillations in order to move, in a way similar to how one parks a car or how an animal changes its shape to locomote. An example of this paradigm is our earlier work in Bullo et al. [4], Cerven and Coverstone-Carroll [5], where local Lie-bracket based methods are proposed for certain classes of vehicles. The classic limitation of Lie bracket methods is their local nature, as only small amplitude motions can be planned satisfactorily. We claim that Lie bracket methods for motion planning rely on *series expansions* as a way to characterize the evolution of a nonlinear system. Indeed, the local nature of Lie-bracket base planners results from the poor convergence properties of series expansions. This is seen in the work of Chen [7], Fliess [8], Sussmann [16].

The central goal of this paper are novel algorithms for the generation of feasible and optimal trajectories for locomotion systems. We aim to design algorithms that would reduce the computational requirements of the commonly used "transcription plus nonlinear programming" approach in numerical optimal control. We plan to accomplish this by overcoming the classic limitations found in Lie-bracket-based trajectory planners. This is based on two fundamental ideas:

(i) In order to obtain simple expressions, easy to implement and fast to evaluate, we restrict the class of admissible nonlinearities. Polynomial approximations were an obvious choice as representative of a large class of systems, although other choices were possible.

(ii) Under the assumption of hyperbolic stability, we employ series expansions that converge over infinite time and for inputs bounded by a computable finite size constant. This is in contrast with ϵ size planners, where by assumption $\epsilon \ll 1$.

The availability of a convergence analysis is a novelty in the context of Lie brackets based motion planners. Exploiting the simpler expressions but more importantly the stronger convergence properties, various algorithms can be designed. It is precisely the strong convergence properties that lead to the design of approximation schemes with desired error bounds. We summarize two methodologies that will be detailed in the next sections:

Existence of local trajectories: Near the optimal solution of the linearized system, one can set up an iterative procedure that proves the existence and uniqueness of a solution for the nonlinear system.

Trajectory optimization via low dimensional nonlinear programming: The optimal control problem (including constraints on inputs) is cast as a low dimensional nonlinear program, to be solved for example via classic sequential quadratic programming approaches.

The paper is organized as follows. Section 2 describes polynomial control systems, and Section 3 presents series expansions for such systems. Section 4 contains the trajectory optimization algorithms. Section 5 illustrates some simulation results for a PVTOL model.

2 Polynomial control systems

As first step, we select a family of nonlinearities that lead to a tractable treatment while retaining expressive power. Polynomial control systems satisfy these two requirements. Specifically, we consider here second order polynomial control systems

$$\dot{x} = P(x,x) + Ax + Bu(t) \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, A and B represent a classic linear system terms, while $P : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a bilinear map (so that the components of $P(x,x)$ are homogeneous polynomials of degree two).

This model and its higher order polynomial extensions describe a large class of mechanical systems. As first example, consider the kinematic equations of a rigid body in two or three dimensional motion. Let the angular and linear position be described by a unit quaternion and a translation vector (q,p). Let (ω,v) be the angular and linear body velocity. The kinematic equation can be written as

$$\dot{q} = \widehat{q}\omega, \qquad \dot{z} = v + z \times \omega$$

where z is the location of the origin expressed in body coordinates. Since the entries of the matrix $\widehat{q}$ depend linearly on q, these equations are polynomial in the state (q,z,ω,v).

A second set of examples is provided by robotics systems with Centripetal and Coriolis forces, gravity forces, linear damping, and gyroscopic terms. If coordinate changes are allowed, these systems often have polynomial structure. For more general systems, a Taylor expansion can be used for conversion to a polynomial system. In this case, bounds on the truncation error are known.

As a final point, note that if the force input does not appear linearly, as in the term Bu, a simple dynamic extensions can be employed, i.e., $\dot{u} = v$ and then v is the new input that appears linearly.

