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# Stabilization of Relative Equilibria for Underactuated Systems on Riemannian Manifolds*

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**Abstract**

This paper describes a systematic procedure to exponentially stabilize relative equilibria of mechanical systems. We review the notion of relative equilibria and their stability in a Riemannian geometry context. Potential shaping and damping control are employed to obtain full exponential stabilization of the desired trajectory. Two necessary conditions are that the effective potential be positive definite over a specified subspace and that the system be linearly controllable. Relevant applications to underwater and aerospace vehicle control are described.

**Keywords:** nonlinear control, mechanical systems with symmetry, exponential stabilization, underwater vehicle

## Contents



*Short versions of this work have appeared in [6, 7].

# 1 Introduction

Control of underactuated mechanical systems is a challenging research area of increasing interest. On the theoretical side, control problems for mechanical systems benefit from the wealth of geometric mechanics tools available. On the other hand, strong motivation for these problems comes from applications to autonomous vehicles design and control.

In this paper, we investigate stabilization techniques for the steady motions called relative equilibria. This family of trajectories is of great interest in theory and applications as they provide a rich family of motions with the simplifying property of having constant body-fixed velocity. Relative equilibria for systems in three dimensional Euclidean space include straight lines, circles, and generic helices.

A wide variety of control techniques have been employed to tackle the kind of stabilization problems of interest in this work. Typically, gain scheduling approaches have been widely employed in applications. These methods include works that rely on linear parameter varying systems; see Wu et al. [44], or that explicitly focus on rigid bidy dynamics; see for example Kaminer et al. [18]. Recently, attention has recently focused on the notion of differential flatness for the purpose of trajectory generation and tracking; see [10, 31]. Related results in this direction include work on flatness for mechanical systems; see [33, 38], and on approximate linearization; see [2, 30].

Stabilization via backstepping and forwarding have encountered success in various nonlinear control fields, see [21, 39], and various applications to ship control are described in the work of Fossen and co-workers [11, 12, 13]. The difficulty in applying these techniques to underactuated mechanical systems is that these systems are typically not in strict feedback form and that the presence of un-stable zero dynamics renders the design of control Lyapunov functions difficult; for various attempts to study this problem see [14, 15]. Nevertheless, Pettersen and Nijmeijer [36, 37] have recently overcome these limitations for a model of underactuated ship and provided a full state tracking controller.

A line of research that has been parallel to the efforts described so far is that on geometric control for mechanical systems. Stabilization of underactuated Hamiltonian systems was originally investigated by van der Schaft [42]; see [34] for a standard treatment. Recently, the emphasis has been on models of vehicles, i.e., on the class of mechanical systems with symmetries. Stability of underwater vehicles is studied in Leonard [24] where symmetry breaking potentials are employed to shape the energy of the closed loop system. Jalnapurkar and Marsden [16] present a framework for the design of dissipative controllers for underactuated mechanical systems. In these treatments the family of input forces is assumed momentum preserving and stability in the reduced space is characterized via the Energy-Momentum method. Finally, recent works have extended the kind of feedback transformations that retain the mechanical structure of the system and extended the applicability

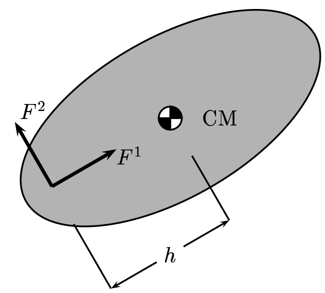


Figure 1: Planar vehicle with two body-fixed forces $\{F^1, F^2\}$ a distance $h$ from the center of mass CM.

of these geometric stabilization methods; see [4, 5].

Within the context of the geometric stabilization techniques, this work focuses on vehicles with generic body forces, including both internal (e.g, momentum wheels and sliding masses) and external ones (e.g., propellers). Typically, these systems move on trajectories that do not belong to a constant momentum level set. A simple idealized example is a planar body, depicted in Figure 1. This model is reminiscent of the V/STOL aircraft studied by Hauser et al. [15], of the surface vessel studied by Pettersen and Nijmeijer [36], and of the underwater submersible studied by Leonard and Woolsey [24, 26]. This particular systems is proven to be differentially flat by Martin et al. [31] when no hydrodynamic forces are present. The investigation by Rathinam and Murray [38] indicates how differential flatness might not be applicable to the class of systems of interest here.

This paper focuses on the exponential stabilization problem (as opposed to Lyapunov or asymptotic stabilization) on the full phase space (as opposed to stabilization on a reduced space). As compared to [16, 24, 42], we investigate potential shaping and damping control in this more general context and present a Riemannian geometry formulation that allows for more general control forces. We require certain stability properties of the unforced system and strengthen these stability properties via feedback.

The main contribution of this paper is the design of a Lyapunov function and of a corresponding controller that stabilize all of the variable of interest, that is, all of the velocity variables and the internal configuration variables. We refer to this notion as stabilization on the full phase space, as opposed to stabilization for only the internal variables or stabilization on a momentum surface. In other words, the stabilization result in this paper focuses on full state tracking control instead of output tracking. The Lyapunov function is designed by taking advantage of the constants of motion of the Lagrangian system and of some concepts from geometric mechanics. The key observation is that the so-called effective Hamiltonian is a positive definite function with respect not only to internal configurations, see [16], but also to all of the velocity variables.

A second theme is the emphasis on exponential as opposed to asymptotic convergence. We bring to bear the full power of dissipation-based stabilization techniques, and in particular we exploit the fact that a dissipative system has exponential convergence rates under the assumption of linear controllability and the existence of a quadratic Lyapunov function. While this fact is known within the nonlinear stabilization literature, see for example [17, 22, 39], it has not been fully exploited within the context of mechanical systems.

We present a brief self-contained treatment of the needed results.

Finally, a third feature of our approach is that we employ a novel Riemannian geometry formalism in describing relative equilibria and their stabilization. One important advantage of this approach is that it is capable of dealing with general (but velocity independent) control forces and this is a useful feature when dealing with models of vehicles such as that in Figure 1; see the following section for a more precise statement. This approach leads to a self contained treatment, all the way from the definition of relative equilibrium to sufficient conditions for stability and stabilization. Lastly, a third motivation for employing this description of Lagrangian systems is the recent increasing success of this approach as testified by the contributions on modeling [3], stabilization [20], controllability [27, 28], interpolation [35, 43] and dynamic feedback linearization [38].

The paper is organized as follows. We present a preliminary and summary description of the content of this paper in the following Section 1.1. This section is written with the intent of being largely accessible. In Section 2 we review some stabilization techniques for nonlinear control systems based on Lyapunov functions. Section 3 introduces formally the notion of mechanical system with symmetry and of relative equilibrium. Here we also make the fundamental observation that the effective Hamiltonian is well suited to play the role of positive definite Lyapunov function on the entire phase space. In Section 4 we extend the notion of proportional derivative control to the current setting. This entails an understanding of potential shaping and of dissipative forces (damping control) along relative equilibria. Section 5 contains an underwater and an aerospace example of the design procedure.

## 1.1 Preliminary example and summary of results

We briefly present the key steps of our design procedure by applying it to the planar vehicle in Figure 1. More specifically, we attempt to design some feedback controls that stabilize a trajectory consisting of a steady motion along a straight line, e.g., the $x$ axis of an inertial reference frame. This motion is called a *relative equilibrium*; we refer to Section 3 for a precise definition.

### Stability of an equilibrium point

We start by reviewing some classic results on stability of mechanical systems about a point. Let $q = [q^1, \dots , q^n]^T$ be the configuration of the systems and let the Hamiltonian be

$$H(q, \dot{q}) = V(q) + \frac{1}{2}\dot{q}^T M(q)\dot{q} =: V(q) + \frac{1}{2}\|\dot{q}\|^2_M. \tag{1}$$

The equations of motion are

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = -\frac{\partial V}{\partial q} + F, \tag{2}$$

where $C(q, \dot{q})$ is the Coriolis matrix and where the resultant force $F = F(q, u)$ can be written as linear combination of $m$ independent control forces $F_i$:

$$F(q, u) = \sum_{i=1}^{m} F_i(q)u_i. \tag{3}$$

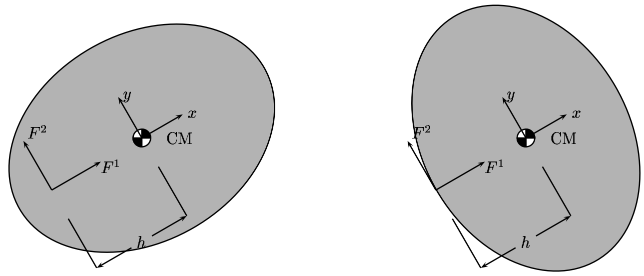


Figure 2: Planar underwater ellipsoids with two body-fixed forces $\{F^1, F^2\}$. The body inertia and the effect of the surrounding fluid are modeled via added masses $m_x$ and $m_y$, with $m_x < m_y$ in the left system and $m_x > m_y$ in the right system.

As mentioned in the Introduction, the $m$ input vectors $F_i(q)$ are arbitrary functions of $q$, but velocity independent. It is of interest to consider the situation where $m$ is strictly less than the degrees of freedom: such systems are called underactuated.

