AVERAGING AND VIBRATIONAL CONTROL OF MECHANICAL SYSTEMS*

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Abstract. This paper investigates averaging theory and oscillatory control for a large class of mechanical systems. A link between averaging and controllability theory is presented by relating the key concepts of averaged potential and symmetric product. Both analysis and synthesis results are presented within a coordinate-free framework based on the theory of affine connections.

The analysis focuses on characterizing the behavior of mechanical systems forced by high amplitude high frequency inputs. The averaged system is shown to be an affine connection system subject to an appropriate forcing term. If the input codistribution is integrable, the subclass of systems with Hamiltonian equal to "kinetic plus potential energy" is closed under the operation of averaging. This result precisely characterizes when the notion of averaged potential arises and how it is related to the symmetric product of control vector fields. Finally, a notion of vibrational stabilization for mechanical systems is introduced and provide sufficient conditions are provided in the form of linear matrix equality and inequality tests.

Key words. mechanical system, averaging, vibrational stabilization, nonlinear controllability

AMS subject classifications. 34C29, 70Q05, 93B05, 93B29, 93D99

1. Introduction. This paper investigates the open loop response of nonlinear mechanical control systems. This topic is studied in different ways by the classic disciplines of averaging and controllability. Relying on tools from both fields, this work characterizes the response of a large class of mechanical systems to high amplitude high frequency forcing. The class of mechanical control systems we consider includes systems with integrable inputs (Hamiltonian systems with conservative forces), as well as systems with more general types of forces and nonholonomic constraints.

Averaging and vibrational stabilization techniques find useful applications in various areas. Within the context of mechanical systems much recent interest has focused on the control of underactuated robotic manipulators and on the analysis and design of robotic locomotion devices. Underactuated robotic manipulators have fewer control inputs than their degrees of freedom due to either design or failure. In both cases the objective is to control the system despite the lack of control authority. Examples of works in this area are [37, 25], where the authors investigate the control via oscillatory inputs for some two and three degrees of freedom planar manipulators.

Robotic locomotion studies the movement patterns that biological systems and mechanical robots undergo during locomotion; see [24]. Typically, cyclic motion in certain internal variables generates displacement in Euclidean space; consider the example of how a snake changes its shape to locomote. Computing the feasible trajectories of a locomotion system is an analytically untractable problem for any nontrivial example. Averaging provides a means of tackling such problems; see for example the contributions on motion planning and trajectory generation documented in [17, 6] and the references therein.

Finally, averaging analysis seems well suited to tackle novel applications in the

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field of micro-electro mechanical systems and vibrational control is being investigated within the context of active control of fluids and separation control. Examples include [7] on the scale dependence in oscillatory control, and [43] on unsteady flow control using oscillatory blowing. In these settings, vibrational stabilization schemes appear advantageous since they require no expensive or complicated sensing.

Literature review. Averaging theory is discussed in a number of textbooks [13, 42, 19]. The control relevance of averaging ideas was underlined in the work on vibrational control by Bentsman and co-workers [9, 10, 11]. These works introduce vibrational stabilization techniques under various types of input forcing (e.g., vector additive, linear and nonlinear multiplicative forcing). The later work by Baillieul [3, 4, 5, 6] extends these techniques to the context of mechanical systems described by specific Lagrangian and Hamiltonian models. In particular, the work in [3] presents two treatments to averaging for mechanical control systems. The first approach relies on a coordinate transformation to bring the system to standard averaging form. The second approach is based on directly averaging the Hamiltonian function and gives rise to the notion of *averaged potential*. Some assumptions restrict how applicable the latter approach is. For example, the control system is assumed to have a cyclic variable and to be single-input with the control input applied to the cyclic variable. Nonetheless, the notion of averaged potential has proven very successful in treating a number of important cases, see for example [50, 51, 7].

Another set of relevant results includes the work on small-time local controllability for mechanical systems. The main references are the original work in Lewis and Murray [32] and the advances in [31]. These works introduce the notion of configuration and equilibrium controllability and provide sufficient conditions to characterize them. The main technical tool is the notion of *symmetric product* as a way to represent certain Lie brackets. Control algorithms that exploit motions along the “symmetric product directions” are presented in [17].

Statement of contributions. This paper contains a number of novel results both on averaging analysis as well as on control design. One key technical contribution is the understanding of the relationship between the symmetric product [32] and the averaged potential [3]. We describe the contributions in the next three paragraphs.

We start by studying the behavior of a large class of mechanical systems forced by high amplitude high frequency inputs. We rely on the notion of *system described by an affine connection* as a generalized way of describing mechanical control systems with simple Hamiltonian, generic non-integrable (non-conservative) forces and nonholonomic constraints. Under mild assumptions, we show how the averaged system is again a system described by an affine connection and subject to an appropriate forcing. Since this forcing term is a certain symmetric product, the result illustrates an instructive connection between controllability and averaging. The averaging analysis relies on a careful application of the variation of constants formula and of the homogeneity property of mechanical systems. The theorem statement and proof are presented in a coordinate-free manner.

We then consider the set of simple mechanical systems, that is, systems with Hamiltonian equal to “kinetic plus potential” and investigate when this subclass is closed under the operation of averaging. A sufficient condition is that the input codistribution be integrable, or in other words, that the control forces be described by conservative fields. Under this assumption, the Hamiltonian function of the average system includes a generalized averaged potential. This result shows how the notion of averaged potential is applicable to a wider set of systems than those considered by

Baillieul [3]. The proof relies on the observation that the averaged potential is related to a certain symmetric product of functions; see [20].

Finally, we focus on the design of open and closed loop controllers based on high amplitude high frequency forcing. We introduce an appropriate notion of vibrational stabilization for mechanical systems, where only the configuration variables are considered. We consider simple systems with integrable forces and assume that the control system is underactuated (i.e., fewer control inputs are available than degrees of freedom). We consider the point stabilization problem and design control Lyapunov functions via the "potential shaping" technique, see the original [46] and a modern account in [47]. Here the closed loop potential energy reflects the presence of both a proportional action as well an oscillatory action. We provide sufficient conditions for stabilizability in the form of a linear matrix equality and inequality test. We illustrate the control design by applying it to an underactuated two-link manipulator.

Organization. The paper is organized as follows. We present a quick summary of averaging and we introduce some tools from chronological calculus in Section 2. In Section 3 we introduce a useful classification of mechanical systems and study their common homogeneous structure. Section 4 contains the averaging analysis. In Section 5 we present the vibrational stabilization results and work out the example.

2. Averaging and the variation of constants formula.

In this section we present some basic results on averaging theory and their coordinate-free interpretation. The averaging results are taken from Sanders and Verhulst [42] and from Guckenheimer and Holmes [22].

Let x, y, x_0 belong to an open subset $D \subset \mathbb{R}^n$, let $t \in \mathbb{R}_+ = [0, \infty)$, and let the parameter ϵ vary in the range $(0, \epsilon_0]$ with $\epsilon_0 \ll 1$. Let $f, g : \mathbb{R}_+ \times D \to \mathbb{R}^n$ be smooth time-varying vector fields. Consider the initial value problem in *standard form*:

$$\frac{dx}{dt} = \epsilon f(t, x), \qquad x(0) = x_0. \tag{2.1}$$

If $f(t, x)$ is a T-periodic function in its first argument, we let *the averaged system* be the initial value problem

$$\frac{dy}{dt} = \epsilon f^0(y), \qquad y(0) = x_0, \tag{2.2}$$
$$f^0(y) = \frac{1}{T}\int_0^T f(t, y)dt.$$

We say that an estimates is *on the time scale* $\delta^{-1}(\epsilon)$, if the estimate holds for all times t such that $0 < \delta(\epsilon)t < L$ with L a constant independent of ϵ. From pages 39 and 71 in [42] and from page 168 in [22], we summarize:

THEOREM 2.1 (First order periodic averaging). *There exist a positive ϵ_0, such that for all $0 < \epsilon \leq \epsilon_0$*

(i) $x(t) - y(t) = O(\epsilon)$ as $\epsilon \to 0$ on the time scale $1/\epsilon$, and

(ii) if the origin is a hyperbolically stable critical point for f^0, then $x(t) - y(t) = O(\epsilon)$ as $\epsilon \to 0$ for all $t \in \mathbb{R}_+$ and the differential equation (2.1) *possesses a unique periodic orbit which is hyperbolically stable and belongs to an $O(\epsilon)$ neighborhood of the origin.*

Next, consider the initial value problem

$$\frac{dx}{dt} = f\left(t/\epsilon, x\right), \qquad x(0) = x_0, \tag{2.3}$$

where $f(t,x)$ is a T-periodic function in its first argument. A time scaling argument shows that the averaged version of this problem is the same as in equation (2.2). Accordingly, Theorem 2.1 implies that $x(t) - y(t) = O(\epsilon)$ as $\epsilon \to 0$ *only* on the time scale 1, unless $y = 0$ is an hyperbolically stable point of f^0.

