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A SERIES DESCRIBING THE EVOLUTION OF MECHANICAL CONTROL SYSTEMS

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Abstract: We compute a series describing the evolution of a mechanical system starting at rest and subject to a time-varying external force. This generalizes various previous results and lays the foundation for the design of motion control algorithms for a large class of autonomous vehicles, robotic manipulators and locomotion devices.

Keywords: mechanical systems, motion planning, nonlinear controllability

1. INTRODUCTION

Underactuated mechanical systems provide a challenging research area of increasing interest in both application and theory. In this paper, we examine an important class of mechanical control systems, including autonomous vehicles and robotic locomotion devices.

A rich literature is available on the motion planning problem for kinematic systems, that is systems without drift. Numerous approaches include chained systems (Murray and Sastry, 1993), systems on Lie groups (Leonard and Krishnaprasad, 1995) and (Kolmanovsky and McClamroch, 1996), and the general solutions proposed in (Lafferriere and Sussmann, 1991). The enabling step in these works is the characterization of the evolution of the control system. In other words, the basic contributions of these papers is the computation of a "input history to final displacement" map. In its most general formulation this problems is solved by the Chen-Fliess-Sussmann series in (Sussmann, 1986) and by the logarithmic series in (Agračhev and Gamkrelidze, 1978). Motion control algorithms are then designed by inverting this "inputs to displacement" map.

Unfortunately, this body of literature is of limited applicability to a large class of mechanical systems that present dynamics. Recently, some progress in this direction has been obtained in (Bullo and Leonard, 1997) and (Bullo *et al.*, 1997). Under the assumption of small amplitude forcing, the authors compute the initial terms of a Taylor series describing the forced evolution. This result is related to the controllability analysis in (Lewis and Murray, 1997).

The main contribution of the paper is a series that describes the evolution of a forced mechanical system starting from rest. Mechanical systems are presented as second order systems on a configuration manifold. Instead of a series on the full phase space ($2n$ dimensional), the evolution is described as a flow on the configuration space Q. The treatment relies on some "chronological calculus" tools, see (Agračhev and Gamkrelidze, 1978), as opposed to the previous work based on the perturbation method. This improvement leads to numerous conceptual and computational advantages, one of which is the series this paper presents.

2. TOOLS FROM CHRONOLOGICAL CALCULUS

We present some basic result useful to describe composition and perturbation of flows of vector fields on manifolds. They are borrowed from the so-called "chronological calculus", as introduced in (Agračhev and Gamkrelidze, 1978), and employed in (Lafferriere and Sussmann, 1991; Kawski and Sussmann, 1997).

We begin by reviewing some notation:

(i) The order of composition of functions is $f(\phi(x)) = (f \circ \phi)(x)$.
(ii) x is a point on the manifold M, $f(x), g(x)$ are vector fields, and $[f,g](x) = \mathrm{ad}_f\, g(x)$ is their Lie bracket.
(iii) A non-autonomous differential equation on M is written as

$$\begin{aligned}\dot{x}(t) &= f(x,t)\\ x(0) &= x_0,\end{aligned}$$

and the solution is denoted by

$$x(t) = \Phi^f_{0,t}(x_0).$$

(iv) If g is a vector field and ϕ is a diffeomorphism on M, the pull-back $\phi^* g$ is a vector field defined by

$$(\phi^* g)(x) \triangleq \left(T_x \phi^{-1} \circ g \circ \phi\right)(x).$$

2.1 *Variation of constants formula*

Consider the non-stationary differential equation

$$\begin{aligned}\dot{x}(t) &= f(x,t) + g(x) \qquad (1)\\ x(0) &= x_0,\end{aligned}$$

where $f(x,t)$ and g are analytic vector fields. It is instructive to regard g as a perturbation to the vector field f and describe the flow of the previous differential equation in terms of a nominal and perturbed flow. As depicted in Figure 1, we have the following relationship between flows of vector fields (i.e., solutions of differential equations):

$$\Phi^{f+g}_{0,t} = \Phi^f_{0,t} \circ \Phi^{(\Phi^f_{0,t})^* g}_{0,t}. \qquad (2)$$

We formally state this result as follows.