A PVTOL model

To render this discussion as concrete as possible, we here present a simple planar vertical takeoff and landing aircraft model based upon that of Martin et al. [13] with added viscous damping forces; see Figure 1. We parametrize its configuration and velocity space via the state variables $\{s,c,x,z,\omega,v_x,v_z\}$. We let x and z be the inertial coordinates of the aircraft, θ be its roll angle and we let

$$s = \sin\theta, \quad c = \cos\theta - 1.$$

The angular velocity is ω and the linear velocities in the body-fixed x (respectively z) axis are v_x (respectively v_z). Explicitly separating the linear from the homogeneous polynomial component, the equations are written as:

$$\begin{bmatrix} \dot{s} \\ \dot{c} \\ \dot{x} \\ \dot{z} \\ \dot{\omega} \\ \dot{v}_x \\ \dot{v}_z \end{bmatrix} = \begin{bmatrix} \omega \\ 0 \\ v_x \\ v_z \\ \frac{-k_1}{J}\omega \\ \frac{-k_2}{m}v_x - gs \\ \frac{-k_3}{m}v_z - gc \end{bmatrix} + \begin{bmatrix} c\omega \\ -s\omega \\ cv_x - sv_z \\ sv_x + cv_z \\ 0 \\ v_z\omega \\ -v_x\omega \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{h}{J} \\ 0 & \frac{1}{m} \\ \frac{1}{m} & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Control u_1 corresponds to the body vertical force minus gravity, while u_2 corresponds to coupled forces on the wingtips with a net horizontal component. The other forces depend upon the constants k_i, which parametrize some damping force, and g, the gravity constant. The constant h is the distance from the center of mass to the wingtip, while m and J are mass and moment of inertia, respectively.

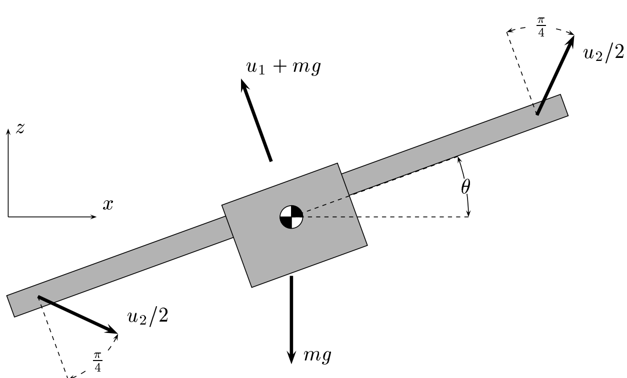

Figure 1: Diagram of the PVTOL model.

3 Series expansions

We present the results in Bullo [3]. Consider the differential equation (1) with initial condition $x(0) = 0$. For an appropriate time length T, the solution $x : I = [0, T] \mapsto \mathbb{R}^n$ is formally written as a series

$$x = \sum_{k=1}^{+\infty} x_k$$

where

$$x_1(t) = \int_0^t e^{A(t-\tau)} B u(\tau) d\tau$$
$$x_k(t) = \sum_{a=1}^{k-1} \int_0^t e^{A(t-\tau)} P(x_a(\tau), x_{k-a}(\tau)) d\tau, \quad k \geq 2.$$

From [10] we recall various norms on tensors

$$\|B\|_\infty = \max_{\|y_1\|_\infty = 1} \|By_1\|_\infty$$
$$\|P\|_\infty = \max_{\|y_1\|_\infty = \|y_2\|_\infty = 1} \|P(y_1, y_2)\|_\infty,$$

where $y_1, y_2 \in \mathbb{R}^n$, and on function spaces

$$\|x(t)\|_{\mathcal{L}_\infty} = \sup_{t \in I} \max_{i=1,\dots,n} |x_i(t)|$$
$$\left\|e^{At}\right\|_{\mathcal{L}_1} = \max_{i=1,\dots,n} \sum_{j=1}^{n} \int_0^T |(e^{At})_{ij}| dt.$$

Without loss of generality we set $\|B\|_\infty = \|P\|_\infty$ and

$$\beta = 2 \left\|e^{At}\right\|_{\mathcal{L}_1} \|P\|_\infty. \tag{2}$$

If $\beta^2 \|u\|_{\mathcal{L}_\infty} < 1$, the series converges absolutely and uniformly in $t \in [0, T]$, and for all integers K the truncation error is bounded by

$$\left\| x - \sum_{k=1}^{K} x_k \right\|_{\mathcal{L}_\infty} \leq \frac{1}{\beta} \operatorname{Remainder}_K \left(1 - \sqrt{1 - \beta^2 \|u\|_{\mathcal{L}_\infty}}\right).$$

where, given any $\eta \mapsto h(\eta)$, $\operatorname{Remainder}_M(h)(\eta)$ is its Taylor remainder of order M.