Recall that $q_0$ is a stable equilibrium point if the first variation of $V$ vanishes and if the second variation is positive definite at $q_0$, that is, if

$$\frac{\partial V}{\partial q}(q_0) = 0, \qquad \text{and} \qquad \frac{\partial^2 V}{\partial q \partial q}(q_0) > 0.$$

**Steady translations of the planar vehicle**

Next, we examine the planar vehicle in Figure 1 with the purpose of stabilizing not a point, but a trajectory described by a straight line. Let $q = [\theta, x, y]^T \in \mathbb{R}^3$ denote the position of the vehicle. Assuming that gravity plays no role, the Hamiltonian is

$$H(q,\dot{q}) = \frac{1}{2}\dot{q}^T \begin{bmatrix} I & 0 & 0 \\ 0 & m_x(\cos\theta)^2 + m_y(\sin\theta)^2 & (m_x - m_y)\cos\theta\sin\theta \\ 0 & (m_x - m_y)\cos\theta\sin\theta & m_y(\cos\theta)^2 + m_x(\sin\theta)^2 \end{bmatrix} \dot{q}, \tag{4}$$

where $\dot{q} = [\dot{\theta}, \dot{x}, \dot{y}]^T$ is the velocity of the body and where $I, m_x, m_y$ are inertial parameters that include the influence of the fluid surrounding the vehicle (e.g., they include the so-called added masses); see [23]. Let $v_{\mathrm{re}} \in \mathbb{R}^3$ denote the velocity of the desired steady motion; e.g., we set $v_{\mathrm{re}} = [0, 1, 0]^T$ to require the vehicle to move at unit speed along the $x$ axis of a reference frame.

The relative magnitude of the added masses along the $x$ and $y$ body fixed axis (i.e., $m_x$ versus $m_y$) plays an important role in stability analysis. As later computations will show, motion along the body-fixed $x$ axis enjoys certain stability properties if $m_x > m_y$. Accordingly, it is simpler to design stabilizing controllers for translations along the short axis (system on the right of Figure 2) as opposed to the long axis (system on the left).

Notice that the Hamiltonian does not depend on the variable $x$, that is $\partial H/\partial x = 0$. As it is known in mechanics, this independence implies that the momentum in the direction $v_{\mathrm{re}}$ is a conserved quantity along the solution of the unforced equations of motion. The latter quantity is defined as

$$\begin{aligned} J_{\mathrm{re}}(q,\dot{q}) &= \dot{q}^T M(q) v_{\mathrm{re}} \\ &= \big(m_x(\cos\theta)^2 + m_y(\sin\theta)^2\big)\dot{x} + \big((m_x - m_y)\sin\theta\cos\theta\big)\dot{y}. \end{aligned}$$

Beside Hamiltonian and momentum, an additional (but not independent) conserved quantity is computed via some algebraic manipulation ("summing the square") as follows:

$$\begin{aligned} H_{\mathrm{re}}(q,\dot{q}) &:= H - J_{\mathrm{re}} \\ &= \frac{1}{2}\|\dot{q} - v_{\mathrm{re}} + v_{\mathrm{re}}\|_M^2 - \dot{q}^T M(q) v_{\mathrm{re}} \\ &= -\frac{1}{2}\|v_{\mathrm{re}}\|_M^2 + \frac{1}{2}\|\dot{q} - v_{\mathrm{re}}\|_M^2. \end{aligned}$$

We call $H_{\mathrm{re}}$ the *effective Hamiltonian*, and accordingly we define the *effective potential* as

$$\begin{aligned} V_{\mathrm{re}}(q) &= -\frac{1}{2}\|v_{\mathrm{re}}\|_M^2 \\ &= -\frac{1}{2}\left(m_x(\cos\theta)^2 + m_y(\sin\theta)^2\right). \end{aligned}$$

The concepts of effective Hamiltonian and potential lead to an elegant parallel between the treatment on stability of an equilibrium point and stability of a steady motion. For example, $H_{\mathrm{re}}$ has a "kinetic energy" component proportional to the velocity error $(\dot{q} - v_{\mathrm{re}})$, as opposed to the usual kinetic energy being proportional to the velocity $\dot{q}$. Additionally, we will show the following results. The steady motion $v_{\mathrm{re}}$ through the point $q_0$ is a solution to the equations of motion if the first variation of $V_{\mathrm{re}}$ vanishes at $q_0$,

$$\frac{\partial V_{\mathrm{re}}}{\partial q}(q_0) = 0,$$

and it is a stable motion if the second variation of $V_{\mathrm{re}}$ restricted to the subspace perpendicular to $v_{\mathrm{re}}$ is positive definite:

$$\frac{\partial^2 V_{\mathrm{re}}}{\partial q \partial q}(q_0) > 0, \qquad \text{restricted to } v_{\mathrm{re}}^{\perp}.$$

Notice how $V_{\mathrm{re}}(q)$ is independent from $x$ and therefore its second variation cannot be positive definite as a 2 form on $\mathbb{R}^3$. In the planar body example, steady translation along the $x$ axis is a solution whenever $\sin 2\theta = 0$. However, this analysis[1] tells us nothing about the stability of this motion since $V_{\mathrm{re}}(q)$ is independent of $y$ and hence its second variation is not positive definite when restricted to $v_{\mathrm{re}}^{\perp}$.

### Proportional derivative control for steady translations

A control technique that improves the stability properties of an equilibrium point is proportional and derivative control. A proportional action $F_P$ is a control force proportional to the first variation of a function $f$:

$$F_P = -\frac{\partial f(q)}{\partial q}.$$

[1] The analysis in [25] would provide us with more information on the stability of the unforced system. This is not of interest today as the use of feedback will be employed to obtain strong stability properties.

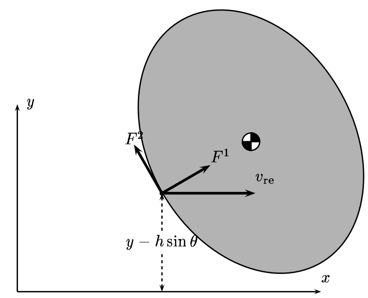


Figure 3: The function $(y - h\sin\theta)$ is constant along a steady translation aligned with $v_{\rm re}$ and its first variation lies in the span of the input force $\mathrm{span}\{F^1, F^2\}$.

Under such a feedback, the closed loop system satisfies an equation of motion of the form (2), where the closed loop potential energy equals $(V + f)$ and the stability of the equilibrium point $q_0$ depends on whether the second variation of this new potential is positive definite. A dissipative, or damping, action $F_D$ is usually defined as a control force proportional to the velocity, and it is employed to turn a stable equilibrium point into an asymptotically stable one. In particular, for an underactuated system (2) one would set:

$$F_D = \sum_{i=1}^{m} F_i \, (-F_i^T \dot{q}).$$

To render the steady translations of the planar vehicle first stable and then asymptotically stable, we adapt proportional derivative control to the current setting. In particular, we have two fundamental constraints in the design of the proportional action. The proportional control must preserve the existence of the steady solution, that is the closed loop Hamiltonian must remain independent of $x$, and must lie in the span of the available forces $\{F^1, F^2\}$; see Figure 1. In other words, we employ a feedback $\partial f/\partial q$ where the function $f$ is required to satisfy $\partial f/\partial x = 0$ and $\partial f/\partial q \in \mathrm{span}\{F^1, F^2\}$.

In Section 4.1 we provide a methodology to design such functions. Within the present context, it suffices to note that in the planar vehicle example $f(q) = (y - h\sin\theta)^2$ satisfies these constraints. Additionally, under the simplifying assumption $m_x > m_y$ the second variation of

$$(V_{\rm re} + f) = -\frac{1}{2}\left(m_x(\cos\theta)^2 + m_y(\sin\theta)^2\right) + (y - h\sin\theta)^2$$

is positive definite when restricted to $v_{\rm re}^{\perp}$. The steady translation $v_{\rm re}$ through $(\theta_0, x_0, y_0) = (0, 0, 0)$ is therefore stabilized by the proportional feedback $-\partial f/\partial q$.

Finally, we employ dissipation (damping control) to render the steady translation exponentially stable. Since the nominal velocity is $\dot{q} = v_{\rm re}$, we expect the correction should be proportional to the velocity error $\dot{q} - v_{\rm re}$. In fact, in Section 4.2, we show how the feedback controls

$$u_i = -F_i^T(\dot{q} - v_{\rm re}).$$

lead the desired exponential convergence rates (under appropriate assumptions). The stabilization will be obtained on the full space modulo $\mathbb{R}$, i.e., the converge will be in terms of $\dot{q} \mapsto v_{\rm re}$ and $(\theta, y) \mapsto (0, 0)$.

Notice how this stabilization result relies on the $m_x > m_y$ assumptions, i.e., it holds for the planar ellipsoid on the right of Figure 2 and not the one on the left. Therefore motion along the body-fixed x axis was already somehow stable - now however we specify what axis and we render the convergence exponential

# 2 Stabilization techniques for nonlinear systems

Before focusing on mechanical control systems and relative equilibria, we review some stabilization results for general nonlinear systems. Let $M$ be a smooth $n$ dimensional manifold and consider the control system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{5}$$

where $f, g_i$ are smooth vector fields and $u_i(t)$ are bounded measurable functions. Let $x_0$ be an equilibrium point for $f$, and let $V : M \to \mathbb{R}_{\geq 0}$ be a smooth function such that $V(x_0) = 0$ and that

$$V(x) > 0, \qquad \forall x \in B(x_0) - \{x_0\}, \tag{6}$$

where $B(x_0)$ is a neighborhood of $x_0$. Additionally, we require $V$ to be proper at $x_0$, i.e., the set $\{x \in B(x_0) |\ V(x) \leq \epsilon\}$ is a compact subset of $B(x_0)$ for each $\epsilon$ small enough.

