American Control Conference
San Diego, CA, 2-4 June 1999

# Stabilization of Relative Equilibria for Systems on Riemannian Manifolds[1]

Francesco Bullo
General Engineering and Coordinated Science Laboratory
University of Illinois at Urbana-Champaign, 1308 W. Main St, Urbana, IL 61801
*tel* (217) 333-0656, *fax* (217) 244-1653, *bullo@uiuc.edu*, *http://motion.csl.uiuc.edu/~bullo*

**Abstract**

This paper describes a systematic procedure to exponentially stabilize relative equilibria of mechanical systems. We review the notion of relative equilibria and their stability in a Riemannian geometry context. Potential shaping and dissipation are employed to obtain full exponential stabilization to the desired trajectory. Two necessary conditions are that the effective potential be positive definite over a specified subspace and that the system be linearly controllable.

**Keywords:** mechanical systems with symmetry, exponential stabilization, underwater vehicle

## 1 Introduction

Control of underactuated mechanical systems is a challenging research area of increasing interest. On the theoretical side, control problems for mechanical systems benefit from the wealth of geometric mechanics tools available. On the other hand, strong motivation for these problems comes from applications to autonomous vehicles design and control. In this paper, we investigate stabilization techniques for the steady motions called relative equilibria. This family of trajectories is of great interest in theory and applications.

Stabilization of underactuated Hamiltonian systems was originally investigated by van der Schaft [12]. Recently, geometric tools have been employed to address the class of mechanical systems with symmetries. Stability of underwater vehicles is studied in Leonard [7] where symmetry breaking potentials were employed to shape the energy of the closed loop system. Jalnapurkar and Marsden [5] present a framework for the design of controllers for underactuated mechanical systems. In these treatment the family of input forces is assumed momentum preserving and stability in the reduced space is characterized via the Energy-Momentum method.

In this work we focus on vehicles with generic body forces, including both internal (e.g, momentum wheels and sliding masses) and external ones (e.g., propellers). Typically, these systems move on trajectories that do not belong to a constant momentum level set. A simple idealized example is an underwater planar body, depicted in Figure 1. This model is reminiscent of the V/STOL aircraft studied by Hauser and co-workers [4], of the surface vessel studied by Godhavn [3], and of the underwater submersible studied by Leonard [7, 8]. This particular systems is proven to be differentially flat by Martin and co-workers in [10] when no hydrodynamic forces are present.

The main contribution of this paper is the design of a Lyapunov function and of a corresponding controller that stabilize all of the variable of interest, i.e., all of the velocity variables and the internal configuration variables. We refer to this notion as stabilization on the full phase space, as opposed to stabilization for only the internal variables or stabilization on a momentum surface.

A second theme is the emphasis on exponential as opposed to asymptotic convergence, and on the full power of dissipation-based stabilization techniques. In particular we exploit the fact that a dissipative systems has exponential convergence rates under the assumption of linear controllability and the existence of a quadratic Lyapunov function. This fact is well known within the nonlinear stabilization literature, see [6, 11], but has not been fully exploited within the context of mechanical systems. Finally, a third feature of our approach is that we employ a novel Riemannian geometry formalism in describing relative equilibria and their stabilization. One advantage of this approach is that it is capable of dealing with general (but velocity independent) control forces, like the ones present in models of vehicles such as that in Figure 1.

### 1.1 Example and summary of result

We briefly present the key steps of our design procedure by applying it to the planar vehicle in Figure 1. More specifically, we attempt to design some feedback controls that stabilize a trajectory consisting of a steady motion along the $x$ axis of the inertial reference frame.

[1] This work is a short version of [1].

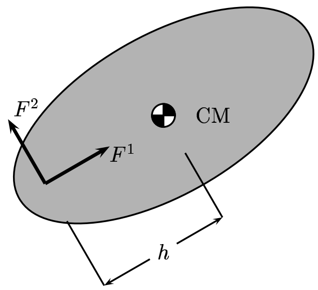


**Figure 1:** Planar underwater vehicle with forces $\{F^1, F^2\}$ a distance $h$ from the center of mass CM. The effect of the fluid is modeled via added masses.