2.1. Variation of constants formula in coordinate-free terms. The variation of constants formula is a means to bring various systems into the standard form in equation (2.1). This tool originates back to Lagrange's work, see [42, page 183], and is presented here in a coordinate-free setting.

Given a diffeomorphism ϕ and a vector field g, the *pull-back of g along ϕ*, denoted $\phi^* g$, is the vector field

$$(\phi^* g)(x) \triangleq \left(\frac{\partial \phi^{-1}}{\partial x} \circ g \circ \phi \right)(x),$$

where the order of composition of functions is $(\varphi \circ \phi)(x) = \varphi(\phi(x))$. A useful diffeomorphism is the flow map $y(t) = \Phi^g_{0,T}(y_0)$ describing the solution at time T to the initial value problem

$$\dot{y} = g(t,y), \quad y(0) = y_0.$$

Next, consider the initial value problem

$$\dot{x}(t) = f(x,t) + g(x,t), \qquad x(0) = x_0. \tag{2.4}$$

We regard f as a perturbation to the vector field g, and we seek to characterize the flow map of $f+g$ in terms of the nominal flow map of g. The answer is provided by the *variation of constants* formula:

$$\Phi^{f+g}_{0,t} = \Phi^g_{0,t} \circ \Phi^{(\Phi^g_{0,t})^* f}_{0,t}. \tag{2.5}$$

In other words, if $z(t)$ is the solution to the initial value problem

$$\dot{z}(t) = \left((\Phi^g_{0,t})^* f \right)(z), \qquad z(0) = x_0, \tag{2.6}$$

the solution $x(t)$ to the initial value problem (2.4) satisfies

$$\dot{x}(t) = g(t,x), \qquad x(0) = z(t). \tag{2.7}$$

We illustrate the formula in Figure 2.1 and provide a self-contained proof in Appendix A.

2.2. Formal expansions for the pull-back of a flow map. Here we study in more detail the differential geometry of the initial value problem (2.6). Such a system is referred to as the "pulled back" or the "adjoint" system, e.g. see [23].

If f and g are time-invariant vector fields, the infinitesimal Campbell-Baker-Hausdorff formula, see [26], provides a means of computing the pull-back

$$(\Phi^g_{0,t})^* f(x) = \sum_{k=0}^{\infty} \operatorname{ad}^k_g f \, \frac{t^k}{k!},$$

where $\operatorname{ad}_g f(x) = [g,f](x)$ is the Lie bracket between g and f, and $\operatorname{ad}^k_g f = \operatorname{ad}^{k-1}_g \operatorname{ad}_g f$.

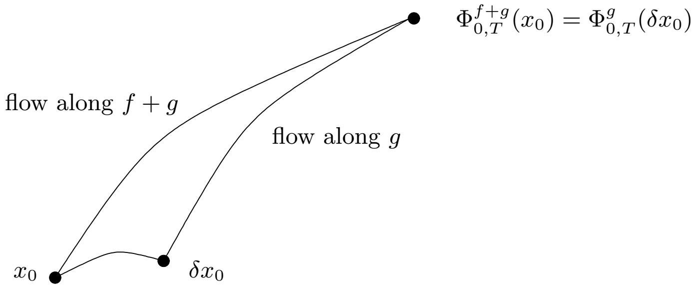

FIG. 2.1. *The flow along $f+g$ with initial condition x_0 equals the flow along g with initial condition δx_0. The variation δx_0 is computed via the variation of constants formula as the flow along $(\Phi^g_{0,t})^* f$ for time $[0,T]$ with initial condition x_0.*

If instead f is time-invariant vector field and g is time-varying vector vector field, we invoke a result from the chronological calculus formalism by Agračhev and Gamkrelidze [2]. It turns out that

$$\left(\left(\Phi^g_{0,t}\right)^* f\right)(t,x) = f(x) + \sum_{k=1}^{\infty} \int_0^t \cdots \int_0^{s_{k-1}} \left(\mathrm{ad}_{g(s_k,x)} \dots \mathrm{ad}_{g(s_1,x)} f(x)\right) ds_k \dots ds_1. \qquad (2.8)$$

The convergence properties for the series expansion in (2.8) are difficult to characterize; see for example a related discussion in [49] on the Campbell-Baker-Hausdorff formula. Nonetheless, sufficient conditions for local convergence are given in [2, Proposition 2.1 and 3.1]. For our analysis, the following simple statement suffices: if the terms $\mathrm{ad}_{g(s_k)} \dots \mathrm{ad}_{g(s_1)} f$ vanish for all k greater than a given N, then the series in equation (2.8) becomes a finite sum.

2.3. Averaging under high magnitude high frequency forcing. We return to the description of averaging results and we focus on a setting of interest in vibrational stabilization problems [9, 10, 11]. Consider the initial value problem

$$\frac{dx}{dt} = f(x) + (1/\epsilon) g\left(t/\epsilon, x\right), \qquad x(0) = x_0, \qquad (2.9)$$

where we assume that $g(t,x)$ is a T-periodic function in its first argument. Let $\Phi^g_{0,t}$ denote the flow map along $g(t,x)$ and define

$$F(t,x) = \left((\Phi^g_{0,t})^* f\right)(x) \qquad (2.10)$$

$$F^0(x) = \frac{1}{T} \int_0^T F(\tau,x) d\tau. \qquad (2.11)$$

Finally, let z and y be solutions to the initial value problems

$$\dot z = F(t/\epsilon, z), \quad z(0) = x_0, \qquad (2.12)$$

$$\dot y = F^0(y), \quad y(0) = x_0. \qquad (2.13)$$

LEMMA 2.2. *Let F be a T-periodic function in its first argument. For $t \in \mathbb{R}_+$, we have*

$$x(t) = \Phi^g_{0,t/\epsilon}(z(t)).$$

As $\epsilon \to 0$ on the time scale 1*, we have*

$$z(t) - y(t) = O(\epsilon)$$

If the origin is a hyperbolically stable critical point for F^0, then $z(t) - y(t) = O(\epsilon)$ as $\epsilon \to 0$ for all $t \in \mathbb{R}_+$ and the differential equation (2.12) *possesses a unique periodic orbit which is hyperbolically stable and belongs to an $O(\epsilon)$ neighborhood of the origin.*

Proof. As first step, we change time scale by setting $\tau = t/\epsilon$. Equation (2.9) becomes

$$\frac{d}{d\tau}x = \epsilon f(x) + g(\tau, x), \quad x(0) = x_0.$$

As second step, we apply the variation of constants formula

$$\begin{aligned} \frac{d}{d\tau}x &= g(\tau, x), \quad x(0) = z(\tau) \\ \frac{d}{d\tau}z &= \epsilon F(\tau, z), \quad z(0) = x_0, \end{aligned}$$

where F is defined according to equation (2.10). As third step, we average the initial value problem in z to obtain

$$\frac{d}{d\tau}y = \epsilon F^0(y), \quad y(0) = x_0,$$

where F^0 is defined according to equation (2.11) and F is assumed to be a T-periodic function. The averaged curve y approximates z over the time scale $\tau = 1/\epsilon$ and over all time according to Theorem 2.1. As fourth step, we change time scale back to $t = \epsilon\tau$ and compute

$$\begin{aligned} \frac{d}{dt}x &= (1/\epsilon) g(t/\epsilon, x), \quad x(0) = z(t) \\ \frac{d}{dt}z &= F(t/\epsilon, z), \quad z(0) = x_0, \\ \frac{d}{dt}y &= F^0(y), \quad y(0) = x_0. \end{aligned}$$

These are the definitions of z and y in equations (2.12) and (2.13). Finally, the equality in $x(t)$ follows by noting that the flow along $(1/\epsilon)g\,(t/\epsilon\,, x)$ for time 1 is equivalent to the flow along $g\,(t, x)$ for time $1/\epsilon$. □

This concludes our geometric presentation of averaging in systems with high magnitude high frequency inputs. These results on averaging and the variation of constants formula are known, see [10, Section III], and they play a key role in the study of vibrational stabilization problems, see also [9, 11]. Novel is the presentation of these results in a coordinate-free fashion: for a large class of mechanical control systems an explicit expression will be provided for the infinite series describing the variation of constants formula.