**Lemma 2.1.** *Let $f, g_i, V$ be as described in equations* (5) *and* (6)*. The following stability results hold for $x \in B(x_0)$.*

*(i) If the Lyapunov function is a first integral of $f$, and $u_i$ are dissipative inputs, that is, if*

$$0 = \mathcal{L}_f V(x),$$
$$u_i(x) = -\mathcal{L}_{g_i} V(x),$$

*then the point $x_0$ is Lyapunov stable in the sense that $V(x(t)) \leq V(x(0))$. If the system satisfies the linear controllability rank condition at all $x \in B(x_0)$, i.e., if*

$$\operatorname{rank}\{g_i, \operatorname{ad}_f g_i, \dots, \operatorname{ad}_f^n g_i,\ \forall i\}(x) = n, \tag{7}$$

*then the point $x_0$ is asymptotically stable in the sense that $\lim_{t\to\infty} x(t) = x_0$.*

*(ii) In addition, if the second variation of $V$ at $x = 0$ is positive definite, i.e., if*

$$\delta^2 V(x_0) > 0,$$

*then the point $x_0$ is exponentially stable in the sense that $V(x(t)) \leq cV(x(0))e^{-\lambda t}$, for some positive $c$ and $\lambda$.*

*Proof.* The results in (i) are well known, beginning with the original contributions in [17, 22]. Exponential convergence is proven by noting two facts: first, the results in (i) apply to the linearized closed loop system with $\delta^2 V(x_0)$ as Lyapunov function, and second, asymptotic stability of the linearized system implies exponential of the nonlinear system. We refer to [39, pp. 212-213 (Corollary 5.30)] for a similar discussion. Furthermore, while exponential convergence of $x(t)$ to $x_0$ is a notion that requires a choice of a coordinate system, exponential convergence of the function $V(x(t))$ is an intrinsic fact. $\square$

These results can be extended to the stabilization problem for 1 dimensional sub-manifolds of $M$. The definition of Lyapunov function is generalized by requiring $V(x)$ to be positive whenever $x$ does not belong to the sub-manifold (in other words, $V(x)$ is now invariant under motions on the sub-manifold). It then remains true that, if the system satisfies the linear controllability condition (7) in a neighborhood of the submanifold, then damping controls defined as in (i) lead to asymptotic stability. By the way, notice that the linear controllability rank condition can be checked at any arbitrary point. Finally, consider the second variation of $V$ restricted to the subspace complementary to the one dimensional sub-manifold. If it is positive definite at each point on the sub-manifold, then $V(x(t))$ converges to zero exponentially fast.

**Remark 2.2.** *One remarkable aspect of the results in Lemma 2.1 is that they are coordinate-free statements on manifolds. This is in apparent disagreement with the discussion in [41] on how the notion of exponential converge is* not *invariant under coordinate changes. However, our stability definitions explicitly depend on the existence of a Lyapunov function $V$. The existence of a Lyapunov function with a positive definite second variation is a very natural assumption in the context of mechanical systems where the Hamiltonian is a natural integral of motion.*

# 3 Mechanical systems with symmetries

We present a coordinate free definition of mechanical control systems based on geometric ideas. We divide the treatment in three parts. First we review some necessary machinery from Riemannian geometry, then we define mechanical systems and finally we treat symmetries and introduce the effective Hamiltonian. We refer to Appendix A for coordinate expressions for all important quantities.

## 3.1 Natural objects and operations on Riemannian manifolds

We review some useful definitions in order to fix some notation; see [9, 19] for a comprehensive treatment. Let $Q$ be a smooth manifold, $q$ be a point on it, $v_q$ be a point on $TQ$, $I \subset \mathbb{R}$ be a real interval and $\gamma : I \to Q$ be a curve on $Q$. On the manifold $Q$, we can define smooth functions $q \mapsto f(q) \in \mathbb{R}$, vector fields $q \mapsto X_q \in T_qQ$, and more general $(r, s)$ tensors fields, that is, real valued multi-linear maps on $(T_qQ^*)^r \times (T_qQ)^s$. We let $C(Q)$ and $\mathfrak{X}(Q)$ denote the set of functions and vector fields on $Q$. We refer to Appendix A for coordinated based expression of these quantities and of the following ones.

A Riemannian metric on a manifold $Q$ is a $(0, 2)$ symmetric, positive definite tensor field, that is a map that associates to each $q \in Q$ an inner product $\langle\langle \cdot\, , \cdot \rangle\rangle_q$ on $T_qQ$. A manifold endowed with a Riemannian metric is said to be a Riemannian manifold. Lie derivatives of functions and Lie brackets between vector fields are denoted by

$$\mathcal{L}_X f, \qquad \text{and} \qquad \mathcal{L}_X Y = [X, Y].$$

An affine connection on $Q$ is a smooth map that assigns to a pair of vector fields $X, Y$ a vector field $\nabla_X Y$ such that for all $f \in C(Q)$ and for all $X, Y, Z \in \mathfrak{X}(Q)$:

(i) $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$,

(ii) $\nabla_X (fY + Z) = (\mathcal{L}_X f)Y + f\nabla_X Y + \nabla_X Z$.

We also say that $\nabla_X Y$ is the covariant derivative of $Y$ with respect to $X$. Given a Riemannian metric on $Q$, there exist a unique affine connection $\nabla$, called the Riemannian (or Levi-Civita) connection, such that for all $X, Y, Z$ on $Q$,

$$[X, Y] = \nabla_X Y - \nabla_Y X, \tag{8}$$

$$\mathcal{L}_X \langle\langle Y\, ,\, Z\rangle\rangle = \langle\langle \nabla_X Y\, ,\, Z\rangle\rangle + \langle\langle Y\, ,\, \nabla_X Z\rangle\rangle. \tag{9}$$

Interestingly, the second equality is the equivalent in this geometric setting of the classic "$\dot{M} - 2C$ is skew symmetric" fact.

Next, we introduce the notion of covariant derivative along a curve. This concepts will be instrumental in writing the Euler-Lagrange equations in the next section. Consider a smooth curve $\gamma = \{\gamma(t) \in Q, t \in [0, 1]\}$, and a vector field $\{v(t) \in T_{\gamma(t)}Q, t \in [0, 1]\}$ defined along $\gamma$. Let $V \in \mathfrak{X}(Q)$ satisfy $V(\gamma(t)) = v(t)$. The covariant derivative of the vector field $v$ along $\gamma$ is defined by

$$\nabla_{\dot\gamma(t)} v(t) = \nabla_{\dot\gamma(t)} V(q)\big|_{q=\gamma(t)}.$$

In what follows, we let $\frac{D}{dt}$ denote the covariant derivative along a curve $\nabla_{\dot\gamma}$.

We conclude this section with the notion of first and second variation of a function. Given a function $f \in C(Q)$, its gradient $\operatorname{grad} f$ is the vector field defined by

$$\mathcal{L}_X f = \langle\langle \operatorname{grad} f\, ,\, X\rangle\rangle,$$

and its Hessian $\operatorname{Hess} f$ is the $(0, 2)$ symmetric tensor field defined by

$$\operatorname{Hess} f(X, Y) = (\mathcal{L}_Y \mathcal{L}_X - \mathcal{L}_{\nabla_Y X}) f, \qquad \text{for all } X, Y \in \mathfrak{X}(Q). \tag{10}$$

Notice that $\operatorname{Hess} f$ maps $T_qQ \times T_qQ$ to the real line and it is therefore a two form; we will often investigate whether this two form is positive definite over certain sub-bundles of $T_qQ$.

### 3.2 Mechanical control systems

We present a geometric notion of mechanical control systems based on the concepts introduced in the previous section. This Lagrangian approach to modeling of vehicles and robotic manipulators is common to a number of recent works; see [3, 20, 28, 38].

A *mechanical control system* is defined by the following objects:

(i) an $n$-dimensional configuration manifold $Q$, with local coordinates $\{q^1, \ldots, q^n\}$,

(ii) a Riemannian metric $M_q$ on $Q$ (the kinetic energy), alternatively denoted by $\langle\langle \cdot\, ,\, \cdot\rangle\rangle$,

(iii) a function $V$ on $Q$ describing the potential energy, and

(iv) an $m$-dimensional codistribution $\mathcal{F} = \operatorname{span}_{C(Q)}\{F^1, \ldots, F^m\}$ defining the input forces.

Let $q \in Q$ be the configuration of the system and $v_q \in T_qQ$ its velocity. The total energy, or *Hamiltonian* $H : TQ \to \mathbb{R}$ is

$$H(v_q) = \frac{1}{2}\langle\langle v_q\, ,\, v_q\rangle\rangle + V(q) = \frac{1}{2}\|v_q\|^2 + V(q).$$

Mechanical systems of this kind are sometimes called "simple," see the references above, or "natural," see [1], because their Hamiltonians are of the form "kinetic + potential."

Let the input vector fields be $Y_i = M_q^{-1} F^i$, and let $\mathcal{Y} = \operatorname{span}_{C(Q)}\{Y_1, \ldots, Y_m\}$ be the input distribution. The Euler-Lagrange equations for the system can be written in a coordinate independent form relying on the Riemannian connection of the metric $M_q$:

$$\frac{D\, v_q}{dt} = -\operatorname{grad} V + Y_i u^i, \tag{11}$$

where the input functions $\{u^i(t), t \in \mathbb{R}^+\}$ are bounded measurable and where the summation convention is enforced here and in what follows. A quick application of equation (9) shows how the Hamiltonian is a conserved quantity along the unforced equations of motion.

**Remark 3.1.** *This definition of mechanical control systems differs in two points from the classic one of "affine Hamiltonian system" presented in Nijmeijer and van der Schaft [34, page 353]. First of all, we explicitly assume that the Hamiltonian is of the form "kinetic plus potential energy." Additionally, our definition does not impose the requirement that the input forces be Hamiltonian vector fields. This is relevant in aerospace and underwater dynamics, because body-fixed forces are generally not exact one-forms, and therefore not representable by Hamiltonian vector fields. Nevertheless, one important limitation of the previous setting is that viscous forces, lift in particular, are not accounted for.*

## 3.3 Symmetries and the effective potential

In this section we present the notion of (infinitesimal) symmetries. We present concepts both from classical Hamiltonian reduction theory, see [29], and from Riemannian geometry, see [9, 19].