**Stability of an equilibrium point:** We start with some classic results on stability of mechanical systems about a point. Let $q = [q^1, \ldots, q^n]^T$ be the configuration of the systems and let the Hamiltonian be

$$H(q, \dot{q}) = V(q) + \frac{1}{2}\dot{q}^T M(q)\dot{q} =: V(q) + \frac{1}{2}\|\dot{q}\|_M^2. \quad (1)$$

The equations of motion are

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} = -\frac{\partial V}{\partial q} + F, \quad (2)$$

where $C(q, \dot{q})$ is the Coriolis matrix and where the resultant force $F$ can be written as linear combination of $m$ independent control forces $F_i$:

$$F = \sum_{i=1}^{m} F_i u_i.$$

If $m$ is strictly less than the degrees of freedom $n$, the system is called underactuated. Finally, $q_0$ is a stable equilibrium point if the first variation of $V$ vanishes and if the second variation is positive definite at $q_0$:

$$\frac{\partial V}{\partial q}(q_0) = 0, \qquad \text{and} \qquad \frac{\partial^2 V}{\partial q \partial q}(q_0) > 0.$$

**Steady translations of the planar vehicle:** Next, we examine the planar vehicle in Figure 1. Let $q = [\theta, x, y]^T \in \mathbb{R}^3$ denote the position of the vehicle. Assuming that gravity is absent, the Hamiltonian is $H(q, \dot{q}) = \dot{q}^T M(q)\dot{q}/2$, where $M(q)$ is

$$\begin{bmatrix} I & 0 & 0 \\ 0 & m_x(\cos\theta)^2 + m_y(\sin\theta)^2 & (m_x - m_y)\cos\theta\sin\theta \\ 0 & (m_x - m_y)\cos\theta\sin\theta & m_y(\cos\theta)^2 + m_x(\sin\theta)^2 \end{bmatrix},$$

and where $I, m_x, m_y$ are inertial parameters that include the influence of the fluid surrounding the vehicle (e.g., they include the so-called added masses); see [7]. Let $v_{\text{re}} \in \mathbb{R}^3$ denote the velocity of the desired steady motion; e.g., $v_{\text{re}} = [0, 1, 0]^T$ to require the vehicle to move at unit speed along the $x$ axis of a reference frame.

Notice that the Hamiltonian does not depend on the variable $x$, i.e., $\partial H/\partial x = 0$. As it is known in mechanics, this independence implies that the momentum in the direction $v_{\text{re}}$ is a conserved quantity along the solution of the equations of motion. The momentum is:

$$J_{\text{re}}(q, \dot{q}) = \dot{q}^T M(q) v_{\text{re}} = \big(m_x(\cos\theta)^2 + m_y(\sin\theta)^2\big)\dot{x} + \big((m_x - m_y)\sin\theta\cos\theta\big)\dot{y}.$$

Beside Hamiltonian and momentum, an additional conserved quantity is computed via some algebraic manipulation ("summing the square") as follows:

$$\begin{aligned} H_{\text{re}}(q, \dot{q}) &:= H - J_{\text{re}} \\ &= \frac{1}{2}\|\dot{q} - v_{\text{re}} + v_{\text{re}}\|_M^2 - \dot{q}^T M(q) v_{\text{re}} \\ &= -\frac{1}{2}\|v_{\text{re}}\|_M^2 + \frac{1}{2}\|\dot{q} - v_{\text{re}}\|_M^2. \end{aligned}$$

We call $H_{\text{re}}$ the *effective Hamiltonian*, and accordingly we define the *effective potential* as

$$V_{\text{re}}(q) = -\frac{1}{2}\|v_{\text{re}}\|_M^2 = -\frac{1}{2}\left(m_x(\cos\theta)^2 + m_y(\sin\theta)^2\right).$$