Lemma 2.1. Let $\{x(t), t \in [0,T]\}$ be the solution to equation (1) and let $\{y(t), t \in [0,T]\}$ be the solution to

$$\begin{aligned}\dot{y}(t) &= \left((\Phi^f_{0,t})^* g\right)(y) \qquad (3)\\ y(0) &= x_0.\end{aligned}$$

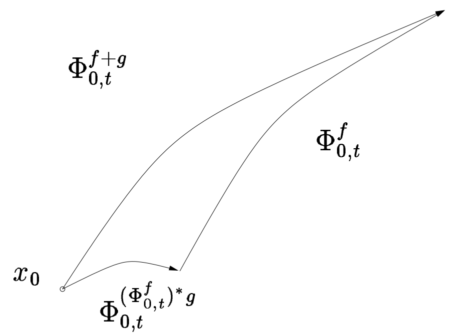

Fig. 1. The flow along $f + g$ is written as the composition of a flow along the pull-back system $(\Phi^f_{0,t})^* g$ and the flow along f.

Then it holds

$$x(T) = \Phi^f_{0,T}(y(T)).$$

2.2 *Formal expansions for the pull-back along a flow*

Motivated by equation (3) we investigate the pull-back of a vector field $g(x)$ along the flow of a time-varying vector field $f(x,t)$. It turns out that

$$\frac{d}{dt}\left(\Phi^f_{0,t}\right)^* g = \left(\Phi^f_{0,t}\right)^* [f(x,t), g(x)], \qquad (4)$$

where the Lie bracket between f and g is computed at t fixed. While this equality is a well-known fact for autonomous vector fields, it also holds for time-varying f. This statement is proved in (Agračhev and Gamkrelidze, 1978), see equation 3.3.

At fixed $x \in M$, we integrate equation (4) from time 0 to t to obtain

$$(\Phi^f_{0,t})^* g = g + \int_0^t \left((\Phi^f_{0,s})^* [f(s), g]\right) ds,$$

where we have dropped the argument x inside the integral in the interest of simplicity. We can iteratively apply the previous equality to obtain the formal expansion:

$$(\Phi^f_{0,t})^* g = g + \sum_{n=1}^{\infty} \int_0^t \cdots \int_0^{s_{n-1}} \left(\mathrm{ad}_{f(s_n)} \ldots \mathrm{ad}_{f(s_1)}\, g\right) ds_n \ldots ds_1. \qquad (5)$$

If the vector field f is time-independent, i.e., $f(x,t) = f(x)$, the previous formula reduces to the classic Campbell-Backer-Hausdorff formula:

$$(\Phi^f_{0,t})^* g = \sum_{n=0}^{\infty} \mathrm{ad}^n_f\, g \frac{t^n}{n!}.$$

3. MECHANICAL SYSTEMS AND HOMOGENEITY

In what follows we model mechanical systems on a configuration manifold Q. Because we are interested in local problems, we restrict our analysis to the case where $Q = \mathbb{R}^n$; we include some comments on the extension to the general manifold case. The fundamental feature of this setting is that we analyse *second order* differential equations, where the input is an acceleration (alternatively a force) and not a velocity.

Let $q = (q^1, \dots, q^n) \in \mathbb{R}^n$ be the configuration. We consider the following control system

$$\ddot{q}^i + \Gamma^i_{jk}(q)\dot{q}^j\dot{q}^k = \sum_{a=1}^{m} Y^i_a(q)u_a(t), \tag{6}$$

where

- (i) the summation convention is in place here and in what follows,
- (ii) $\Gamma^k_{jk}(q)$ are n^3 analytic functions called the Christoffel symbols,
- (iii) for $a = 1, \dots, m$, $Y_a(q)$ is an input vector fields and $Y^i_a(q,t)$ is its i^{th} component with respect to the usual basis on $\mathbb{R}^n$. In what follows, we let

$$Y(q,t) = \sum_{a=1}^{m} Y_a(q)u_a(t).$$

Equation (6) describes a large class of mechanical systems with Lagrangian equal to kinetic energy, see (Lewis and Murray, 1997), with symmetries, see (Bullo *et al.*, 1997) and with nonholonomic constraints (Lewis, 1998; Lewis, 1997). For example, should the mechanical system be a robotic manipulator, then the Christoffel symbols can be easily computed via a well-known combination of partial derivatives of the inertia tensor.