Given a metric tensor $M$ on the manifold $Q$, a vector field $X$ is said to be an *infinitesimal isometry* if[2] the tensor field $\nabla X : Y \to \nabla_Y X$ is skew symmetric with respect to the metric tensor $M$, that is,

$$\langle\langle Y\,,\, \nabla_Z X\rangle\rangle + \langle\langle Z\,,\, \nabla_Y X\rangle\rangle = 0. \tag{12}$$

We call $X$ an *infinitesimal symmetry* for the mechanical control system $(Q, M, V, \mathcal{F})$, if it is an infinitesimal isometry and if it satisfies $\mathcal{L}_X V = 0$ and $\mathcal{L}_X Y_i = 0$ for all $i = 1, \ldots, m$.

Typically we focus on isometries that arise from group actions. In other words, we assume that the manifold $Q$ can be written at least locally as $Q = N \times \mathbb{R}$, so that the configuration is $q = (r, x) \in N \times \mathbb{R}$ with $N$ being a smaller dimensional manifold. We then say that the mechanical system $(Q, M, V, \mathcal{F})$ has a $\mathbb{R}$ symmetry, if the metric tensor, the potential energy and the forces are invariant under the action of $\mathbb{R}$, e.g., if $\partial V/\partial x = 0$. The vector field $X = \partial/\partial x$ is the infinitesimal symmetry.

An infinitesimal isometry gives rise to a conserved quantity. The *momentum* $J_X : TQ \to C(Q)$ is defined by

$$J_X(Y) = \langle\langle X\,,\, Y\rangle\rangle, \qquad Y \in \mathfrak{X}(Q).$$

The well-known Noether theorem states that the momentum of $v_q$, that is, $J_X(v_q)$ is constant along the solutions to the un-forced mechanical system.

[2]Equivalently, for all $t$, the metric tensor is invariant under the time-$t$ flow map of the vector field $X$, i.e., $\mathcal{L}_X M = 0$. We refer to [19] for the definition of Lie derivative of a tensor field.

**Proposition 3.2.** *Let $(Q, M, V, \mathcal{F})$ be a mechanical control system with equations of motion* (11)*, and let $X$ be an infinitesimal symmetry. Along the un-forced solutions, that is, at $u_i = 0$, we have*

$$\frac{D}{dt} J_X(v_q) = 0. \tag{13}$$

*Proof.* Using tools from equations (9) and (12), we compute:

$$\begin{aligned} \frac{D}{dt} J_X(v_q) = \frac{D}{dt} \langle\!\langle v_q\,,\, X \rangle\!\rangle &= \langle\!\langle \frac{D}{dt} v_q\,,\, X \rangle\!\rangle + \langle\!\langle v_q\,,\, \frac{D}{dt} X \rangle\!\rangle \\ &= \langle\!\langle -\operatorname{grad} V(q)\,,\, X \rangle\!\rangle + \langle\!\langle v_q\,,\, \nabla_{v_q} X \rangle\!\rangle. \end{aligned}$$

The result follows from $\mathcal{L}_X V = 0$ and from the skew symmetry of $\nabla X$. □

Last, we present a useful way of devising an integral of motion by combining Hamiltonian and momentum. We shall call *effective Hamiltonian* the map $H_X : TQ \to \mathbb{R}$ defined by

$$H_X(v_q) = H(v_q) - J_X(v_q). \tag{14}$$

We recall from [29] the famous "summing the square" computation:

$$\begin{aligned} H_X(v_q) &= V(q) + \frac{1}{2}\|v_q\|^2 - \langle\!\langle v_q\,,\, X \rangle\!\rangle \\ &= V(q) - \frac{1}{2}\|X\|^2 + \frac{1}{2}\|X\|^2 + \frac{1}{2}\|v_q\|^2 - \langle\!\langle v_q\,,\, X \rangle\!\rangle \\ &= \left( V - \frac{1}{2}\|X\|^2 \right)(q) + \frac{1}{2}\|v_q - X\|^2. \end{aligned}$$

Correspondingly, we call *effective potential* the map $V_X : Q \to \mathbb{R}$ defined by

$$V_X(q) = V(q) - \frac{1}{2}\|X\|^2(q).$$

The effective Hamiltonian is sum of two terms a potential energy like term and a kinetic energy like term. The latter term is a modified kinetic energy where the argument is a "velocity error" $(v_q - X)$. The effective Hamiltonian is therefore a positive definite function in the velocity error $(v_q - X)$. This is a key requirement for a candidate Lyapunov function that will be used to analyze stability of a relative equilibrium with infinitesimal symmetry $X$. A related notion of velocity error is discussed in detail in [8].

### 3.4 Relative equilibria and their stability

We present a quick review of various definitions and results; see [29] for a more extensive treatment. The effective potential plays a crucial role in characterizing relative equilibria and their stability.

A curve $\gamma : I \subset \mathbb{R} \to Q$ is called *relative equilibrium* if is a solution to the equations of motion (11) and if it is an integral curve of the infinitesimal isometry $X$, that is,

$$\frac{d}{dt}\gamma(t) = X(\gamma(t)). \tag{15}$$

In the next two propositions we characterize the existence and stability of relative equilibria.

**Proposition 3.3 (Existence of relative equilibria).** *Let $(Q, M, V, \mathcal{F})$ be a mechanical control system and let $X$ be an infinitesimal symmetry. An integral curve $\gamma : I \to Q$ of the vector field $X$ is a relative equilibrium, that is it satisfies the equations of motion* (11), *if $\gamma(0) = q_0$ is a critical point for the effective potential, that is, if*

$$\operatorname{grad} V_X(q_0) = 0. \tag{16}$$

*Proof.* Along the curve $\gamma(t)$, the Euler-Lagrange equation (11) are satisfied since

$$0 = \nabla_{\dot\gamma}\dot\gamma + \operatorname{grad} V = \nabla_X X + \operatorname{grad} V = \operatorname{grad}\big(-\tfrac{1}{2}\|X\|^2 + V\big),$$

where the equality

$$\nabla_X X = -\frac{1}{2}\operatorname{grad}\|X\|^2, \tag{17}$$

follows from the skew symmetry of $\nabla X$:

$$\langle\!\langle Z\,,\,\nabla_X X\rangle\!\rangle = -\langle\!\langle X\,,\,\nabla_Z X\rangle\!\rangle = -\frac{1}{2}\mathcal{L}_Z\|X\|^2 = -\frac{1}{2}\langle\!\langle Z\,,\,\operatorname{grad}\|X\|^2\rangle\!\rangle.$$

□

**Proposition 3.4 (Stability of relative equilibria).** *Let $(Q, M, V, \mathcal{F})$ be a mechanical control system and let $X$ be an infinitesimal symmetry. The relative equilibrium $\gamma : I \to Q$, $\gamma(0) = q_0$, is Lyapunov stable if the Hessian of the effective potential is positive definite over variations perpendicular to $X$, that is, if*

$$\operatorname{Hess} V_X(Y, Y)(q_0) > 0, \tag{18}$$

*for all $Y \in T_{q_0}Q$ such that $\langle\!\langle Y\,,\,X\rangle\!\rangle = 0$.*

*Proof.* If the effective potential $V_X$ has positive definite second variation over all variations perpendicular to $X$, then $H_X$ is a Lyapunov function for the sub-manifold $\{\dot\gamma(t), t \in I\} \subset TQ$. □

The fundamental fact in the last proposition is that the effective Hamiltonian $H_X$ is a map $TQ \to \mathbb{R}$ that has positive definite Hessian in all but one directions on the phase space $TQ$. This observation is key to later developments, where $H_X$ will be the candidate Lyapunov function for the stabilization problem.

**Remark 3.5 (Comparison with the Energy-Momentum method).** *The stability test in equation* (18) *in the previous proposition is only sufficient. The Energy–Momentum method in [29, 40] provides a sharper, more detailed analysis. In fact, it is not necessary to ask for the second variation of $V_X$ to be positive definite in every direction, since not every direction on the tangent space to $TQ$ is compatible with the momentum constraint. The more detailed analysis investigates this aspect. However, within the context of the following sections we will explicitly require the effective potential $V_X$ to be positive definite in all directions and therefore we consider the Energy-Momentum test not appropriate. This latter statement is justified since a (fully) positive definite Lyapunov function is required in order to use the nonlinear stabilization concepts described in Section 2.*

# 4 Stabilization of relative equilibria on the phase space

In what follows, we design controllers that stabilize relative equilibria of a mechanical control system with a symmetry. The goal is to achieve exponential convergence rates in all error variables. As described in the Introduction, this is to be contrasted to the situation where the forces are momentum-preserving as in the case of internally actuated vehicles and therefore the value of the momentum remains unchanged in the closed loop. The treatment relies on certain stability properties of the open loop system that are formalized in Proposition 4.4.

## 4.1 Potential shaping along a relative equilibrium

Numerous works have studied the application of a so-called proportional feedback action; see [16, 20, 24, 34]. The effect of such controls is described in terms of a potential energy "shaping" of the closed loop. This freedom in designing the potential energy of the closed loop is then used to advantage in stabilization problems. We illustrate this phenomena in the following lemma.

**Lemma 4.1.** *Given the mechanical control system* $(Q, M, V, \mathcal{F})$ *with equations of motion*

$$\frac{D\,v_q}{dt} = -\operatorname{grad} V + Y_i u^i. \tag{19}$$

*Assume there exists a function* $\phi : Q \to \mathbb{R}$ *such that*

$$\operatorname{grad}\phi = \sum_{i=1}^{m} c^i(q) Y_i,$$

*and set the inputs*

$$u^i = -c^i(q)\phi + v^i.$$

*Then the closed loop system is the mechanical control system* $(Q, M, (V + \frac{1}{2}\phi^2), \mathcal{F})$.