The concepts of effective Hamiltonian and potential lead to an elegant parallel between the treatment on stability of an equilibrium point and stability of a steady motion. For example, $H_{\text{re}}$ has a "kinetic energy" component proportional to the velocity error $(\dot{q} - v_{\text{re}})$, as opposed to the usual kinetic energy being proportional to the velocity $\dot{q}$. Additionally, we will show the following results. The steady motion $v_{\text{re}}$ through the point $q_0$ is a solution to the equations of motion if the first variation of $V_{\text{re}}$ vanishes at $q_0$,

$$\frac{\partial V_{\text{re}}}{\partial q}(q_0) = 0,$$

and it is a stable motion if the second variation of $V_{\text{re}}$ restricted to the subspace perpendicular to $v_{\text{re}}$ is positive definite:

$$\frac{\partial^2 V_{\text{re}}}{\partial q \partial q}(q_0) > 0, \qquad \text{restricted to } v_{\text{re}}^{\perp}.$$

In the planar body example, steady translation along the $x$ axis is a solution whenever $\sin 2\theta = 0$. However, we know nothing about the stability of this motion since $V_{\text{re}}(q)$ is independent of $y$ and its second variation is not positive definite when restricted to $v_{\text{re}}^{\perp}$.

**Proportional derivative control for steady translations:** A proportional action $F_P$ is a control force proportional to the first variation of a function $f$:

$$F_P = -\frac{\partial f(q)}{\partial q}.$$

Under such a feedback, the closed loop system satisfies an equation of motion of the form (2), where the closed loop potential energy equals $(V + f)$ and the stability of the equilibrium point $q_0$ depends on whether the second variation of this new potential is positive definite. A dissipative action $F_D$ is usually defined as a control force proportional to the velocity. In particular, for an underactuated system (2) one would set:

$$F_D = -\sum_{i=1}^{m} F_i \, (F_i^T \dot{q}).$$

To render the steady translations of the planar vehicle first stable and then asymptotically stable, we adapt proportional derivative control to the current setting. In particular, we have two fundamental constraints in the design of the proportional action. The latter quantity must preserve the existence of the steady solution, and must lie in the span of the available forces $\{F^1, F^2\}$; see Figure 1. In other words, we employ a feedback $\partial f/\partial q$ where the function $f$ is required to satisfy $\partial f/\partial x = 0$ and $\partial f/\partial q \in \operatorname{span}\{F^1, F^2\}$. In Section 3.1 we provide a methodology to design such functions. For now, it suffices to note that $f(q) = (y - h\sin\theta)^2$ satisfies these constraints in the planar vehicle example. For $m_x > m_y$ the second variation of

$$f + V_{\text{re}} = (y - h\sin\theta)^2 - \tfrac{1}{2}\left(m_x(\cos\theta)^2 + m_y(\sin\theta)^2\right)$$

is positive definite when restricted to $v_{\text{re}}^{\perp}$. The steady translation $v_{\text{re}}$ through $(\theta_0, x_0, y_0) = (0,0,0)$ is therefore stabilized by the proportional feedback $-\partial f/\partial q$.

Finally, we employ dissipation to render the steady translation exponentially stable. Since the nominal velocity is $\dot{q} = v_{\text{re}}$, we expect the correction should be proportional to the velocity error $\dot{q} - v_{\text{re}}$. In fact, in Section 3.2, we show how the feedback controls

$$u_i = -F_i^T(\dot{q} - v_{\text{re}}).$$

lead to the desired exponential convergence rates.

## 2 Mechanical systems with symmetries

We present a coordinate free definition of mechanical control systems based on geometric ideas. Because of space limitations, this section is a very short summary of the treatment in [1].