3. Mechanical control systems and their homogeneous structure. In this section we present three different types of mechanical systems and a geometric formalism that leads to a unified modeling framework. Also we present some results on the Lie algebraic structure common to these systems and to generic second order control systems, where the input is an acceleration (alternatively a force). To present an accessible treatment, we assume the configuration space to be $Q = \mathbb{R}^n$. However, Remark 3.1 and Section 3.1 provide the key ideas necessary to develop a coordinate-free treatment over manifolds.

Let $q = (q^1, \dots, q^n) \in \mathbb{R}^n$ be the configuration of the mechanical system. We consider the following control system

$$\ddot{q}^i + \Gamma^i_{jk}(q)\dot{q}^j\dot{q}^k = Y^i_0(q) + Y^i_a(q)u^a(t) + R^i_j(q)\dot{q}^j, \tag{3.1}$$

where the summation convention is in place here and in what follows, the indices j, k run from 1 to n, the index a runs from 1 to m (the number of input fields) and where:

(i) the Γ^i_{jk} are $n^2(n+1)/2$ arbitrary scalar functions on $\mathbb{R}^n$ called the Christoffel symbols (they satisfy the symmetric relationship $\Gamma^i_{jk} = \Gamma^i_{kj}$),

(ii) $q \mapsto Y_a(q)$ for $a = 1, \dots, m$ are vector fields characterizing configuration-dependent forces applied to the system. Y_0 for example might include the effect of a conservative forces such as gravity.

(iii) the functions $t \mapsto u^a(t)$ are integrable and describe the control magnitude applied along the input Y_a. The i^{th} component of Y_a is Y^i_a. We also let

$$Y(q,t) = Y_a(q)u^a(t),$$

(iv) $R(q)\dot{q}$ describes a generic force linearly proportional to velocity.

All quantities are assumed smooth functions of their arguments.

Equation (3.1) describes a large class of mechanical systems with Hamiltonian equal to kinetic plus potential energy, with symmetries and with nonholonomic constraints. A slightly loose but instructive classification follows.

Simple systems with integrable forces These systems have Hamiltonian equal to "kinetic plus potential energy" and are subject to integrable (conservative) input forces. For example, should the mechanical system be a robotic manipulator with motors at joints, then the appropriate Christoffel symbols are computed via a well-known combination of partial derivatives of the inertia tensor, see the definition of Coriolis matrix in [36] for example. Only for this kind of systems can one write a Hamiltonian function that includes the effect of forces; the treatment in Chapter 14 of [38] relies on this assumption.

Simple systems with non-integrable forces This class is a superset of the previous, where however *non-integrable* input forces are allowed. For example, the force applied by a thruster of a satellite, hovercraft or underwater vehicle is in general a non-integrable force. Simplified equations of motion can be written if the system has symmetries, i.e, if the system's configuration belongs to the group of rigid displacements (or one of its subgroups) and its Hamiltonian is independent of the configuration.

Systems with nonholonomic constraints This set includes systems from the previous two subclasses and additionally subject to nonholonomic constraints. Two very interesting locomotion devices called snakeboard and roller racer are described in recent papers [40] and [29]. Two methodology to write the equations of motions for these systems into form (3.1) are discussed in [30, 31, 12]. While the the description "nonholonomic" is commonly used to refer to wheeled robots and while such systems are usually driftless[1], we consider here nonholonomic systems with drift.

[1] Driftless control systems have the characterizing property that $u_i = 0$ implies $\dot{x} = 0$, where x is the state and u_i are the inputs.

Three remarks are appropriate. First, the model relies on no specific structure on the Γ^i_{jk} functions. In the classic Hamiltonian system case, these functions are readily computed from the inertia matrix. By leaving these functions unspecified our analysis includes systems with nonholonomic constraints. We refer to [12, 31] for a through treatment of this point.

Second, the distinctions between these three sets of mechanical systems have various instructive implications. For example, the notion of "actuated degree of freedom" is well defined only in systems subject to integrable forces. This simple fact is neglected even in recent literature on mechanical control systems.

Third, a more complete definitions of the various quantities above should include transformation rules under changes of coordinates. For example, the Christoffel symbols $\{\Gamma^i_{jk}, i, j, k = 1, \dots, n\}$ obey relatively surprising transformation rules, if the correct equations of motion are to be computed. If $\overline{q} = (\overline{q}^1, \dots, \overline{q}^n) \in \mathbb{R}^n$ are the transformed coordinates, the transformation rule for the Γ^i_{jk} is:

$$\overline{\Gamma}^k_{ij} = \frac{\partial q^p}{\partial \overline{q}^i} \frac{\partial q^m}{\partial \overline{q}^j} \frac{\partial \overline{q}^k}{\partial q^r} \Gamma^r_{pm} + \frac{\partial \overline{q}^k}{\partial q^l} \frac{\partial^2 q^l}{\partial \overline{q}^i \partial \overline{q}^j}. \tag{3.2}$$

We refer to [35, Section 7.5] for a more complete discussion.

3.1. Control systems described by an affine connection. Equations (3.1) are the Euler-Lagrange equations for a simple mechanical system. Numerous methodologies are available to write these equations in vector or in abstract formats. The theory of affine connections is a convenient formalism that formalizes the Euler-Lagrange equations as well as more general second order control systems (including systems with nonholonomic constraints).

An easily accessible treatment to the theory of affine connections is given by Do Carmo [21]. Early references on mechanical control systems on Riemannian manifolds is the work by Crouch [20]. The use of Riemannian concepts is encountering increasing success as testified by the contributions on modeling [12], decompositions [33], controllability [32], stabilization [28], tracking [18], interpolation [39] and (static and dynamic) feedback linearization [8, 41].

A smooth affine connection ∇ is a collection of n^3 smooth functions Γ^i_{jk} that satisfy the transformation rule in equation (3.2). An affine connection induces an operation between vector fields as follows. Let the vector fields X and Y have components

$$X(q) = X^i(q) \frac{\partial}{\partial q^i}, \qquad \text{and} \qquad Y(q) = Y^i(q) \frac{\partial}{\partial q^i}.$$

The covariant derivative of Y along X is the vector field $\nabla_X Y$ defined by

$$\nabla_X Y = \left(\frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial q^i}.$$

Similarly, an affine connection induces an operation between a curve $\gamma : [0,1] \mapsto \mathbb{R}^n$ and a vector field Y. The covariant derivative of Y along γ is a vector field along γ defined by

$$\nabla_{\dot\gamma} Y = \left(\frac{dY^i(\gamma(t))}{dt} + \Gamma^i_{jk} \dot\gamma^j Y^k \right) \frac{\partial}{\partial q^i}.$$

Whenever the reference curve is uniquely determined, we let $\nabla_{\dot\gamma} Y = \dfrac{DY}{dt}$. The two definitions of covariant derivative have similarities, however $\dfrac{DY}{dt}$ is not a vector field over $\mathbb{R}^n$, but it is only defined on the trajectory $\gamma : [0,1] \mapsto \mathbb{R}^n$. We refer to [21] for a more complete treatment of affine connections and of manifolds.