*Proof.* Simple manipulations shows that

$$\begin{aligned}\frac{D\,v_q}{dt} &= -\operatorname{grad} V + Y_i(-c^i(q)\phi + v^i)\\ &= -\operatorname{grad} V - \phi\operatorname{grad}\phi + Y_i v^i = -\operatorname{grad}\left(V + \frac{1}{2}\phi^2\right) + Y_i v^i.\end{aligned}$$

□

We apply this idea to the present context: we attempt to "shape" the effective potential $V_X$ while preserving the existence of the relative equilibrium through the point $q_0$. In equivalent words, we ask for the existence of a function $\phi : Q \to \mathbb{R}$ with $\phi(q_0) = 0$ and such that

(i) $\mathcal{L}_X\phi = 0$, and

(ii) $\operatorname{grad}\phi \in \mathcal{Y}$, where we recall that $\mathcal{Y} = \text{span}_{C(Q)}\{Y_1, \dots, Y_m\}$ is the input distribution. Before proceeding, we introduce the following notation. Given the distribution $\mathcal{Y}$, we let $\mathcal{Y}^\perp$ denote its orthogonal complement, i.e.,

$$\mathcal{Y}^\perp = \{Z \in \mathfrak{X}(Q) | \ \langle\!\langle Z \,,\, Y_i \rangle\!\rangle = 0, \ \ i = 1, \dots, m\},$$

and we let $\text{Lie}(\mathcal{Y}^\perp)$ denote its involutive closure.[3] The following result characterizes the set of functions $\phi$ that satisfy the requirements (i) and (ii).

**Proposition 4.2.** *Let $X$ be an infinitesimal isometry on the Riemannian manifold $Q$. Let $\mathcal{Y}$ be an $m$ dimensional distribution invariant under the action of $X$. The distribution $\text{Lie}(\mathcal{Y}^\perp) + \text{span}_{C(Q)}\{X\}$ is involutive, has dimension $(n-p) \geq (n-m)$, and its $p$ integral functions $\phi_1, \dots, \phi_p$ satisfy*

$$\mathcal{L}_X \phi_j = 0, \qquad \textit{and} \qquad \operatorname{grad} \phi_j \in \mathcal{Y}, \qquad \forall j = 1, \dots, p.$$

*Finally, given any $q_0 \in Q$ these $p$ functions can be chosen so that $\phi_j(q_0) = 0$ for all $j$.*

*Proof.* We start from the definition of gradient and we note the basic equivalence:

$$\mathcal{L}_Z \phi_j = 0 \iff \operatorname{grad} \phi_j \in \text{span}\{Z\}^\perp.$$

Next, we consider the requirement (i): from $\operatorname{grad} \phi \in \mathcal{Y}$ we have $\operatorname{grad} \phi \perp \mathcal{Y}^\perp$ and therefore $\mathcal{L}_Z \phi = 0$ for all $Z \in \mathcal{Y}^\perp$. From requirement (ii) we have $\mathcal{L}_X \phi = 0$, and therefore $\mathcal{L}_Z \phi = 0$ for all $Z \in \mathcal{Y}^\perp + \text{span}\{X\}$. But then $\phi$ must be invariant under the involutive closure of the sum of these two distributions: $\mathcal{L}_Z \phi = 0$ for all $Z \in \text{Lie}\left(\{X\} + \mathcal{Y}^\perp\right)$.

Next, we invoke Frobenius Theorem to assert that if the dimension of the distribution $\text{Lie}\left(\{X\} + \mathcal{Y}^\perp\right)$ is $(n-p)$, then there exist $p$ integral functions $\phi_j$ such that $\mathcal{L}_Z \phi_j = 0$ for all $Z$ in this distribution. Additionally, notice that $\text{Lie}\left(\mathcal{Y}^\perp + \text{span}\{X\}\right) = \text{Lie}\left(\mathcal{Y}^\perp\right) + \text{span}\{X\}$, because of the invariance properties of $\mathcal{Y}$ and $M$. The last statement in the lemma follows immediately. □

Given[4] these $p$ functions $\phi_j$, we can now employ the potential shaping technique and provide sufficient conditions for stabilization of relative equilibria. Let $k_1, \dots, k_p$ be positive scalars and apply the feedback controls $u^i(q)$ defined implicitly via

$$\sum_{i=1}^m Y_i(q) u^i(q) = -\sum_{j=1}^p k_j \phi_j \operatorname{grad} \phi_j. \tag{20}$$

As described in Lemma 4.1, the closed loop system is still mechanical. Additionally, the closed loop system posses the same infinitesimal symmetry $X$ as the system before feedback. Accordingly, we define the effective Hamiltonian for the closed loop as

$$\begin{aligned} \widehat{H}_X(v_q) &= H(v_q) + \frac{1}{2}\sum_{j=1}^p k_j \phi_j^2 - \langle\!\langle X \,,\, v_q \rangle\!\rangle \\ &= \left( V_X + \frac{1}{2}\sum_{j=1}^p k_j \phi_j^2 \right) + \frac{1}{2}\|v_q - X\|^2, \end{aligned}$$

[3] The perpendicular distribution to $\{M^{-1}F^1, \dots, M^{-1}F^m\}$ can be computed as the annihilator of the co-distribution $\{F^1, \dots, F^m\}$.

[4] Computing integral functions for involutive distribution of arbitrary dimension and codimension is generally as difficult a providing explicit solutions to a set of ordinary differential equations.

and the effective potential for the closed loop as

$$\widehat{V}_X = V_X + \frac{1}{2}\sum_{j=1}^{p} k_j \phi_j^2.$$

Recall from Proposition 3.4, that stability of a relative equilibrium depends from the positive definiteness of the Hessian of $V_X$. The following lemma will be instrumental in describing when it is possible to render $\widehat{V}_X$ positive definite via the potential shaping feedback.

**Lemma 4.3.** *Let $P$ be an $n \times n$ symmetric matrix and let $C$ be a surjective $p \times n$ matrix. There exist $p$ positive constants $k_j$ such that $P + C^T \operatorname{diag}\{k_1, \dots, k_p\} C > 0$ if and only if $P$ restricted to $\operatorname{Ker} C$ is positive definite.*

We refer to [42] for a proof of this statement. Finally, we formalize the discussion in this section via the following result.

**Proposition 4.4 (Stabilization of relative equilibria).** *Let $(Q, M, V, \mathcal{F})$ be a mechanical control system, let $X$ be an infinitesimal symmetry and let $\{\gamma : I \to Q, \gamma(0) = q_0\}$ be a relative equilibrium. Let $\phi_1, \dots, \phi_p$ be $p$ functions obtained as in Lemma 4.2 so that, without loss of generality, we set*

$$Y_j = \operatorname{grad} \phi_j, \qquad j = 1, \dots, p \leq m.$$

*Assume that the effective potential $V_X$ is positive definite over variations perpendicular to the subspace $\{X, \operatorname{grad} \phi_1, \dots, \operatorname{grad} \phi_p\}$, that is,*

$$\operatorname{Hess} V_X(Y, Y)(q_0) > 0, \tag{21}$$

*for all $Y$ perpendicular to $\{X, \operatorname{grad} \phi_1, \dots, \operatorname{grad} \phi_p\}_{q_0}$.*

*Then there exist positive constants $k_1, \dots, k_p$ such that the feedback controls*

$$u^j(q) = -k_j \phi_j(q),$$

*render the relative equilibrium $\{\gamma : I \to Q, \gamma(0) = q_0\}$ Lyapunov stable.*

*Proof.* The proof is a application of Proposition 3.4 to the closed loop system and follows in spirit the original proof in [42]. Notice that the feedback controls in the statement of the theorem coincide with the one defined in equation (20). Therefore, by Lemma 4.1 the closed loop system is a mechanical control system with infinitesimal symmetry $X$ and relative equilibrium $\{\gamma : I \to Q, \gamma(0) = q_0\}$. The effective potential of the closed loop is $\widehat{V}_X$ and, according to Proposition 3.4 the relative equilibrium is stable if the Hessian of $\widehat{V}_X$ is positive definite when restricted to the perpendicular to $X$.

From the definition of Hessian in Section 3, we compute

$$\begin{aligned}
\operatorname{Hess}\left(\frac{1}{2}\phi^2\right)(Z_1, Z_2) &= \left(\mathcal{L}_{Z_1}\mathcal{L}_{Z_2} - \mathcal{L}_{\nabla_{Z_1} Z_2}\right)\left(\frac{1}{2}\phi^2\right) \\
&= \mathcal{L}_{Z_1}\phi \mathcal{L}_{Z_2}\phi - \phi \mathcal{L}_{\nabla_{Z_1} Z_2}\phi \\
&= \left(\mathcal{L}_{Z_1}\phi\right)\left(\mathcal{L}_{Z_2}\phi\right) + \phi\left(\mathcal{L}_{Z_1}\mathcal{L}_{Z_2}\phi - \mathcal{L}_{\nabla_{Z_1} Z_2}\phi\right),
\end{aligned}$$

so that

$$\begin{aligned}
\operatorname{Hess} \widehat{V}_X(Z_1, Z_2) &= \operatorname{Hess} V_X(Z_1, Z_2) + \operatorname{Hess}\left(\frac{1}{2}\sum_{j=1}^{p} k_j \phi_j^2\right)(Z_1, Z_2) \\
&= \operatorname{Hess} V_X(Z_1, Z_2) + \sum_{j=1}^{p} k_j \,\langle\!\langle \operatorname{grad}\phi_j \,,\, Z_1 \rangle\!\rangle \langle\!\langle \operatorname{grad}\phi_j \,,\, Z_2 \rangle\!\rangle,
\end{aligned}$$

where the last equality holds at $q = q_0$ since $\phi_j(q_0) = 0$.