Note that if the matrix A is Hurwitz, then the norm $\left\|e^{At}\right\|_{\mathcal{L}_1}$ is finite also over the infinite horizon. Therefore, the series converges for all time provided the input is bounded according to $\beta^2 \|u\|_{\mathcal{L}_\infty} < 1$.

4 Trajectory design

Exploiting the characterization just obtained, we formulate a novel efficient ways to design feasible and optimal trajectories. For simplicity, we consider the minimum energy optimal control problem with bounded input:

$$\text{minimize} \quad \frac{1}{2} \int_0^T \|u(t)\|^2 dt \tag{3}$$
$$\text{subject to} \quad x(0) = 0, \;\; x(T) = x_{\text{target}} \tag{4}$$
$$\dot{x} = P(x, x) + Ax + Bu \tag{5}$$
$$|u_i| \leq 1. \tag{6}$$

A feasible trajectory is one that satisfies the constraint (4) (5), and (6). Rate constraints on u are also often present, but we neglect them here.

The series expansion reduces the feasibility constraints (4) and (5) that involve both the signal x and u, into a constraint only on the input u. Justified by the truncation bound above, we truncate the series expansion and call a trajectory feasible if

$$x_{\text{target}} = \sum_{k=1}^{K} x_k(T). \tag{7}$$

The accuracy as well as the complexity of the representation in equation (7) increase proportionally to the truncation order K. In what follows, we set $K = 2$ and illustrate how efficient these approximations can be in the design of various trajectory generation schemes. A reliable implementation of these algorithms would monitor the magnitude of the truncation error.

Base functions for input

The feasibility equation (7) is now a constraint on the input functions u (all the terms x_k are computed as a function of u). This constraint can be discretized (transcribed) into a finite dimensional equation via a collection of base functions $\{\psi_i(t) : i = 1, \dots, N_u\}$, where we assume $N_u \geq n$. In other words, if we write

$$u(t) = \sum_{i=1}^{N_u} c_i \psi_i(t),$$

and numerically compute (e.g., via FFT techniques)

$$\Psi^i(t) = \int_0^t e^{A(t-\tau)} B \psi_i(\tau) d\tau,$$
$$\Psi^{ij}(t) = \int_0^t e^{A(t-\tau)} P\left(\Psi^i,\ \Psi^j\right) d\tau,$$

then equation (7) turns into

$$\begin{aligned} x_{\text{target}} &= \sum_{i=1}^{N_u} c_i \Psi^i(T) + \sum_{i,j=1}^{N_u} c_i c_j \Psi^{ij}(T) \\ &= \Psi_1 c + \Psi_2(c, c), \end{aligned} \tag{8}$$

where we have defined the tensors Ψ_1 and Ψ_2.

Some remarks are in order. As base functions we might select shape functions, see Žefran and Kumar [17], as they lead to simple expressions for magnitude and rate constraints (and are common choice in finite element methods). A more refined choice would be piecewise cubic Hermite polynomials, that are quite successful in classic optimal control settings [1]. More general base functions can be considered for unilateral or quantized control inputs. In the context of real-time applications, the computation of the tensors Ψ_1, Ψ_2 can be performed off-line through FFT or other techniques.

4.1 Existence of local trajectories

In general no analytic solution to the nonlinear equation (8) appears available. In this section we provide sufficient conditions for the existence of one solution and design an iterative algorithm guaranteed to converge to it. The key tool is the contraction mapping theorem [10].

The tensor $\Psi_1 \in \mathbb{R}^{n \times N_u}$ characterizes the behavior of the linearized system. Roughly speaking it describes the first order, linear controllability effects, whereas the tensor Ψ_2 describes the second order nonlinear effects.