### 2.1 Natural operations on manifolds

We review some definitions in order to fix some notation; see [2] for a comprehensive treatment. Let $Q$ be a smooth manifold, $q$ be a point on it, $v_q$ be a point on $TQ$, $I \subset \mathbb{R}$ be a real interval and $\gamma : I \to Q$ be a curve on $Q$. On the manifold $Q$, we can define smooth functions $q \mapsto f(q) \in \mathbb{R}$, vector fields $q \mapsto X_q \in T_qQ$, and more general $(r,s)$ tensors fields, that is, real valued multi-linear maps on $(T_qQ^*)^r \times (T_qQ)^s$. We let $C(Q)$ and $\mathfrak{X}(Q)$ denote the set of functions and vector fields on $Q$.

A Riemannian metric on a manifold $Q$ is a $(0,2)$ positive definite tensor field, that is a map that associates to each $q \in Q$ an inner product $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle_q$ on $T_qQ$. A manifold endowed with a Riemannian metric is said to be a Riemannian manifold. An affine connection on $Q$ is a smooth map that assigns to a pair of vector fields $X, Y$ a vector field $\nabla_X Y$ such that for all $f \in C(Q)$ and for all $X, Y, Z \in \mathfrak{X}(Q)$,
(1) $\nabla_{fX+Y} Z = f\nabla_X Z + \nabla_Y Z$, and
(2) $\nabla_X(fY + Z) = (\mathcal{L}_X f)Y + f\nabla_X Y + \nabla_X Z$.
We also say that $\nabla_X Y$ is the covariant derivative of $Y$ with respect to $X$. A Riemannian metric on $Q$ induced an affine connection connection called Riemannian by means of the Levi-Civita theorem, see [2].

Next, we introduce the notion of covariant derivative along a curve. Consider a smooth curve $\gamma = \{\gamma(t) \in Q, t \in [0,1]\}$, and a vector field $\{v(t) \in T_{\gamma(t)}Q, t \in [0,1]\}$ defined along $\gamma$. The covariant derivative of the vector field $v$ along $\gamma$ is denoted by $\dfrac{Dv(t)}{dt}$.

The notion of first and second variation of a function are introduced as follows. Given a function $f \in C(Q)$, its gradient is the vector field defined by

$$\mathcal{L}_X f = \langle\!\langle \operatorname{grad} f\, , \, X \rangle\!\rangle,$$

and its Hessian is the $(0,2)$ tensor field defined by

$$\operatorname{Hess} f(X,Y) = (\mathcal{L}_Y \mathcal{L}_X - \mathcal{L}_{\nabla_Y X}) f. \tag{3}$$

Notice that $\operatorname{Hess} f$ maps $T_qQ \times T_qQ$ to the real line and it is therefore a two form; we will often investigate whether this two form is positive definite over certain sub-bundles of $T_qQ$.

### 2.2 Mechanical control systems

A *mechanical control system* is defined by the following objects: (1) an $n$-dimensional configuration manifold $Q$, with local coordinates $\{q^1, \ldots, q^n\}$, (2) a Riemannian metric $M_q$ on $Q$ (the kinetic energy), also denoted by $\langle\!\langle \cdot\, , \cdot \rangle\!\rangle$, (3) a function $V$ on $Q$ describing the potential energy, and (4) an $m$-dimensional codistribution $\mathcal{F} = \operatorname{span}\{F^1, \ldots, F^m\}$ defining the input forces.

Let $q \in Q$ be the configuration of the system and $v_q \in T_qQ$ its velocity. The total energy, *Hamiltonian*, $H$ is

$$H(v_q) = \frac{1}{2}\langle\!\langle v_q\, , \, v_q \rangle\!\rangle + V(q) = \frac{1}{2}\|v_q\|^2 + V(q).$$

Let the input vector fields be $Y_i = M_q^{-1} F^i$, and let $\mathcal{Y} = \operatorname{span}_{C(Q)}\{Y_1, \ldots, Y_m\}$ be the input distribution. The Euler-Lagrange equations are

$$\frac{D\, v_q}{dt} = -\operatorname{grad} V + \sum_{i=1}^{m} Y_i u^i, \tag{4}$$

where the input functions $\{u^i(t), t \in \mathbb{R}^+\}$ are bounded measurable.