We are finally ready to rewrite equation (3.1) in a coordinate-free fashion. According to the definition of covariant derivative along a curve, the generalized Euler-Lagrange equations are

$$\frac{D\,\dot q}{dt} = Y_0(q) + R(q)\dot q + Y_a(q)u^a(t), \tag{3.3}$$

where the covariant derivative of $\dot q$ is computed along the curve $q(t)$, i.e., $D\dot q/dt = \nabla_{\dot q}\dot q$.

3.2. Lie algebraic structure. The fundamental structure of the control system in equation (3.1) (and accordingly (3.3)) is the polynomial dependence of the various vector fields on the velocity variable $\dot q$. This structure affects the Lie brackets computations involving input and drift vector fields, see related ideas in [32, 45]. We start by rewriting the system (3.1) as a first order differential equation. We write

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot q\end{bmatrix} = \begin{bmatrix} \dot q \\ -\Gamma(q,\dot q) + Y_0(q) + R(q)\dot q\end{bmatrix} + \begin{bmatrix} 0 \\ Y_a\end{bmatrix} u^a(t)$$

where $\Gamma(q,\dot q)$ is the vector with ith component $\Gamma^i_{jk}(q)\dot q^j \dot q^k$. Also, we let $x = (q,\dot q)$,

$$Z_g(x) = \begin{bmatrix} \dot q \\ -\Gamma(q,\dot q)\end{bmatrix}, \quad Y_a^{\text{lift}}(x) \triangleq \begin{bmatrix} 0 \\ Y_a(q)\end{bmatrix}, \quad \text{and} \quad R^{\text{lift}}(x) \triangleq \begin{bmatrix} 0 \\ R(q)\dot q\end{bmatrix},$$

so that the control system is rewritten as

$$\dot x = Z_g(x) + Y_0^{\text{lift}}(x) + R^{\text{lift}}(x) + Y_a^{\text{lift}}(x)u^a(t).$$

Let $h_i(q,\dot q)$ be the set of scalar functions on $\mathbb{R}^{2n}$, which are arbitrary functions of q and homogeneous polynomials in $\{\dot q^1,\dots,\dots q^n\}$ of degree i. Let $\mathcal{P}_i$ be the set of vector fields on $\mathbb{R}^{2n}$ whose first n components belong to h_i and whose second n components belong to h_{i+1}. It is easily seen that

$$Z_g \in \mathcal{P}_1, \quad R^{\text{lift}} \in \mathcal{P}_0, \quad \text{and} \quad Y_a^{\text{lift}} \in \mathcal{P}_{-1}.$$

Direct computations show that the sets $\{\mathcal{P}_i\}$ have the following properties:

(i) $[\mathcal{P}_i, \mathcal{P}_j] \subset \mathcal{P}_{i+j}$, i.e., the Lie bracket between a vector field in $\mathcal{P}_i$ and a vector field in $\mathcal{P}_j$ belongs to $\mathcal{P}_{i+j}$.

(ii) $\mathcal{P}_k = \{0\}$ for all $k \le -2$,

(iii) if $k \ge 1$, then $X(q,0) = 0$ for all $X(q,\dot q) \in \mathcal{P}_k$.

Given these properties, we investigate the Lie brackets between the vector fields Z_g and Y_a^{lift}. A few useful brackets are:

$$[Z_g, Y_a^{\text{lift}}] \in \mathcal{P}_0, \qquad [Y_a^{\text{lift}}, Y_b^{\text{lift}}] = 0,$$
$$[Y_b^{\text{lift}}, [Z_g, Y_a^{\text{lift}}]] \in \mathcal{P}_{-1}.$$

Of particular interest is the Lie bracket $[Y_b^{\text{lift}}, [Z_g, Y_a^{\text{lift}}]]$. Since this vector field belongs to $\mathcal{P}_{-1}$, there must exist a vector field on $\mathbb{R}^n$, which we denote $\langle Y_a : Y_b\rangle$, such that

$$\langle Y_a : Y_b\rangle^{\text{lift}} = [Y_b^{\text{lift}}, [Z_g, Y_a^{\text{lift}}]].$$

We call this vector field the *symmetric product* between Y_b and Y_a. Some straightforward computations in coordinates show that $\langle Y_a : Y_b \rangle = \langle Y_b : Y_a \rangle$ and that

$$\langle Y_b : Y_a \rangle^i = \frac{\partial Y_a^i}{\partial q^j} Y_b^j + \frac{\partial Y_b^i}{\partial q^j} Y_a^j + \Gamma_{jk}^i \left(Y_a^j Y_b^k + Y_a^k Y_b^j \right)$$
$$\langle Y_b : Y_a \rangle = \nabla_{Y_a} Y_b + \nabla_{Y_b} Y_a.$$

REMARK 3.1. *While the results in this sections are presented in coordinates, it is possible to turn them into coordinate-free statements on manifolds. The enabling concepts are the operation of vertical lift and symmetric product between vector fields, see [32]; the notion of geometric homogeneity, see [27]; and the intrinsic definition of the Liouville vector field; see [34, page 64].*

4. Averaging for mechanical systems under high amplitude high frequency forcing. This section contains the main result of the paper. We consider systems described by an affine connection and subject to high amplitude high frequency forcing. We show how the average system is again described by the same affine connection subject to an appropriate forcing term. Additionally, we show how the subclass of systems subject to integrable forces and without nonholonomic constraints is also closed under the operation of averaging.

The approach we take differs substantially from the classic averaging of Hamiltonian systems; see Chapter 4 in [22]. In that setting, the Hamiltonian system is integrable, and the variation of constants formula is applied by treating the ϵ size forcing as perturbation. In our setting, it is the Hamiltonian dynamics that plays the role of the perturbation to the dominant high amplitude high frequency forcing. Finally, it is important to note that while the accelerations driving the systems are high amplitude, the generated displacements are typically small in magnitude.

4.1. Systems described by affine connections. Consider a control system described by an affine connection as in equation (3.3)

$$\begin{aligned} &\frac{D\,\dot{q}}{dt} = Y_0(q) + R(q)\dot{q} \;+\; Y_a(q)(1/\epsilon)v^a(t/\epsilon) \\ &q(0) = q_0, \quad \dot{q}(0) = v_0, \end{aligned} \tag{4.1}$$

where $u^a(t) = v^a(t/\epsilon)/\epsilon$, and $\{v^1, \ldots, v^m\}$ are T-periodic functions that satisfy

$$\int_0^T v^a(s_1) ds_1 = 0, \tag{4.2}$$

$$\int_0^T \int_0^{s_2} v^a(s_1) ds_1 ds_2 = 0. \tag{4.3}$$

Also, let $v(t) = [v^1(t), \ldots, v^m(t)]'$ and define the matrix Λ according to:

$$\Lambda = \frac{1}{2T} \int_0^T \left(\int_0^{s_1} v(s_2) ds_2 \right) \left(\int_0^{s_1} v(s_2) ds_2 \right)' ds_1. \tag{4.4}$$

Finally, define the time-varying vector field

$$\Xi(t, q) = \left(\int_0^t v^a(s) ds \right) Y_a(q),$$

and the curve

$$z(t) = \big(q(t),\ \dot q(t) - \Xi\big(t/\epsilon, q(t)\big)\big)\,. \tag{4.5}$$

THEOREM 4.1. *Let $q(t)$ be the solution to the initial value problem in equation* (4.1) *and let $r(t)$ be the solution to*

$$\begin{aligned} \frac{D\dot r}{dt} &= Y_0(r) + R(r)\dot r - \sum_{a,b=1}^{m} \Lambda_{ab}\left\langle Y_a : Y_b\right\rangle(r) \\ r(0) &= q_0, \quad \dot r(0) = v_0. \end{aligned} \tag{4.6}$$

There exist a positive ϵ_0, such that for all $0<\epsilon\leq\epsilon_0$

$$\begin{aligned} q(t) &= r(t) + O(\epsilon) \\ \dot q(t) &= \dot r(t) + \Xi(t/\epsilon, q(t)) + O(\epsilon) \end{aligned} \tag{4.7}$$

as $\epsilon\to 0$ on the time scale 1.