Let us now write coordinate expressions for the tangent space $T_{q_0}Q$. With respect to a basis, we let $\operatorname{Hess} V_X = P \in \mathbb{R}^{n\times n}$, we let $\langle\!\langle \operatorname{grad}\phi_j \,,\, Z_i \rangle\!\rangle = c_j z_i$, where $c_j \in \mathbb{R}^{1\times n}$ and $z_i \in \mathbb{R}^{n\times 1}$. According to these definition we have

$$\sum_{j=1}^{p} k_j \langle\!\langle \operatorname{grad}\phi_j \,,\, Z_1 \rangle\!\rangle \langle\!\langle \operatorname{grad}\phi_j \,,\, Z_2 \rangle\!\rangle = z_1^T C^T \operatorname{diag}\{k_1, \ldots, k_p\} C z_2,$$

where $C = [c_1 \ldots\ c_p]^T$. Notice that the subspace $\ker C$ is spanned by vectors orthogonal to $\operatorname{grad}\phi_j$ for all $j$. The assumption in equation (21) states that $P$ is positive definite when restricted to $\ker C + X_{q_0}^{\perp}$. Therefore the assumptions of Lemma 4.3 hold and there exist positive constants $k_1, \ldots, k_p$ such that

$$P + C^T \operatorname{diag}\{k_1, \ldots, k_p\} C > 0 \qquad \text{restricted to} \quad \{X\}^{\perp}.$$

Therefore the Hessian of $\widehat{V}_X$ is positive definite when restricted to the perpendicular to $X$ and the relative equilibrium is stable. □

## 4.2 Damping control along a relative equilibrium

In this section we employ the classic notion of damping (or dissipative) feedback to achieve asymptotic and exponential stability. Within the context of relative equilibria stabilization, the key observation is that the effective Hamiltonian $H_X$ is positive definite in all but one direction on the tangent space to $TQ$.

We start by computing the time derivative of the effective Hamiltonian along the closed loop system. Instead of computing Lie derivatives on the tangent space to $TQ$, we take advantage of the geometric formalism described in Section 3 and state the following lemma.

**Lemma 4.5.** *Consider the mechanical control system $(Q, M, V, \mathcal{F})$ with infinitesimal symmetry $X$ and with effective Hamiltonian*

$$H_X(v_q) = \left(V - \frac{1}{2}\|X\|^2\right) + \frac{1}{2}\|v_q - X\|^2.$$

*It holds*

$$\frac{D\, H_X(v_q)}{dt} = \langle\!\langle v_q - X \,,\, Y_i \rangle\!\rangle u^i.$$

*Proof.* We compute

$$\begin{aligned}
\frac{1}{2}\frac{D}{dt}\|v_q - X\|^2 &= \langle\!\langle v_q - X \,,\, \frac{D\, v_q}{dt} - \frac{DX}{dt} \rangle\!\rangle \\
&= \langle\!\langle v_q \,,\, -\operatorname{grad} V - \nabla_{v_q} X \rangle\!\rangle - \langle\!\langle X \,,\, -\operatorname{grad} V - \nabla_{v_q} X \rangle\!\rangle + \langle\!\langle v_q - X \,,\, Y_i \rangle\!\rangle u^i \\
&= -\langle\!\langle v_q \,,\, \operatorname{grad} V \rangle\!\rangle - \langle\!\langle v_q \,,\, \nabla_X X \rangle\!\rangle + \langle\!\langle v_q - X \,,\, Y_i \rangle\!\rangle u^i,
\end{aligned}$$

and

$$\frac{D}{dt}\left(V - \frac{1}{2}\|X\|^2\right) = \langle\!\langle v_q \, , \, \operatorname{grad} V - \frac{1}{2} \operatorname{grad} \|X\|^2 \rangle\!\rangle,$$

so that

$$\frac{1}{2}\frac{D}{dt}H_X = -\langle\!\langle v_q \, , \, \frac{1}{2} \operatorname{grad} \|X\|^2 + \nabla_X X \rangle\!\rangle + \langle\!\langle v_q - X \, , \, Y_i \rangle\!\rangle u^i.$$

The conclusion follows from equation (17). □

Finally, we can state the concluding result:

**Proposition 4.6 (Exponential stabilization of relative equilibria).** *Let $(Q, M, V, \mathcal{F})$ be a mechanical control system with equations of motion*

$$\frac{D\, v_q}{dt} = -\operatorname{grad} V + Y_i u^i.$$

*Let $X$ be an infinitesimal symmetry, let $\{\gamma : I \to Q, \gamma(0) = q_0\}$ be a relative equilibrium, and let $\phi_1, \ldots, \phi_p$ be $p$ functions obtained as in Lemma 4.2 so that, without loss of generality, we set*

$$Y_j = \operatorname{grad} \phi_j, \qquad j = 1, \ldots, p \leq m.$$

*Assume that the system (with all $m$ input forces) satisfies the linear controllability condition (7) at $v_q = X(q)$ and $q = q_0$, and assume that*

$$\operatorname{Hess} V_X(Y, Y)(q_0) > 0,$$

*for all $Y$ perpendicular to $\{X, \operatorname{grad} \phi_1, \ldots, \operatorname{grad} \phi_p\}_{q_0}$.*

*Then, there exist positive constants $k_1, \ldots, k_p$ and $d_1, \ldots, d_m$ such that the feedback controls*

$$\begin{aligned} u^j(v_q) &= -k_j \phi_j(q) - d_j \dot{\phi}_j(v_q), && 1 \leq j \leq p \\ u^i(v_q) &= -d_i \langle\!\langle v_q - X \, , \, Y_i \rangle\!\rangle, && p < i \leq m, \end{aligned}$$

*render the relative equilibrium $\{\gamma : I \to Q, \gamma(0) = q_0\}$ exponentially stable.*

*Proof.* For $1 \leq j \leq p$, the expression for the damping controls are justified by

$$\langle\!\langle v_q - X \, , \, Y_j \rangle\!\rangle = \langle\!\langle v_q - X \, , \, \operatorname{grad} \phi_j \rangle\!\rangle = \mathcal{L}_{(v_q - X)} \phi_j = \dot{\phi}_j,$$

since $\mathcal{L}_X \phi_j = 0$.

From Lemma 2.1 we know that a damping controls $u^i = -\mathcal{L}_{g_i} V$ lead to exponential convergence under two assumption: the system is linearly controllable and the Lyapunov function has positive definite second variation. But linear controllability is an assumption and Proposition 3.4 shows that the effective Hamiltonian with the feedback correction described has in fact a positive definite second variation. □

**Remark 4.7 (Positive definiteness and controllability assumptions).** *Loosely speaking, the assumption in equation (21) requires the unforced system to be stable in certain directions. This is illustrated by the examples in the following section.*

*Certain vehicles are actuated only by means of internal moving masses or spinning rotors. Such forces are called "internal" because they do not affect the value of the momentum. Mechanical systems endowed with only internal actuation are necessarily not linear controllable (since momentum is conserved along forced equations of motion).*

# 5 Applications to Vehicle Control

We present two design examples for models of vehicles. While both the underwater planar body and the satellite have a full SE(2) (the group of planar displacements) and SO(3) (the group of rotations in three dimensional space) symmetry, we focus on the stabilization of a one-dimensional symmetry: translation along the major axis and rotation about a specific axis. Our analysis on the effective potential agrees with the results in [23, 24, 29].

This is a brief summary of the results presented below. With regards to the planar vehicles depicted in Figure 2 stabilization is obtained only for the system on the right with $m_x > m_y$. With regards to the satellite example, spin axis stabilization about the first axis (with actuation on the first and second axis) is obtained so long as $I_1 > I_2$; no assumption is made on $I_3$, i.e., first and second axis need not be the first and second principal axis of the rigid body.

## 5.1 A Planar Underwater Body with Two Forces

In this section we study in more detail the motivating example presented in the Introduction; see Figure 1 at page 3. The control objective is to stabilize a steady translation of the body.

The four objects describing the planar body are the following. The configuration manifold is SE(2), that is the group of rotations and translations on the plane. Coordinates for this manifold are $q = (\theta, x, y)$. The inertia tensor is $M$ as described in equation (1). With respect to the basis of vector fields $\{\frac{\partial}{\partial\theta}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$, the inertia has the matrix expression in equation (4). No potential is present, $V = 0$, and the forces are $F^1 = \cos\theta \mathrm{d}x + \sin\theta \mathrm{d}y$ and $F^2 = -\sin\theta \mathrm{d}x + \cos\theta \mathrm{d}y - h\mathrm{d}\theta$, that is, in vector notation with respect to the basis $\{\mathrm{d}\theta, \mathrm{d}x, \mathrm{d}y\}$:

$$F^1 = \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix}, \qquad \text{and} \qquad F^2 = \begin{bmatrix} -h \\ -\sin\theta \\ \cos\theta \end{bmatrix}.$$

The input vector fields can be computed by inverting $M$:

$$Y_1 = \begin{bmatrix} 0 \\ \cos\theta/m_x \\ \sin\theta/m_x \end{bmatrix}, \qquad \text{and} \qquad Y^2 = \begin{bmatrix} -h/I \\ -\sin\theta/m_y \\ \cos\theta/m_y \end{bmatrix}.$$

In this set of coordinates the un-forced Euler-Lagrange equations are:

$$\begin{aligned}
\ddot\theta &= \frac{m_y - m_x}{2I}\Big((\dot x^2 - \dot y^2)\sin(2\theta) - 2\dot x\dot y\cos(2\theta)\Big) \\
\ddot x &= \frac{m_y - m_x}{2m_x m_y}\Big(\dot\theta\dot y(m_y\cos^2\theta - m_x\sin^2\theta) - \dot\theta\dot x(m_x + m_y)\sin(2\theta)\Big) \\
\ddot y &= \frac{m_y - m_x}{2m_x m_y}\Big(\dot\theta\dot x(m_x\cos^2\theta + m_y\sin^2\theta) + \dot\theta\dot y(m_x + m_y)\sin(2\theta)\Big).
\end{aligned}$$