Given a metric tensor $M$ on the manifold $Q$, a vector field $X$ is said to be an *infinitesimal isometry* if the tensor field $\nabla X : Y \to \nabla_Y X$ is skew symmetric with respect to the metric tensor $M$, that is,

$$\langle\!\langle Y\,,\,\nabla_Z X\rangle\!\rangle + \langle\!\langle Z\,,\,\nabla_Y X\rangle\!\rangle = 0. \tag{5}$$

We call $X$ an *infinitesimal symmetry* for the mechanical control system $(Q, M, V, \mathcal{F})$, if it is an infinitesimal isometry and if it satisfies $\mathcal{L}_X V = 0$ and $\mathcal{L}_X Y_i = 0$ for all $i = 1, \ldots, m$. An infinitesimal isometry gives rise to a conserved quantity. The *momentum* $J_X : TQ \to C(Q)$ is defined by

$$J_X(Y) = \langle\!\langle X\,,\,Y\rangle\!\rangle, \qquad Y \in \mathfrak{X}(Q),$$

and, along the solutions to the equations of motion (4) at $u_i = 0$, it holds $\frac{D}{dt} J_X(v_q) = 0$.

Last, we devise an integral of motion by combining Hamiltonian and momentum. The *effective Hamiltonian* is the map $H_X : TQ \to \mathbb{R}$ defined by $H_X(v_q) = H(v_q) - J_X(v_q)$. The "summing the square" computation [9] shows that

$$H_X(v_q) = \left(V - \frac{1}{2}\|X\|^2\right)(q) + \frac{1}{2}\|v_q - X\|^2,$$

and, accordingly, we call *effective potential* the map $V_X : Q \to \mathbb{R}$ defined by

$$V_X(q) = V(q) - \frac{1}{2}\|X\|^2(q).$$

Hence, the effective Hamiltonian is sum of two terms: a potential and a kinetic energy-like term. The latter term is a modified kinetic energy, where the argument is a "velocity error" $(v_q - X)$.

Next, we present a quick review of various definitions and results; see [9] for a more extensive treatment. A curve $\gamma : I \subset \mathbb{R} \to Q$ is called *relative equilibrium* if it solves the equations of motion (4) and if it is a flow of the infinitesimal isometry $X$, that is,

$$\frac{d}{dt}\gamma(t) = X(\gamma(t)). \tag{6}$$

**Proposition 2.1 (Existence and Stability)** *Let $(Q, M, V, \mathcal{F})$ be a mechanical control system and let $X$ be an infinitesimal symmetry. A solution $\gamma : I \to Q$ to the equations of motion (4) is a relative equilibrium if $\gamma(0) = q_0$ is a critical point for the effective potential:*

$$\operatorname{grad} V_X(q_0) = 0. \tag{7}$$

*Additionally, the relative equilibrium $\gamma$ is Lyapunov stable if the Hessian of the effective potential is positive definite over variations perpendicular to $X$:*

$$\operatorname{Hess} V_X(Y, Y)(q_0) > 0, \tag{8}$$

*for all $Y \in T_{q_0}Q$ such that $\langle\!\langle Y\,,\,X\rangle\!\rangle = 0$.*

The fundamental fact in the last proposition is that the effective Hamiltonian $H_X$ is a map $TQ \to \mathbb{R}$ that has positive definite Hessian in all but one directions on the phase space $TQ$. This observation is key to later developments, where $H_X$ will be the candidate Lyapunov function for the stabilization problem.

## 3 Stabilization on the phase space

In what follows, we design controllers that stabilize relative equilibria of a mechanical control system with a symmetry. Because of space limitations, this section is a very short summary of the treatment in [1].

### 3.1 Potential shaping that preserves the relative equilibrium

As discussed in the Introduction, numerous works have studied the application of a so-called proportional feedback action. The effect of such controls is described in terms of the effective potential energy "shaping" of the closed loop.