Furthermore, let $(r,\dot r)=(q_1,0)$ be a hyperbolically stable critical point for (4.6), *and let its region of attraction contain the initial condition (q_0,v_0). Then the approximations in* (4.7) *are valid for all $t\in\mathbb{R}_+$, and the curve $z(t)$ is the solution to an initial value problem which possesses a unique, hyperbolically stable, periodic orbit belonging to an $O(\epsilon)$ neighborhood of $(q_1,0)$.*

Justified by the approximations in (4.7), we call the initial value problem in equation (4.6) the *averaged mechanical system* of the initial value problem in equation (4.1).

Proof. The proof brings together the analysis in Subsection 2.3 and in Subsection 3.2. As first step, we translate the second order equation (4.1) into the first order format in equation (2.9). We let $x=(q,\dot q)$ and

$$\begin{aligned} f(x) &= Z_g(x) + Y_0^{\mathrm{lift}}(x) + R^{\mathrm{lift}}(x), \\ g(t,x) &= \sum_{a=1}^{m} Y_a^{\mathrm{lift}}(x) v^a(t). \end{aligned}$$

Next, we compute the vector field F according to equation (2.10)

$$F(t,y) = \left((\Phi^g_{0,t})^* f\right)(y) = \left(\Phi_{0,t}^{\sum Y_a^{\mathrm{lift}}(y)v^a(t)}\right)^* (Z_g(y) + Y_0^{\mathrm{lift}}(y) + R^{\mathrm{lift}}(y)).$$

and we study its expression according to the series expansion in Section 2.2

$$(\Phi^g_{0,t})^* f = f + \sum_{k=1}^{\infty}\int_0^t \cdots \int_0^{s_{k-1}} \left(\mathrm{ad}_{g(s_k)} \ldots \mathrm{ad}_{g(s_1)} f\right) ds_k \ldots ds_1.$$

The Lie algebraic structure unveiled in Section 3.2 leads to remarkable simplifications:

$$\begin{aligned} \mathrm{ad}^k_{Y_a^{\mathrm{lift}}}(Z_g(y) + Y_0^{\mathrm{lift}}(y) + R^{\mathrm{lift}}(y)) &= 0, \qquad \forall k\geq 3, \\ \mathrm{ad}_{Y_b^{\mathrm{lift}}}\mathrm{ad}_{Y_a^{\mathrm{lift}}}(Z_g(y) + Y_0^{\mathrm{lift}}(y) + R^{\mathrm{lift}}(y)) &= -\left\langle Y_a : Y_b\right\rangle^{\mathrm{lift}}. \end{aligned}$$

With a little book-keeping we exploit these equalities and compute

$$
\begin{aligned}
&\left(\Phi_{0,t}^{\sum Y_a^{\mathrm{lift}}(y)v^a(t)}\right)^* \left(Z_g(y)+Y_0^{\mathrm{lift}}(y)+R^{\mathrm{lift}}(y)\right)\\
&\quad= \left(Z_g+Y_0^{\mathrm{lift}}+R^{\mathrm{lift}}\right)+\sum_{a=1}^m\left(\int_0^t v^a(s_1)ds_1\right)\,[Y_a^{\mathrm{lift}},\left(Z_g+Y_0^{\mathrm{lift}}+R^{\mathrm{lift}}\right)]\\
&\qquad+\sum_{a,b=1}^m\left(\int_0^t\int_0^{s_b} v^b(s_b)v^a(s_a)ds_a ds_b\right)\,[Y_b^{\mathrm{lift}},[Y_a^{\mathrm{lift}},\left(Z_g+Y_0^{\mathrm{lift}}+R^{\mathrm{lift}}\right)]]\\
&\quad= \left(Z_g+Y_0^{\mathrm{lift}}+R^{\mathrm{lift}}\right)+\sum_{a=1}^m\left(\int_0^t v^a(s_1)ds_1\right)\,[Y_a^{\mathrm{lift}},\left(Z_g+R^{\mathrm{lift}}\right)]\\
&\qquad\qquad-\sum_{a,b=1}^m\left(\int_0^t\int_0^{s_b} v^b(s_b)v^a(s_a)ds_a ds_b\right)\langle Y_a:Y_b\rangle^{\mathrm{lift}}.
\end{aligned}
$$

An integration by parts and the symmetry of the symmetric product lead to

$$
\begin{aligned}
&\sum_{a,b=1}^m\left(\int_0^t\int_0^{s_b} v^b(s_b)v^a(s_a)ds_a ds_b\right)\langle Y_a:Y_b\rangle\\
&\qquad=\frac{1}{2}\sum_{a,b=1}^m\left(\int_0^t v^b(s_b)ds_b\int_0^t v^a(s_a)ds_a\right)\langle Y_a:Y_b\rangle,
\end{aligned}
$$

so that we have

$$
\begin{aligned}
F(t,y)&=\left(Z_g+Y_0^{\mathrm{lift}}+R^{\mathrm{lift}}\right)+\sum_{a=1}^m\left(\int_0^t v^a(s_1)ds_1\right)\,[Y_a^{\mathrm{lift}},\left(Z_g+R^{\mathrm{lift}}\right)]\\
&\quad-\frac{1}{2}\sum_{a,b=1}^m\left(\int_0^t v^b(s_b)ds_b\int_0^t v^a(s_a)ds_a\right)\langle Y_a:Y_b\rangle^{\mathrm{lift}}.
\end{aligned}
\tag{4.8}
$$

Assumption (4.2) implies that the function F is T-periodic, so that we can compute its average F^0 according to equation (2.11). Given the assumption on v^a in equation (4.3) and the definition of Λ in equation (4.4), we have

$$
F^0(y)=\left(Z_g+Y_0^{\mathrm{lift}}+R^{\mathrm{lift}}\right)-\sum_{a,b=1}^m\Lambda_{ab}\langle Y_a:Y_b\rangle^{\mathrm{lift}}.
$$

This is precisely the vector field that describes the evolution of $(r,\dot r)$. This proves that $y=(r,\dot r)$. Let $\widehat{z}=(p,\dot p)$ be the flow of the vector field F starting from (q_0,v_0). Lemma 2.2 implies that over the appropriate time scale

$$
\begin{aligned}
x(t)&=\Phi_{0,t/\epsilon}^g(\widehat{z}(t))\\
\widehat{z}(t)&=y(t)+O(\epsilon),
\end{aligned}
$$

and that, should $(q_1,0)$ be a hyperbolically stable critical point for F^0, the vector field F possesses a unique, hyperbolically stable, periodic orbit in an $O(\epsilon)$ neighborhood of $(q_1,0)$.

Finally, we verify that the curve $\widehat{z}$ defined via the equality $x(t) = \Phi^g_{0,t/\epsilon}(\widehat{z}(t))$ is equal to the curve z defined in equation (4.5). In coordinates we have

$$\frac{d}{ds}\begin{bmatrix} q(s) \\ \dot{q}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ Y_a(q(s))v^a(s) \end{bmatrix}, \qquad \big(q(0), \dot{q}(0)\big) = \Phi^g_{0,t/\epsilon}\big(p(t), \dot{p}(t)\big),$$

so that at final time $s = t/\epsilon$ we compute $q(t) = q(0) = p(t)$ and

$$\dot{q}(t) = Y_a(q(0)) \int_0^{t/\epsilon} v^a(s)ds + \dot{q}(0) = \Xi(t/\epsilon, q(t)) + \dot{p}(t).$$

□

The coordinate-free treatment and the use of the Lie algebraic structure underline the connection between these results on averaging and the treatment on controllability in [32] and on motion planning in [17]. To quickly recall the first of these references, consider the control system in equation (3.3) where $Y_0 = R = 0$. If the family of vector fields $\{Y_a, \langle Y_a : Y_b\rangle, a, b = 1, \ldots, m\}$ is full rank in a neighborhood of q_0, then the control system (3.3) is small-time locally accessible from $(q_0, 0)$. Similar in these works is the key observation that a mechanical control system subject to a force Y approximately moves in the direction spanned by $\langle Y : Y\rangle$.