If the system is linearly controllable, then the map $\Psi_1 : \mathbb{R}^{N_u} \to \mathbb{R}^n$ is full rank under mild assumptions on the family of base functions $\{\psi_i\}$. We let $\Psi_1^\dagger$ be its pseudo-inverse, let $\chi \in \mathbb{R}^n$ and write

$$c = \Psi^\dagger \chi$$

so that equation (8) can be rewritten as

$$\chi = x_{\text{target}} - \Psi_2\left(\Psi_1^\dagger \chi,\ \Psi_1^\dagger \chi\right) \quad \triangleq T(\chi). \tag{9}$$

Lemma 4.1 *Let $S = \{\chi : \|\chi - x_{\text{target}}\| \leq \|x_{\text{target}}\|\}$. If*

$$4\,\|\Psi_2\|\,\|\Psi_1^\dagger\|^2\,\|x_{\text{target}}\| < 1,$$

there exists a unique vector χ^* satisfying $\chi^* = T(\chi^*)$, and it can be computed by iterating the map T starting from any initial condition in S.

Proof: If $\chi \in S$, then $\|\chi\| \leq 2\,\|x_{\text{target}}\|$. The set S is invariant under T since:

$$\begin{aligned} \|T(\chi) - x_{\text{target}}\| &= \left\|\Psi_2\left(\Psi_1^\dagger \chi,\ \Psi_1^\dagger \chi\right)\right\| \\ &\leq \|\Psi_2\|\,\|\Psi_1^\dagger\|^2\,\|\chi\|^2 \\ &\leq \|\Psi_2\|\,\|\Psi_1^\dagger\|^2 4\,\|x_{\text{target}}\|^2 < \|x_{\text{target}}\|. \end{aligned}$$

Over S, the map T is a contraction since

$$\begin{aligned} &\|T(\chi_1) - T(\chi_2)\| \\ &= \left\|\Psi_2\left(\Psi_1^\dagger \chi_1,\ \Psi_1^\dagger \chi_1\right)\right\| - \left\|\Psi_2\left(\Psi_1^\dagger \chi_2,\ \Psi_1^\dagger \chi_2\right)\right\| \\ &= \left\|\Psi_2\left(\Psi_1^\dagger(\chi_1 + \chi_2),\ \Psi_1^\dagger(\chi_1 - \chi_2)\right)\right\| \\ &\leq \|\Psi_2\|\,\|\Psi_1^\dagger\|^2\,\|\chi_1 + \chi_2\|\,\|\chi_1 - \chi_2\| \\ &\leq \|\Psi_2\|\,\|\Psi_1^\dagger\|^2 4\,\|x_{\text{target}}\|\,\|\chi_1 - \chi_2\| < \|\chi_1 - \chi_2\|. \end{aligned}$$

The statement follows from the contraction mapping theorem. ■

Three remarks are in order. First, our pseudo inverse procedure mimicks the computation of *minimum energy control* for linear systems; see [6, page 557]. If the base functions $\{\psi_i\}$ are chosen according to the treatment in [6], then the solution to equation (9) is the optimal solution for the linearized system, or a nearby solution for $\Psi_2 \neq 0$. Second, the setup above assumes linear controllability of the given system. In fact, more similar methods can be used in the case of a system that is controllable via low order Lie brackets, i.e., via the tensor Ψ_2. Finally, note that even if iteration (9) is guaranteed to find the unique solution, a steepest descent or a Newton's algorithm might be preferred for better convergence speed.

4.2 Trajectory optimization via nonlinear programming

In this section, we use the series expansion representation of feasibility directly inside the optimal control setup. We define

$$Q_{ij} = \int_0^T \psi_i(t)' \psi_j(t) dt,$$

and rewrite the optimal control problem in equations (3)-(6) as

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} c' Q c \\ \text{subject to} \quad & x_{\text{target}} = \Psi_1 c + \Psi_2(c, c), \\ & |c| \leq 1, \end{aligned}$$

where the component-wise inequality constraint $|c| \leq 1$ is the transcription of the magnitude constraints on u (i.e., we use triangular base functions that lead to this simple transcription). The problem is a finite dimensional nonlinear program (specifically, a quadratic cost, quadratically constrained program), and can be tackled numerically via sequential programming techniques, as in the classic numerical optimal control setting.

In short, this approach is similar to the standard "collocation plus nonlinear programming" technique, with one key difference. The proposed methodology only parameterizes (discretizes) the input u and therefore the nonlinear program is defined over a limited number of variables (N_u). Collocation is one of the classic transcription methods that discretizes also the value of the state variables x, so that it needs a arguably much larger number of variables ($N_u + N_x$). In applications where the dimension of the state space is much larger than the number of inputs, this limited dimensionality is very advantageous.