**Lemma 3.1** *Given the mechanical control system $(Q, M, V, \mathcal{F})$ with equations of motion (4). Assume there exists a function $\phi : Q \to \mathbb{R}$ such that*

$$\operatorname{grad}\phi = \sum_{i=1}^{m} c^i(q) Y_i,$$

*and set the inputs $u^i = -c^i(q)\phi + v^i$. Then the closed loop system is the mechanical control system $(Q, M, (V + \frac{1}{2}\phi^2), \mathcal{F})$.*

We apply this idea to the present context: we attempt to "shape" the effective potential $V_X$ while preserving the existence of the relative equilibrium through the point $q_0$. In equivalent words, we ask for the existence of a function $\phi : Q \to \mathbb{R}$ with $\phi(q_0) = 0$ and such that

$$\mathcal{L}_X \phi = 0,$$

and that

$$\operatorname{grad}\phi \in \mathcal{Y},$$

where $\mathcal{Y} = \operatorname{span}\{Y_1, \ldots, Y_m\}$ is the input distribution.

**Proposition 3.2** *Let $X$ be an infinitesimal isometry on the Riemannian manifold $Q$. Let $\mathcal{Y}$ be an $m$ dimensional distribution invariant under the action of $X$, and let $\mathcal{Y}^\perp$ denote its orthogonal complement and let* $\operatorname{Lie}(\mathcal{Y}^\perp)$ *denote its involutive closure.*

*Then, the distribution* $\operatorname{Lie}(\mathcal{Y}^\perp) + \operatorname{span}_{C(Q)}\{X\}$ *is involutive, has dimension $(n-p) \geq (n-m)$, and its $p$ integral functions $\phi_1, \ldots, \phi_p$ satisfy*

$$\mathcal{L}_X \phi_j = 0, \qquad \text{and} \qquad \operatorname{grad}\phi_j \in \mathcal{Y}, \qquad \forall j = 1, \ldots, p.$$

Given these $p$ functions $\phi_j$, let $k_1, \ldots, k_p$ be positive scalars and apply the feedback controls $u^i(q)$ defined implicitly via

$$\sum_{i=1}^{m} Y_i(q) u^i(q) = -\sum_{j=1}^{p} k_j \phi_j \operatorname{grad} \phi_j. \qquad (9)$$

According to Lemma 3.1, we define the effective Hamiltonian for the closed loop as

$$\begin{aligned} \widehat{H}_X(v_q) &= H(v_q) + \frac{1}{2}\sum_{j=1}^{p} k_j \phi_j^2 - \langle\!\langle X\,,\, v_q \rangle\!\rangle \\ &= \left( V_X + \frac{1}{2}\sum_{j=1}^{p} k_j \phi_j^2 \right) + \frac{1}{2}\|v_q - X\|^2, \end{aligned}$$

and the effective potential for the closed loop as

$$\widehat{V}_X = V_X + \frac{1}{2}\sum_{j=1}^{p} k_j \phi_j^2.$$

Recall from Proposition 2.1, that stability of a relative equilibrium depends from the positive definiteness of the Hessian of $V_X$.

**Proposition 3.3 (Stabilization)** *Let* $(Q, M, V, \mathcal{F})$ *be a mechanical control system, let* $X$ *be an infinitesimal symmetry and let* $\{\gamma : I \to Q, \gamma(0) = q_0\}$ *be a relative equilibrium. Let* $\phi_1, \ldots, \phi_p$ *be* $p$ *functions obtained as in Lemma 3.2 so that, without loss of generality, we set*

$$Y_j = \operatorname{grad} \phi_j, \qquad j = 1, \ldots, p \leq m.$$

*Assume that the effective potential* $V_X$ *is positive definite over variations perpendicular to the subspace* $\{X, \operatorname{grad}\phi_1, \ldots, \operatorname{grad}\phi_p\}$*, that is,*

$$\operatorname{Hess} V_X(Y, Y)(q_0) > 0, \qquad (10)$$

*for all* $Y$ *perpendicular to* $\{X, \operatorname{grad}\phi_1, \ldots, \operatorname{grad}\phi_p\}_{q_0}$.