As noticed in the Introduction, the vector field $X = v_0\frac{\partial}{\partial x} = [0\ v_0\ 0]^T$ is an infinitesimal isometry for this mechanical system. According to the various definitions, we compute the momentum as

$$J_X(\theta, x, y; \dot\theta, \dot x, \dot y) = v_0(m_x(\cos\theta)^2 + v_0 m_y(\sin\theta)^2)\dot x + v_0((m_x - m_y)\sin\theta\cos\theta)\dot y,$$

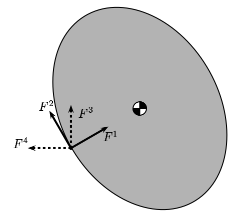


Figure 4: Change of basis for the set of input forces. We let $Y_i = M^{-1}F^i$.

and the effective potential as

$$V_X(\theta, x, y) = -\frac{v_0^2}{2}\left(m_x(\cos\theta)^2 + m_y(\sin\theta)^2\right).$$

Relative equilibria for this system along the infinitesimal symmetry $X$ correspond to critical points of $V_X$:

$$\frac{\partial V_X}{\partial \theta} = 0 \qquad \Longleftrightarrow \qquad \sin 2\theta = 0.$$

Notice that since $V_X$ depends only on the variable $\theta$, its second variation cannot be positive definite in a two dimensional subspace. Therefore we cannot apply Proposition 3.4 to assess stability.

To design a Lyapunov function and obtain stability via feedback we employ the potential shaping technique as described in Lemma 4.2 and Proposition 4.4. The distribution $\mathcal{Y}^\perp$ has dimension one and it is spanned by the vector field

$$Y_\perp = \begin{bmatrix} 1/h \\ -\sin\theta \\ \cos\theta \end{bmatrix}.$$

Additionally, we notice that the distribution spanned by $X$ and $Y_\perp$ is involutive and that therefore there must exist a function $\phi$ that satisfies the requirements in Lemma 4.2. As mentioned in the Introduction, the function $\phi(\theta, x, y) = y - h\sin\theta$ satisfies $\mathcal{L}_X\phi = \mathcal{L}_{Y_\perp}\phi = 0$. Accordingly, we compute

$$\operatorname{grad}\phi = \begin{bmatrix} -h\cos\theta/I \\ (m_y - m_x)\cos\theta\sin\theta/m_x m_y \\ (\cos\theta)^2/m_y + (\sin\theta)^2/m_x \end{bmatrix} \equiv (\sin\theta)Y_1 + (\cos\theta)Y_2.$$

Before defining the feedback controls, it is convenient to perform a change of basis for the input distribution. According to the statement in Proposition 4.4, we let

$$\begin{aligned} Y_3 &= (\sin\theta)Y_1 + (\cos\theta)Y_2 \\ Y_4 &= -(\cos\theta)Y_1 + (\sin\theta)Y_2, \end{aligned}$$

so that we have $u^1Y_1 + u^2Y_2 = u^3Y^3 + u^4Y_4$. In Figure 4 we illustrate what this change of basis means with respect to the force co-vectors $F^i$. Next, we proceed to apply the feedback controls described in Proposition 4.4. We set $u^4 = 0$ and

$$u^3(\theta, x, y) = -k(y - h\sin\theta). \tag{22}$$

To check the stability of the closed loop we compute the second variation of $V_X$ over the perpendicular subspace to $\{X, Y_3\}$. Some straightforward algebra leads to $\{X, Y_3\}^\perp = \operatorname{span}\{Y_5\}$, where

$$Y_5 = \begin{bmatrix} \sec\theta/h \\ \frac{(m_x - m_y)\cos\theta\sin\theta}{m_x(\cos\theta)^2 + m_y(\sin\theta)^2} \\ 1 \end{bmatrix}.$$

According to the definition in equation (10) (see also the coordinate expression in equation (24)), the Hessian of $V_X$ at the critical point $\theta = 0$ is

$$\operatorname{Hess} V_X(Y_5, Y_5) = v_0^2(m_x - m_y)/h^2.$$

In what follows we assume $m_x > m_y$ and we focus on the relative equilibrium described by $(\theta, y) = (0, 0)$. Since the second variation of the Hessian is positive definite, this relative equilibrium is stabilizable. Given the feedback in equation (22), the closed loop effective potential is

$$\widehat{V}_X = -\frac{v_0^2}{2}\left(m_x(\cos\theta)^2 + m_y(\sin\theta)^2\right) + \frac{1}{2}k(y - h\sin\theta)^2,$$

and its Hessian is positive definite on the subspace $X^\perp$ for all $k > 0$.

Exponential stability is obtained by invoking Proposition 4.6. We compute the damping (or dissipative) action as:

$$\begin{aligned} u^3(\theta, x, y; \dot\theta, \dot x, \dot y) &= -k(y - h\sin\theta) - d_3(\dot y - h\dot\theta\cos\theta) \\ u^4(\theta, x, y; \dot\theta, \dot x, \dot y) &= d_4(v_0 - \dot x - h\dot\theta\sin\theta). \end{aligned}$$

Since it can be verified that the linearized system is controllable, the convergence is exponential for all $h \neq 0$, $m_x > m_y$ and $v_0 > 0$.

In Figure 5 we present a simulation of the closed loop system. Numerical values are expressed in SI units as: $I = 1.5$, $m_x = 1.5$, $m_y = .5$, $h = .4$, $v = .2$, $k = .25$, $d_3 = .125$, $d_4 = .5$. The initial condition has zero velocity and and error of $\pi/6$ rad in orientation and 1m in $y$ position. The simulation lasts for 80 seconds and one sample is displayed every 5 seconds.

## 5.2 A Satellite with Two Thrusters

As a second example design, we consider the model of a satellite with two thrusters. The classic model is written as follows; see [29]. The attitude and the body fixed velocity are $(R, \Omega)$, the kinetic energy is $\frac{1}{2}\Omega^T\mathbb{I}\Omega$, no potential is present, $V = 0$, and the two inputs consist of torques about the first and second axes. The equations of motion are:

$$\begin{aligned} \dot R &= R\widehat{\Omega} \\ \mathbb{I}\dot\Omega &= \mathbb{I}\Omega \times \Omega + \mathsf{e}_1 u_1(t) + \mathsf{e}_2 u_2(t), \end{aligned} \tag{23}$$

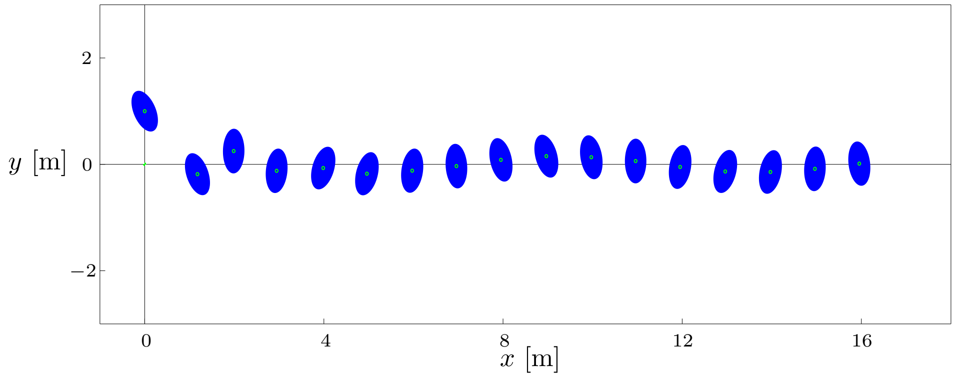


Figure 5: Stabilization of steady forward motion for the planar body example.

where $\widehat{\Omega} y = \Omega \times y$ and where $\{\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3\}$ denotes the standard basis on $\mathbb{R}^3$, i.e., $\mathsf{e}_1 = (1,0,0)$, $\mathsf{e}_2 = (0,1,0)$ and $\mathsf{e}_3 = (0,0,1)$. We assume $\mathbb{I} = \operatorname{diag}\{I_1, I_2, I_3\}$ with $I_1 > I_2$ and we leave $I_3$ free to have any value.

Because the later computations are better performed in coordinates, we now choose a convenient parameterization and obtain various coordinate expressions. Recall that the exponential map $\exp : \mathbb{R}^3 \to \mathrm{SO}(3)$ is defined as (Rodriguez formula; see [32]):

$$\exp(\widehat{x}) = I_3 + \sin\|x\| \frac{\widehat{x}}{\|x\|} + (1 - \cos\|x\|) \frac{\widehat{x}^2}{\|x\|^2},$$

where $I_3$ is the unity matrix. We write $R$ as

$$R(\alpha, \beta, \gamma) = \exp(\alpha\,\widehat{\mathsf{e}}_1) \exp(\beta\,\widehat{\mathsf{e}}_2) \exp(\gamma\,\widehat{\mathsf{e}}_3),$$

that is, we parameterize[5] SO(3) by a set of Euler angles $(\alpha, \beta, \gamma)$, that is singular at $\beta = \pm\pi/2$. The unusual order of rotation is well-suited to the relative equilibrium and to the set of input vector fields at hand. The Jacobian relating Euler angles rates and body fixed velocity is

$$\Omega = \begin{bmatrix} \cos\beta\cos\gamma & \sin\gamma & 0 \\ -\cos\beta\sin\gamma & \cos\gamma & 0 \\ \sin\beta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot\alpha \\ \dot\beta \\ \dot\gamma \end{bmatrix} \triangleq J(\alpha,\beta,\gamma) \begin{bmatrix} \dot\alpha \\ \dot\beta \\ \dot\gamma \end{bmatrix},$$

and, the inertia matrix with respect to this basis is $M(\alpha,\beta,\gamma) = J^T(\alpha,\beta,\gamma)\mathbb{I}J(\alpha,\beta,\gamma)$. The forces are $F^1 = J^T(\alpha,\beta,\gamma)\mathsf{e}_1$, $F^2 = J^T(\alpha,\beta,\gamma)\mathsf{e}_2$, and the input vectors are $Y_1 = J^{-1}(\alpha,\beta,\gamma)\mathbb{I}\mathsf{e}_1$, $Y_2 = J^{-1}(\alpha,\beta,\gamma)\mathbb{I}\mathsf{e}_2$.