The novel proof methodology should facilitate further research into higher order averaging. Indeed, the work in [16] indicates that the exact solution of a mechanical control system can be written as a series expansion with terms including iterated symmetric products and time integrals.

4.2. Averaged potential for simple systems with integrable inputs. The textbook [22] presents the classic result that "the average of a Hamiltonian system forced by a bounded high frequency perturbation can be computed by averaging its Hamiltonian." For the case of high magnitude high frequency forces, the various insightful works by Baillieul [3, 4, 6] introduce the notion of *averaged potential*[2] as a means to characterize the average behavior.

In this section we assume that the original forced system is "simple," i.e, no nonholonomic constraints are present, and we answer the questions "when is the averaged system again simple?" and "what assumptions lead to the definition of an averaged potential?" Incidentally, the answer to these questions involves the relationships between various definitions of symmetric product that go back to the early treatment by Crouch [20].

We quickly review some basic concepts in simple mechanical control systems and refer to the textbooks [21, 35] for a more detailed presentation. In a mechanical system without constraints, the total energy is defined as sum of potential $V(q)$ and kinetic $\frac{1}{2}\langle\langle \dot{q}\,,\, \dot{q}\rangle\rangle = \frac{1}{2}\dot{q}^T M(q)\dot{q}$, where we denote with both $\langle\langle \cdot\,,\, \cdot\rangle\rangle$ and M the metric associated with the kinetic energy. The tensor R is weakly dissipative if $\langle\langle \dot{q}\,,\, R\dot{q}\rangle\rangle \leq 0$; it is strictly quadratically dissipative if there exists a positive constant β such that

$$\langle\langle \dot{q}\,,\, R\dot{q}\rangle\rangle \leq -\beta\langle\langle \dot{q}\,,\, \dot{q}\rangle\rangle. \tag{4.9}$$

If integrable forces are present, they are written as $Y_a(q) = \operatorname{grad}\varphi_a(q)$ for $a =$

[2]More precisely, in Baillieul's work the inputs are assumed to be high frequency bounded magnitude velocities. It is therefore very similar to our setting with high magnitude high frequency accelerations.

$1, \ldots, m$, where a gradient vector field reads in coordinates:

$$(\operatorname{grad} \varphi_a)^i = M^{ij} \frac{\partial \varphi_a}{\partial q_j}.$$

According to the treatment in [38, Chapter 12], the controlled Hamiltonian is

$$H(q, p, u) = V(q) + \frac{1}{2} p' M(q)^{-1} p - \sum_{a=1}^{m} \varphi_a(q) u^a, \tag{4.10}$$

where the momentum $p = M(q)\dot{q}$. The affine connection is the Levi-Civita connection of the metric M. The Christoffel symbols are computed according to the usual

$$\Gamma_{ij}^k = \frac{1}{2} M^{mk} \left(\frac{\partial M_{mj}}{\partial q^i} + \frac{\partial M_{mi}}{\partial q^j} - \frac{\partial M_{ij}}{\partial q^m} \right).$$

The equations of motion (4.6) take the specific form

$$\frac{D\,\dot{q}}{dt} = -\operatorname{grad} V(q) + R(q)\dot{q} \;+\; \operatorname{grad} \varphi_a(q)\, u^a(t). \tag{4.11}$$

Next we present a useful result on the symmetric product of gradient vector fields:

LEMMA 4.2 (Symmetric products of functions). *Let φ_1, φ_2 be two smooth scalar functions. The symmetric product $\langle \operatorname{grad} \varphi_1 : \operatorname{grad} \varphi_2 \rangle$ is again a gradient vector field. Additionally, if one defines a symmetric product of functions according to*

$$\langle \varphi_i : \varphi_j \rangle \triangleq \frac{\partial \varphi_i}{\partial q} M^{-1} \frac{\partial \varphi_j}{\partial q} = \langle\!\langle \operatorname{grad} \varphi_i \,,\, \operatorname{grad} \varphi_j \rangle\!\rangle, \tag{4.12}$$

then

$$\langle \operatorname{grad} \varphi_1 : \operatorname{grad} \varphi_2 \rangle = \operatorname{grad} \langle \varphi_1 : \varphi_2 \rangle\,.$$

This result was originally proven by Crouch in [20], where this symmetric product of functions was presented under the name of Beltrami bracket. It is interesting to note how, in contrast to the treatment in [20], this symmetric operation is here relevant in a Hamiltonian system context.

Finally, we are ready to apply Theorem 4.1 to the setting of simple systems.

THEOREM 4.3. *Consider the simple mechanical control system in equation* (4.11) *with Hamiltonian in equation* (4.10). *Let $u^a(t) = v^a(t/\epsilon)/\epsilon$ and let the functions v^a satisfy the condition in equation* (4.3). *It follows that the averaged system is a simple mechanical system subject to no force and with Hamiltonian*

$$H_{\text{averaged}}(q, p) = V_{\text{averaged}}(q) + \frac{1}{2} p' M(q)^{-1} p,$$

where the averaged potential *is defined as*

$$V_{\text{averaged}}(q) \triangleq V(q) + \sum_{a,b=1}^{m} \Lambda_{ab} \langle \varphi_a : \varphi_b \rangle (q). \tag{4.13}$$

Accordingly, the equations of motions for the averaged system are

$$\frac{D\,\dot{q}}{dt} = -\operatorname{grad}(V_{\text{averaged}}) + R(q)\dot{q}.$$

The result follows directly from Lemma 4.2 and Theorem 4.1. Theorem 4.3 can be used as follows. In order to stabilize a mechanical control system we design oscillatory inputs that render V_{averaged} positive definite about the desired equilibrium point. The next section presents this idea in detail.

5. Vibrational stabilization of mechanical systems. In this section we apply the averaging results to stabilization problems. We focus on simple mechanical systems, consider the point stabilization problem via oscillatory inputs, and rely on the averaged Hamiltonian as candidate control Lyapunov function; see [44].

We start by presenting the notion of vibrational stabilization according to the treatments in [9, 10, 11]. Consider the control system

$$\frac{dx}{dt} = f(x) + g_a(x)u^a(t). \tag{5.1}$$

A critical point x_1 of f is said to be *vibrationally stabilizable*[3] if for any $\delta > 0$ there exist almost-periodic zero-average inputs $u^a(t)$ such that the system in equation (5.1) has an asymptotically stable almost periodic solution $x^*(t)$ characterized by

$$\|\overline{x}^* - x_1\| \leq \delta, \qquad \overline{x}^* = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^*(s)ds.$$

REMARK 5.1. *We refer to [9, 10, 11] for the vibrational stabilization theory for systems controlled by vector additive, linear and nonlinear multiplicative forcing. Adopting these definitions, the vibrational stabilization problem we consider corresponds to a nonlinear multiplicative setting; see [11]. In that paper, the ith component of the vibrational forcing depends only on the ith state variable. This requirement is removed here and the structure of the nonlinearities we consider is more general.*

5.1. Stabilization in systems with integrable inputs. Once more, consider the control system in equation (4.11):

$$\frac{D\,\dot{q}}{dt} = -\operatorname{grad} V(q) + R(q)\dot{q} \ + \ \operatorname{grad} \varphi_a(q)u^a(t). \tag{5.2}$$

We present a notion of vibrational stabilization tailored to mechanical systems. A configuration q_1 is said to be *vibrationally stabilizable* if for any $\delta > 0$ there exist almost-periodic zero-average inputs $u^a(t)$ such that the system in equation (5.2) has an asymptotically stable almost periodic solution $q^*(t)$ characterized by

$$\|\overline{q}^* - q_1\| \leq \delta, \qquad \overline{q}^* = \lim_{T\to\infty} \frac{1}{T}\int_0^T q^*(s)ds. \tag{5.3}$$

This definition is weaker than the general one above since no requirement is imposed on the behavior of the velocity variables $\dot{q}$.