*Then, there exist positive constants* $k_1, \ldots, k_p$ *such that the feedback controls* $u^j(q) = -k_j \phi_j(q)$*, render the relative equilibrium* $\gamma$ *Lyapunov stable.*

### 3.2 Dissipation

In this section we employ the classic notion of dissipative feedback to achieve asymptotic and exponential stability. Within the context of relative equilibria stabilization, the key observation is that the effective Hamiltonian $H_X$ is positive definite in all but one direction on the tangent space to $TQ$.

We start by computing the time derivative of the effective Hamiltonian along the closed loop system. Given the mechanical control system $(Q, M, V, \mathcal{F})$ with infinitesimal symmetry $X$ and with effective Hamiltonian $H_X$, it holds

$$\frac{D\, H_X(v_q)}{dt} = \langle\!\langle v_q - X\,,\, Y_i \rangle\!\rangle u^i.$$

**Proposition 3.4 (Exponential stabilization)** *Under the same assumptions as in Proposition 3.3, there exist positive constants* $k_1, \ldots, k_p$ *and* $d_1, \ldots, d_m$ *such that the feedback controls*

$$\begin{aligned} u^j(v_q) &= -k_j \phi_j(q) - d_j \dot{\phi}_j(v_q), \qquad && 1 \leq j \leq p \\ u^i(v_q) &= -d_i \langle\!\langle v_q - X\,,\, Y_i \rangle\!\rangle, && p < i \leq m, \end{aligned}$$

*render the relative equilibrium* $\{\gamma : I \to Q, \gamma(0) = q_0\}$ *exponentially stable.*

**Acknowledgements**

It is a pleasure to thank Rodolphe Sepulchre for helpful discussions and the anonymous reviewers for numerous comments and corrections.

## References

[1] F. Bullo. Stabilization of relative equilibria for underactuated systems on Riemannian manifolds. Submitted, *Automatica*. An early version to appear in *Proc MTNS*, Padova, Italy 1998.

[2] M. P. Do Carmo. *Riemannian Geometry.* Birkhäuser, Boston, MA, 1992.

[3] J.-M. Godhavn. Nonlinear tracking of underactuated surface vessels. In *Proc CDC*, Kobe, Japan, December 1996.

[4] J. E. Hauser, S. S. Sastry, and G. Meyer. Nonlinear control design for slightly nonminimum phase systems: application to V/STOL aircraft. *Automatica*, 28(4):665–679, 1992.

[5] S. M. Jalnapurkar and J. E. Marsden. Stabilization of relative equilibria. Technical Report CDS 98–007, California Institute of Technology, July 1998. Available at `http://www.cds.caltech.edu`.

[6] K. K. Lee and A. Arapostathis. Remarks on smooth feedback stabilization of nonlinear systems. *Syst. & Control L.*, 10:41–44, 1988.

[7] N. E. Leonard. Stabilization of underwater vehicle dynamics with symmetry-breaking potentials. *Syst. & Control L.*, 32:35–42, 1997.

[8] N. E. Leonard and C. A. Woolsey. Internal actuation for intelligent underwater vehicle control. In *Tenth Yale Workshop on Adaptive and Learning Systems*, pages 295–300, June 1998.

[9] J. E. Marsden. *Lectures on Mechanics.* Cambridge University Press, New York, NY, 1992.

[10] P. Martin, S. Devasia, and B. Paden. A different look at output tracking: Control of a VTOL aircraft. *Automatica*, 32(1):101–107, 1996.

[11] R. Sepulchre, M. Janković, and P. Kokotović. *Constructive Nonlinear Control.* Springer Verlag, New York, NY, 1997.

[12] A. J. van der Schaft. Stabilization of Hamiltonian systems. *Nonl. Analysis, Theory, Methods & App.s*, 10(10):1021–1035, 1986.