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On Perturbation Methods for Mechanical Control Systems

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Abstract

In this note we investigate open-loop control of underactuated mechanical systems and draw connections between averaging and controllability theory. Two sets of results are presented: averaging under high-amplitude high-frequency forcing, and series expansions for the evolution of a forced mechanical system starting at rest.
Keywords: mechanical control systems, averaging, controllability theory

1 Introduction

Perturbation methods for mechanical systems are a classic topic at the center of the attention of numerous mathematicians as well as practitioners. This note reviews two sets of results recently obtained on mechanical systems subject to time-varying forcing. The results build on the contributions on vibrational stabilization via the *averaged potential* in (Baillieul, 1993) and on configuration controllability via the *symmetric product* operation in (Lewis and Murray, 1997).

First, we study the behavior of mechanical systems forced by high amplitude and highly oscillatory inputs. The averaged system is shown to be again a mechanical system subject to an appropriate forcing. By investigating the class subclass of simple systems, i.e., systems with "Hamiltonian equal to kinetic plus potential energy," we precisely characterizes how the averaged potential is related to the symmetric products of certain vector fields. We refer to (Bullo, 1999*b*) for the application of these results to vibrational stabilization problems.

Next, we present a series expansion that describes the evolution of a mechanical system starting at rest and subject to a time-varying external force. We provide a first order description to the solutions of a second order initial value problem. Simplified expressions can be written for simple mechanical systems or systems defined on Lie groups, see (Bullo, 1999*a*).

2 Modeling Mechanical Systems via Affine Connections

The notion of affine connection provides a coordinate-free mean of describing various types of mechanical systems, see (Lewis and Murray, 1997). We write the Euler-Lagrange equations for a system subject to a time-varying force as:

$$\nabla_{\dot q}\dot q = Y(q,t). \tag{1}$$

Alternatively, if m input forces, potential and damping forces are present, we write

$$\nabla_{\dot q}\dot q = Y_0(q) - D(q)\dot q + \sum_{a=1}^{m} Y_a(q)u^a(t). \tag{2}$$

We assume $q(0)=q_0$, and $\dot q(0)=v_0$. We assume the affine connection, the input fields and the input forcing are smooth functions of their respective arguments.

Affine connections are instrumental in defining a key operation, the symmetric product of two vector fields, that is: $\langle Y_1 : Y_2\rangle = \nabla_{Y_1}Y_2 + \nabla_{Y_2}Y_1$.

3 Averaging under high amplitude highly oscillatory forcing

Introduce an $\epsilon>0$, and let $u^a(t) = v^a(t/\epsilon)/\epsilon$, where the v^a are T-periodic functions that satisfy

$$\int_0^T v^a(s_1)ds_1 = 0, \qquad \int_0^T\!\!\int_0^{s_1} v^a(s_2)ds_1ds_2 = 0.$$

Let $v(t) = [v^1(t),\dots,v^m(t)]'$ and define the matrix V according to:

$$V = \frac{1}{2T}\int_0^T\left(\int_0^{s_1} v(s_2)ds_2\right)\left(\int_0^{s_1} v(s_2)ds_2\right)' ds_1.$$

Theorem 3.1. *Consider the initial value problem*

$$\nabla_{\dot r}\dot r = Y_0(r) - D(r)\dot r - \sum_{a,b=1}^{m} V_{ab}\left\langle Y_a : Y_b \right\rangle (r),$$

with initial conditions $r(0) = q_0$, *and* $\dot r(0) = v_0$. *Then* $q(t) - r(t) = O(\epsilon)$ *as* $\epsilon \to 0$ *on the time scale* 1, *and* $q(t) - r(t) = O(\delta(\epsilon))$ *as* $\epsilon \to 0$ *for all* t, *if* $(r, \dot r) = (0, 0)$ *is an asymptotically stable critical point.*

Remark 3.2. *Classic averaging in Hamiltonian systems typically relies on the assumption that the system is integrable and that the force is of size* ϵ. *Here instead it is the Hamiltonian dynamics that is negligible in the first approximation.*

Next, we focus on simple systems with integrable inputs, and to expedite the treatment, we assume the configuration space to be $\mathbb{R}^n$. Such systems are completely characterized by their Hamiltonian:

$$H(q,p,u) = V(q) + \frac{1}{2}p' M(q)^{-1} p - \sum_{a=1}^{m} \varphi_a(q) u^a,$$

where M is the inertia matrix, V the potential energy and φ_a are m arbitrary smooth functions. Neglecting the dissipation term, Hamilton's equation are equivalent to the formulation in equation (2).

Gradient vector fields play a natural key role in this setting. Let φ_1, φ_2 be two functions and define a symmetric product between functions via

$$\langle \varphi_i : \varphi_j \rangle \triangleq \frac{\partial \varphi_i}{\partial q} M^{-1} \frac{\partial \varphi_j}{\partial q}.$$

Remarkably, $\langle \operatorname{grad}\varphi_1 : \operatorname{grad}\varphi_2 \rangle = \operatorname{grad}\langle \varphi_1 : \varphi_2 \rangle$.

Theorem 3.3. *Consider a simple mechanical control system with Hamiltonian defined above. Then the averaged system is a simple mechanical system subject to no input forces and with Hamiltonian*

$$H_{\text{averaged}}(q,p) = V_{\text{averaged}}(q) + \frac{1}{2}p' M(q)^{-1} p,$$

where the averaged potential *is defined as*

$$V_{\text{averaged}}(q) \triangleq V(q) + \sum_{a,b=1}^{m} V_{ab}\left\langle \varphi_a : \varphi_b \right\rangle (q).$$

4 A series expansion for the forced evolution from rest

The procedure underlying the averaging results in the previous section can be iterated. Assuming zero initial velocity and dropping the damping force), the evolution of the second order initial value problem in equation (1) can be described via a first order differential equation. Precise statements and proof are available in (Bullo, 1999*a*).

Theorem 4.1. *Define recursively the time-varying vector fields* V_k*:*

$$V_1(q,t) = \int_0^t Y(q,s)ds$$

$$V_k(q,t) = -\frac{1}{2}\sum_{j=1}^{k-1} \int_0^t \Big\langle V_j(q,s) : \; V_{k-j}(q,s) \Big\rangle ds.$$

The solution $t \to q(t)$ *to equation* (1) *satisfies*

$$\dot q(t) = \sum_{k=1}^{+\infty} V_k(q(t),t),$$

where the series $(q,t) \mapsto \sum_{k=1}^{\infty} V_k(q,t)$ *converges absolutely and uniformly in a neighborhood of* q_0 *and for* $t \in [0,T]$.

5 Conclusion

This brief note brings together a number of exciting recent results. Point stabilization, analysis of locomotion gaits, and motion planning for underactuated systems will provide plenty of challenges.

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References

Baillieul, J. (1993). Stable average motions of mechanical systems subject to periodic forcing. In: *Dynamics and Control of Mechanical Systems: The Falling Cat and Related Problems* (M. J. Enos, Ed.). Vol. 1. pp. 1–23. Field Institute Communications.

Bullo, F. (1999*a*). Series expansions for the evolution of mechanical control systems. *SIAM Journal of Control and Optimization*. Submitted.

Bullo, F. (1999*b*). Vibrational control of mechanical systems. *SIAM Journal of Control and Optimization*. Submitted.

Lewis, A. D. and R. M. Murray (1997). Configuration controllability of simple mechanical control systems. *SIAM Journal of Control and Optimization* **35**(3), 766–790.