5 Simulations

The iterative algorithm in Section 4.2 was applied to the PVTOL model described in Section 2. Matlab$^{\text{TM}}$ was chosen for its built-in functions: 'lsim' computes the evolution of a linear system, and 'fmincon' finds the constrained minimum of several variables (indeed implements an SQP algorithm).

The Matlab$^{\text{TM}}$ simulation results are illustrated in Figures 2 and 3 for approximations of order $K = 1$ and $K = 2$. As base functions we used triangular shape functions described in [17]. Parameter values were chosen to be $J = 10, m = 10, h = 5, g = 9.81, k_1 = 4, k_2 = 5, k_3 = 6$, with control parameters of $T = 4$ seconds, and $N_u = 9$. The state desired in Figure 2 is $(0, 0, 5, 0.1, 0, 0, 0)$. The optimization routines took in the order of 10 seconds on a 350Mhz Pentium computer.

6 Conclusion

We have provided a methodology and some algorithms for the trajectory generation and optimization problems. Our semi-analytical solution seeks a midpoint between the analytical requirements of flatness and the brute force numerical approach of nonlinear programming. Series expansions are the key technical tool, see the companion paper [3] for more details on this topic. These novel algorithms aim at the design of adaptive, reconfigurable, real-time control schemes for autonomous vehicles, locomotion devices and high degrees of freedom multibody systems.

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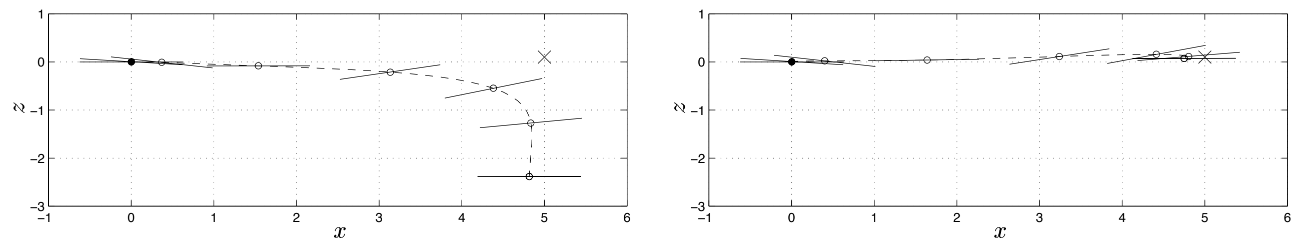

Figure 2: Trajectory generation via first (left) and second order (right) approximations for the PVTOL model: the desired motion is horizontal translation from left to right without any vertical or rotational displacement. Note: 1) the second order approximation already achieves a much higher precision as opposed to the first order approximation, 2) *unlike* traditional Lie-brackets based planners, there are no small-amplitude oscillations.

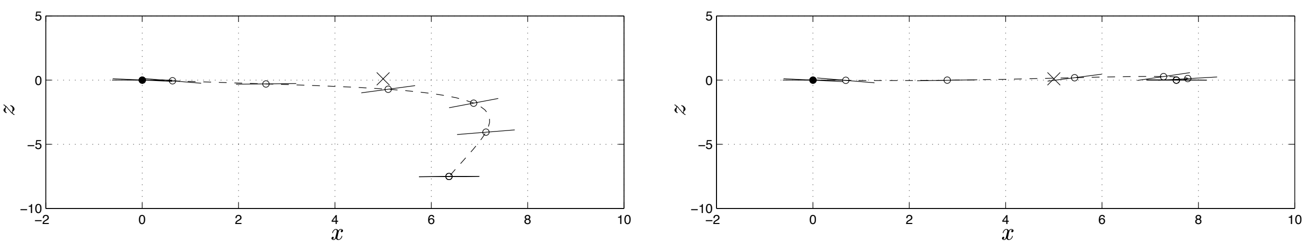

Figure 3: Unsuccessful trajectory generation run. The desired motion is horizontal translation of the same magnitude as above. However the desired final velocity is required to be horizontal leftward. The large final error illustrates how first and second order approximations have a limited validity range. Indeed, even higher order approximation may not converge if the target location exceeds the limits put forth by the analysis in Section 3 and Section 4.1.

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