We attempt to stabilize rotation about the first principal axis; this problem is often referred to as "spin axis stabilization." This correspond to the vector field $X = \omega_0 \frac{\partial}{\partial\alpha} =$

[5]This local chart of SO(3) is *one* set of Euler angles and not the usual one, see [32]. Coordinate systems obtained via repeated single exponentials are referred to as "exponential coordinates of the second kind."

$[\omega_0 \ 0 \ 0]^T$. Accordingly, we compute:

$$\begin{aligned} J_X(\alpha,\beta,\gamma;\dot\alpha,\dot\beta,\dot\gamma) &= \omega_0\big(I_3(\sin\beta)^2 + (\cos\beta)^2(I_1(\cos\gamma)^2 + I_2(\sin\gamma)^2)\big)\dot\alpha \\ &\quad + \omega_0\big((I_1 - I_2)\cos\beta\cos\gamma\sin\gamma\big)\dot\beta + \omega_0\big(I_3\sin\beta\big)\dot\gamma, \\ V_X(\alpha,\beta,\gamma) &= -\frac{1}{2}\omega_0^2\left(I_1(\cos\beta)^2(\cos\gamma)^2 + I_2(\cos\beta)^2(\sin\gamma)^2 + I_3(\sin\beta)^2\right). \end{aligned}$$

The relative equilibrium of interest is described by $(\beta,\gamma) = (0,0)$.

Next, we follow the same procedure as in the general case and in the previous example. It turns out that $Y_\perp = [0,0,1]^T$, the function described in Lemma 4.2 is $\phi(\alpha,\beta,\gamma) = \beta$ and the vector $Y_5 = [-I_3\sin(\beta)/(I_3(\sin\beta)^2 + (\cos\beta)^2(I_1(\cos\gamma)^2 + I_2(\sin\gamma)^2)),\ 0,\ 1]^T$. Finally, the second variation of the effective potential $V_X$ satisfies:

$$\operatorname{Hess} V_X(Y_5, Y_5)(0,0) = \omega_0^2(I_1 - I_2).$$

For $I_1 > I_2$, the design procedure can be applied. In brief summary, the control law

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{bmatrix} \begin{bmatrix} -d_4(\dot\alpha - \omega_0)\cos(\beta) \\ -k\beta - d_3\dot\beta, \end{bmatrix}$$

achieves local exponential stability for positive values of $d_3, d_4$ and for a sufficiently large $k$.

# 6 Conclusions

We have presented an extension of the works in [16, 24, 42] to full exponential stabilization of relative equilibria. The proposed control design is coordinate independent and provides a constructive procedure. Lyapunov stabilization via potential shaping (i.e., proportional action) requires the positive definiteness of the effective potential in certain "uncontrolled" directions. Exponential stabilization is proven under a linear controllability assumption. In brief, our treatment relies on certain stability properties of the unforced system and obtains strong convergence properties of the closed loop. In the work on "Controlled Lagrangians" by Bloch, Leonard and Marsden [4, 5], a complementary approach is introduced to overcome this limitation.

These results are nicely dual to the classic results in van der Schaft [42]. Effective Hamiltonian and effective potential play a similar role for relative equilibria as usual Hamiltonian and potential play in the point stabilization problem. We claim that the adoption of the effective potential as candidate Lyapunov function is appropriate in control applications where a fully positive definite function is sought.

Numerous research avenues provide future challenges. Viscous forces has been neglected in this treatment, whereas they play an important role in applications. Backstepping and I/O linearization techniques have also been successfully applied to vehicles stabilization problems, and it might be rewarding to understand the relationship between these methods and the field of geometric control for mechanical systems. Finally, a wide variety of motion planning and trajectory generation problems are open, such as for example the design of provably stable switching maneuvers from one relative equilibrium to another.


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# A Coordinate expressions

In this first Appendix, we present coordinate expressions for various quantities defined intrinsically in Section 3. Let $(q^1, \ldots, q^n)$ be a local coordinate system about the point $q_0$. A vector field $X$ is written as

$$X(q) = X^i(q)\frac{\partial}{\partial q^i},$$

where the summation convention is enforced here and in what follows.

We write a covariant derivatives in coordinates by defining its Christoffel symbols $\Gamma^i_{ij}$

$$\nabla_{\frac{\partial}{\partial q^i}}\left(\frac{\partial}{\partial q^j}\right) = \Gamma^k_{ij}\frac{\partial}{\partial q^k}.$$

The Christoffel symbols of a Riemannian connection are defined and computed as follows. Let $M$ be a matrix representation of the metric; in other words let $M_{ij} = \langle\!\langle \frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j} \rangle\!\rangle$. We have

$$\Gamma^k_{ij} = \frac{1}{2}M^{mk}\left(\frac{\partial M_{mj}}{\partial q^i} + \frac{\partial M_{mi}}{\partial q^j} - \frac{\partial M_{ij}}{\partial q^m}\right),$$

where $M^{ij}$ is the inverse of the tensor $M_{ij}$. The covariant derivative of a vector field is then written as

$$\nabla_X Y = \left(\frac{\partial Y^i}{\partial q^j}X^j + \Gamma^i_{jk}X^jY^k\right)\frac{\partial}{\partial q^i}.$$

In local coordinates the forced Euler-Lagrange equations are

$$\frac{dv^i_q}{dt} + \Gamma^i_{jk}v^j_qv^k_q = M^{ij}\left(-\frac{\partial V}{\partial q^j} + F^i_j u_i\right),$$

where $\Gamma^i_{jk}(q)$ are the Christoffel symbols of the metric $M_q$ and where $M^{ij}(q)$ is the inverse tensor to $M_{ij}$. As a final detail, note that whenever $\operatorname{grad} f(q) = 0$, the Hessian of a function $f$ is written in coordinates as:

$$\operatorname{Hess} f\left(X^i\frac{\partial}{\partial q^i}, Y^j\frac{\partial}{\partial q^j}\right)(q) = \frac{\partial^2 f}{\partial q^i \partial q^j}X^iY^j(q). \tag{24}$$

# B Implementation in Symbolic Software

The following Mathematica code illustrates the results presented in Section 5.1.

```
(* Francesco Bullo, June 1999     *
 * Planar underwater body example *)

(* Routines: *)
Grad[f_, g_, x_] := Inverse[g]. Table[D[f,x[[i]]],{i,Length[x]}];
Hessian[f_,X1_,X2_,x_] := Sum[D[f,x[[i]],x[[j]]] X1[[i]] X2[[j]],
    {i,Length[x]}, {j,Length[x]}];
Kinetic2Inertia[KE_,Vel_] := Module[{n}, n=Dimensions[Vel][[1]];
```

```
    2 Table[ If[i==j, Coefficient[KE, Vel[[i]], 2],
    Coefficient[KE, Vel[[i]] Vel[[j]]]/2 ] ,{i,1,n},{j,1,n}] ];

(* Configuration and Velocity *)
Conf = {th,x,y}; Vel = {thd,xd,yd};

(* Kinetic Energy and Inertia
 * easily written in body frame first, transformed afterwards *)
rule = {om -> thd, vx -> Cos[th] xd + Sin[th] yd,
                   vy-> -Sin[th] xd + Cos[th] yd };
KineticEnergy = (II om^2 + mx vx^2 + my vy^2)/2 /. rule;
Inertia = Simplify[Kinetic2Inertia[KineticEnergy, Vel]];

(* Conserved quantities *)
Hamiltonian = KineticEnergy;
X = {0,v0,0};
JX = X.Inertia.Vel ;
VX = - X.Inertia.X / 2;
HX = Hamiltonian - JX;

(* Existence of Relative Equilibria *)
RelEqR = Solve[ Simplify[ Grad[VX, Inertia, Conf]]=={0,0,0}, th];

(* Forces as one forms *)
F1 = {0,Cos[th],Sin[th]};
F2 = {-h, -Sin[th], Cos[th]};
(* and as input vector fields *)
Y1 = Simplify[Inverse[Inertia] . F1];
Y2 = Simplify[Inverse[Inertia] . F2];

(* Compute the perpendicular distribution to {Y1,Y2}
 * and its Lie closure - (it is trivially involutive)  *)
Yperp = {a,b,d};
rule = Solve[ {Y1.Inertia.Yperp==0, Y2.Inertia.Yperp==0}, Yperp];
Yperp = Flatten[Simplify[Yperp /. rule /. d-> Cos[th]]];

(* By Lemma 4.2, there exist an appropriate function phi *)
phi = y-h Sin[th];
Y3 = Simplify[Grad[ phi, Inertia, Conf]];
(* and arbitrarily: *)
Y4 = Simplify[-Cos[th] Y1 + Sin[th] Y2];

(* Compute the subspace perpendicular to {X,Y3} = span Y5 *)
Y5 = {a,b,d};
rule = Solve[ {Y3.Inertia.Y5==0, X.Inertia.Y5==0}, Y5];
Y5 = Flatten[Simplify[Y5 /. rule /. d->1]];

(* Check that VX has positive definite Hessian on Y5 *)
```

```
Simplify[ Hessian[VX, Y5, Y5, Conf] /. RelEqR ]

(* Dissipative Input u4 *)
u4 = d4 Simplify[ Y4. Inertia . (Vel - X)];

(* End of File *)
```