Next, we design vibrationally stabilizing control laws. The following useful lemma focuses on "inverting" the definition of $\Lambda = \Lambda(v^1, \ldots, v^m)$ in equation (4.4).

LEMMA 5.2 (Design of vibrations). *Let $t \in [0, T]$ and define a vector-valued function of time $v(t) = [v^1(t), \ldots, v^m(t)]'$ that satisfies equations* (4.2) *and* (4.3)*. Any matrix Λ computed according to equation* (4.4) *is symmetric and positive semidefinite. Vice versa, given any symmetric positive semidefinite matrix Λ, there exists a vector-valued function of time v that satisfies equations* (4.2)*,* (4.3)*, and* (4.4)*.*

Proof. Obviously Λ is symmetric, and for any vector $x \in \mathbb{R}^m$, one has

$$x'\Lambda x = \int_0^T \left(\int_0^{s_1} \big(x'v(s_2)\big)ds_2 \right)^2 ds_1 \geq 0.$$

[3] Baillieul and Lehman [6] assume both the inputs and the asymptotically stable solution x^* to be T-periodic.

Given any symmetric positive semidefinite Λ, we design inputs that satisfy equations (4.2), (4.3) and (4.4). First, we introduce the T-periodic base functions

$$\psi_i(t) = \frac{4\pi i}{T} \cos\left(\frac{2\pi\, i}{T} t\right), \quad i \in \mathbb{N}.$$

Any linear combination of the $\{\psi_i\}$ satisfies equations (4.2), (4.3) and

$$\frac{1}{2T} \int_0^T \left(\int_0^{s_1} \psi_i(s_2) ds_2 \right) \left(\int_0^{s_1} \psi_j(s_2) ds_2 \right) ds_1 = \delta_{ij},$$

where δ_{ij} is the Kronecker delta. Next, we diagonalize Λ via an orthogonal similarity transformation W. Assuming the rank of Λ is $p \leq m$, we have

$$\Lambda = W \operatorname{diag}([\lambda_1, \ldots, \lambda_p, 0, \ldots, 0])\, W' = \sum_{i=1}^{p} (\sqrt{\lambda_i} W e_i)(\sqrt{\lambda_i} W e_i)',$$

where $\operatorname{diag}([\lambda_1, \ldots, \lambda_p, 0, \ldots, 0])$ is the diagonal matrix with non-vanishing elements $\{\lambda_1, \ldots, \lambda_p\}$, and where $\{e_i, \ldots, e_n\}$ is the usual basis for $\mathbb{R}^n$. Since the vectors $(\sqrt{\lambda_i} W e_i)$ are uniquely determined by Λ, we define

$$w(t, \Lambda) = \sum_{i=1}^{p} (\sqrt{\lambda_i} W e_i)\ \psi_i(t). \tag{5.4}$$

By construction, $v(t) = w(t, \Lambda)$ satisfies equations (4.2), (4.3) and (4.4). $\square$

Introduce the control gains $k_1 \in \mathbb{R}^m$, $K_2, K_3 \in \mathbb{R}^{m \times m}$, subject to $K_2 = K_2' \geq 0$ and $K_3 = K_3' \geq 0$. To simplify notation, let $\varphi = [\varphi^1, \ldots, \varphi^m]$ and let the $m \times m$ matrix $\langle \varphi : \varphi \rangle (q)$ have (a, b) component $\langle \varphi_a : \varphi_b \rangle (q)$. Let the control input be the sum of open (feedforward) and closed loop (feedback) terms

$$u(t, \epsilon) = -k_1 - K_2 \varphi + (1/\epsilon) w(t/\epsilon, K_3), \tag{5.5}$$

where w is defined in equation (5.4). According to Theorem 4.3 and to the lemma above, the averaged controlled system is Hamiltonian with potential energy given by

$$V_{\text{control}}(q) = V(q) + k_1' \varphi(q) + \frac{1}{2} \varphi(q)' K_2 \varphi(q) + \operatorname{Trace}\big(K_3 \langle \varphi : \varphi \rangle \big), \tag{5.6}$$

where the Trace operation is equivalent to the summation in equation (4.13). It is useful to note that V_{control} depends linearly on the control gains k_1, K_2, K_3.

Existence and stability of equilibrium points are analyzed according to the classic potential energy criterion. The configuration q_1 is an equilibrium point if it is a critical point for the averaged controlled potential energy V_{control}; it is locally/globally stable if V_{control} has a local/global minimum at q_1. Of course, the point is stable only in the average approximation. We make this point precise in the following theorem.

THEOREM 5.3 (Vibrational stabilization of configurations). *Consider the control system in equation* (5.2), *assume the tensor R is strictly quadratically dissipative. Let $q_1 \in \mathbb{R}^n$ and consider the following set of linear matrix equality and inequalities in the free variables k_1, K_2, K_3:*

$$\begin{aligned} K_2 = K_2' &\geq 0, & K_3 = K_3' &\geq 0 \\ \frac{\partial V_{\text{control}}}{\partial q}(q_1) &= 0, & \frac{\partial^2 V_{\text{control}}}{\partial q^2}(q_1) &> 0. \end{aligned} \tag{5.7}$$

If the convex problem (5.7) *is feasible, the configuration* q_1 *is vibrationally stabilizable and there exists an* $\epsilon_0 > 0$ *such that stabilizing controls are computed according to equation* (5.5)*, with* $0 < \epsilon \leq \epsilon_0$ *and with* k_1, K_2, K_3 *solutions to the system of equations* (5.7)*.*

Proof. As first step, we prove that $(q_1, 0)$ is a locally exponentially stable point for the averaged controlled system. We follow a well-known procedure, see [47], and rely on Theorem 4.3 and Lemma 5.2. At $q = q_1$, the function V_{control} in equation (5.6) and its gradient vanish, while its Hessian is positive definite. The total energy $H_{\text{control}}(q, \dot{q}) \triangleq V_{\text{control}}(q) + \frac{1}{2}\dot{q}' M \dot{q}$ is therefore positive definite about $(q_1, 0)$. Because R is strictly quadratically dissipative, there exists a $\beta > 0$ such that along the solutions of the averaged controlled system

$$\dot{H}_{\text{control}} = -\beta \langle\langle \dot{q} \,,\, \dot{q} \rangle\rangle.$$

The function H_{control} is a Lyapunov function for the averaged controlled system and $(q_1, 0)$ is a stable equilibrium point. Asymptotic stability follows from an application of LaSalle's lemma; exponential stability follows from a linearization argument.

As second step, we prove that the controlled system has a unique periodic exponentially stable solution $q(t)$ in a neighborhood of q_1. We follow a well-know procedure, see [10], and rely on Theorem 4.1. Since the averaged system has an exponentially stable point, the curve $z(t)$ is a solution to a differential equation which possess a unique periodic orbit, say $z^\star(t)$ which is exponentially stable and belongs to an $O(\epsilon)$ neighborhood of $(q_1, 0)$. The same statement can be made for the first component of $z(t)$, that is, the curve $q(t)$. We call this perioc orbit $q^\star(t)$, and its average $\overline{q}^\star$, as defined in equation (5.3). Since $q^\star(t)$ lives in a $O(\epsilon)$ neighborhood of q_1, so does $\overline{q}^\star$. Therefore there must exist ϵ_0 such that $\|\overline{q}^* - q_1\| \leq \delta$, for any $\delta > 0$. □

The stability result relies on the open-loop system having full rank dissipation, i.e., the tensor R is required to be strictly quadratically dissipative. This requirement can be weakened by augmenting the control input with a "derivative action" (a term negatively proportional to the velocity). Asymptotic stability is then guaranteed under a linear controllability like condition; see [47, 15].

The location of the poles of the linearized model about q_1 affects the behavior of the controlled system. Given that a large oscillatory signal is superimposed, better performance is achieved when these poles are far to the left of the imaginary axis. This and related performance requirements can be addressed within the linear matrix equality and inequality formulation; see the surveys in [48, 14].

5.2. Vibrational stabilization of an underactuated two-link manipulator. We present a simple example of vibrational stabilization. We consider a planar two-link manipulator as depicted in Figure 5.1: no potential energy is present. We assume the manipulator is subject to damping forces at both angles.

The configuration of the system is described by the pair (θ_1, θ_2), where θ_1 is the angle between the first link and the horizontal axis, and θ_2 is the relative angle between the two links. Both angles are measured counterclockwise. The links' physical parameters are: length ℓ, mass m and moment of inertia I. We let $\ell_1 = 3, \ell_2 = 4$ and $m_1 = I_1 = \ell_1^2$ and $m_2 = I_2 = \ell_2^2$. A known procedure provides the inertia matrix:

$$M(q) = \begin{bmatrix} \frac{1013}{4} + 192\cos(\theta_2) & 16(5 + 6\cos(\theta_2)) \\ 16(5 + 6\cos(\theta_2)) & 80 \end{bmatrix}.$$

We assume the system is subject to the damping force $(-.2\dot{\theta}_1, -.2\dot{\theta}_2)$, and to a single control input, i.e., a torque τ applied at the first joint. Accordingly, the force can

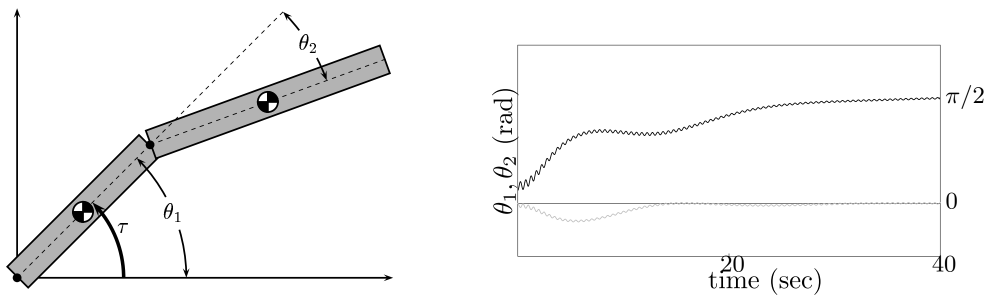

FIG. 5.1. *Two-link manipulator: θ_1 and θ_2 are measured counterclockwise. In the right figure, the gray line is θ_1, the black line is θ_2. Despite the superimposed oscillatory behavior the variables (θ_1, θ_2) converge to the global minimum of the averaged controlled potential energy.*

be described by the function $\varphi(q) = \theta_1$. The symmetric product is easily computed according to Lemma 4.2:

$$\langle \varphi : \varphi \rangle (q) = \frac{20}{2313 - 1152\cos(2\theta_2)}.$$

We adopt the control law in equation (5.5) and compute the averaged controlled potential according to equation (5.6):

$$V_{\text{control}}(q) = k_1\theta_1 + \frac{1}{2}k_2\theta_1^2 + k_3\frac{20}{2313 - 1152\cos(2\theta_2)}.$$

At $k_1 = 0$ and for any positive k_2 and k_3, the function V_{control} has two global minima at $(\theta_1, \theta_2) = (0, \pm\pi/2)$.

We run the simulation as follows. We design the control law parameters as $\epsilon = .5$, $T = 1$, $k_2 = 15$, and $k_3 = 150$. At initial time, the manipulator is at rest with angles $(\theta_1(0), \theta_2(0)) = (0, \pi/16)$. This initial condition is in the domain of attraction of the minimum $(\theta_1, \theta_2) = (0, \pi/2)$. The differential equation solver `NDSolve` within Mathematica generated the simulation results reported in Figure 5.1.

We conclude the example with a final remark. The stabilization result is not surprising and it intuitively agrees with the classic example in [6], where the controlled variable is the speed of the joint connected to the second link, and where the joint itself is constrained to move vertically.

6. Conclusions. This paper provides a systematic study of high magnitude high frequency averaging for mechanical systems. The averaging extends the results of earlier works in two directions. First, the analysis applies to the multi-input setting where controls are not necessarily applied to cyclic variables. Instead, forces are described as generic one-forms. Additionally, our analysis applies to the case of mechanical systems with nonholonomic constraints. From a control design viewpoint, the improved analysis leads to sufficient tests for an appropriate notion of vibrational stabilization.

At the heart of the proposed approach is a detailed analysis of the Lie algebraic structure of mechanical systems (with or without constraints, with or without nonintegrable forces). It is this structure that enables closed form expressions for the averaging analysis. Furthermore, it is this same structure that underlies the controllability analysis in [32]. Our analysis provides a missing link between the notions of averaged potential [3] and symmetric product [32].

Numerous extensions appear promising. First, one could pursue generalizations to high order averaging and applications in the field of robotic motion planning; see [17, 16]. Second, the setting of distributed parameter systems with Lagrangian structure might provide a number of interesting applications and further theoretical challenges. Finally, the tools developed here might be shed new light on the problem of existence and stability of limit cycles in the study of animal and robotics locomotion.

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Appendix A. The variations of constants formula in geometric terms.

Lemma A.1. *Let f, g be smooth time-varying vector fields on $\mathbb{R}^n$. Let $x_0 \in \mathbb{R}^n$, and let $T \in \mathbb{R}^n$ be small enough so that the flow map $\Phi^g_{0,T}$ is a local diffeomorphism* *in a neighborhood of* x_0*. The final value* $x(T) = \Phi^{f+g}_{0,T}(x_0)$ *can be written as*

$$x(T) = \Phi^g_{0,T}(z(T)), \tag{A.1}$$

$$\dot{z}(t) = \left(\left(\Phi^g_{0,t}\right)^* f\right)(z), \quad z(0) = x_0. \tag{A.2}$$

Additionally, we have the formal equality

$$\begin{aligned}\left(\left(\Phi^g_{0,t}\right)^* f\right)(t,x) &= f(x)\\ &+ \sum_{k=1}^{\infty} \int_0^t \cdots \int_0^{s_{k-1}} \left(\mathrm{ad}_{g(s_k,x)} \ldots \mathrm{ad}_{g(s_1,x)} f(x)\right) ds_k \ldots ds_1.\end{aligned} \tag{A.3}$$

Proof. Let $x(T) = \Phi^{f+g}_{0,T}(x_0)$ and let $y(T) = \Phi^g_{0,T}(z(T))$, where $z(t)$ is computed via equation (A.2). We compute

$$\begin{aligned}\dot{z} &= \left((\Phi^g_{0,t})^* f\right)(z) = \left(T_z \left(\Phi^g_{0,t}\right)^{-1} \circ f \circ \Phi^g_{0,t}\right)(z)\\ &= \left(T_z \Phi^g_{0,t}\right)^{-1} \circ f(y(t),t),\end{aligned}$$

so that

$$\begin{aligned}\dot{y}(t) &= \frac{d}{dt}\left(\Phi^g_{0,t}(z(t))\right) = g\left(\Phi^g_{0,t}(z(t)),t\right) + \left(T_z \Phi^g_{0,t}(z(t))\right)\dot{z}\\ &= g\left(y(t),t\right) + \left(T_z \Phi^g_{0,t}(z(t))\right)\dot{z} = g\left(y(t),t\right) + f(y(t),t).\end{aligned}$$

Therefore, $y(t)$ obeys the same differential equation as $x(t)$. Since it is also clear that $x(0) = y(0)$, the curves x and y must be equal.

Next, we investigate the pull-back of f along the flow of g. We assume f to be time-invariant and g time-varying. The following statement is proved in [1, Theorem 4.2.31] and in [2, equation 3.3]:

$$\frac{d}{dt}\left(\left(\Phi^g_{0,t}\right)^* f\right)(t,x) = \left(\Phi^g_{0,t}\right)^* [g(t,x), f(x)],$$

where the Lie bracket between g and f is computed at t fixed. At fixed $x \in \mathbb{R}^n$, we integrate the previous equation from time 0 to t to obtain

$$\left(\left(\Phi^g_{0,t}\right)^* f\right)(t,x) = f(x) + \int_0^t (\Phi^g_{0,s})^* [g(s,x), f(x)] ds.$$

The formal expansion in equation (A.3) follows from iteratively applying the previous equality. ∎