3.1 *Lie algebraic structure*

We here review the Lie algebraic properties of the system (6). We start by rewriting the system in vector notation as

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -\Gamma(q,\dot{q}) \end{bmatrix} + \sum_{a=1}^{m} \begin{bmatrix} 0 \\ Y_a \end{bmatrix} u_a(t)$$

where $\Gamma(q,\dot{q})$ is the vector with components $\Gamma^i_{jk}(q)\dot{q}^j\dot{q}^k$. If we define $x = (q, \dot{q})$, and

$$Z_g = \begin{bmatrix} \dot{q} \\ -\Gamma(q,\dot{q}) \end{bmatrix} \quad \text{and} \quad Y^{\text{lift}}_a \triangleq \begin{bmatrix} 0 \\ Y_a \end{bmatrix},$$

then we can rewrite the system as

$$\dot{x} = Z_g(x) + \sum Y_a(x)^{\text{lift}} u_a.$$

Let $h_i(q,\dot{q})$ be the set of scalar functions on $\mathbb{R}^{2n}$, which are arbitrary functions of q and which are homogeneous polynomials in $\{\dot{q}^1, \dots, \dots q^n\}$ of degree i. Let $\mathcal{P}_i$ be the set of vector fields on $\mathbb{R}^{2n}$ which have the first n components in h_i and the second n components in h_{i+1}. It can be directly verified that

$$Z_g \in \mathcal{P}_1 \quad \text{and} \quad Y^{\text{lift}}_a \in \mathcal{P}_{-1}.$$

This set of vector fields has the useful properties:

- (i) $\{[X,Y] \text{ s.t. } X \in \mathcal{P}_i, Y \in \mathcal{P}_j\} \subset \mathcal{P}_{i+j}$,
- (ii) if $k \leq -2$, then $\mathcal{P}_k = \{0\}$,
- (iii) if $k \geq 1$, then $X(q,0) = 0$ for all $X(q,\dot{q}) \in \mathcal{P}_k$.

On the base of these properties we investigate the Lie brackets between the drift Z_g and the inputs Y^{lift}_a. Iterating these brackets we have:

1. $Z_g \in \mathcal{P}_1$ and $Y^{\text{lift}}_a \in \mathcal{P}_{-1}$.
2. $[Z_g, Y^{\text{lift}}_a] \in \mathcal{P}_0$ and $[Y^{\text{lift}}_a, Y^{\text{lift}}_b] = 0$.
3. $[Z_g, [Z_g, Y^{\text{lift}}_a]] \in \mathcal{P}_1$ and $[Y^{\text{lift}}_b, [Z_g, Y^{\text{lift}}_a]] \in \mathcal{P}_{-1}$,

and so forth, as the number of Lie brackets increases.

Remark 3.1. While the results in this section are presented in a coordinate dependent fashion, see also (Sontag and Sussmann, 1986), it is possible to turn them into geometric statements. Key concept is the notion of geometric homogeneity described in (Kawski, 1995) (mechanical systems are homogeneous with respect to the Liouville vector field).

3.2 *The symmetric product*

In this section we focus on the Lie bracket $[Y^{\text{lift}}_b, [Z_g, Y^{\text{lift}}_a]]$. Since this vector field belongs to $\mathcal{P}_{-1}$, there must exist a vector field on $\mathbb{R}^n$, which we denote $\langle Y_a : Y_b \rangle$, such that

$$\langle Y_a : Y_b \rangle^{\text{lift}} = [Y^{\text{lift}}_b, [Z_g, Y^{\text{lift}}_a]].$$

Such a vector field is called *symmetric product* between Y_b and Y_a and a direct computation shows that it satisfies

$$\langle Y_b : Y_a \rangle^i = \frac{\partial Y^i_a}{\partial q^j} Y^j_b + \frac{\partial Y^i_b}{\partial q^j} Y^j_a + \Gamma^i_{jk}\left(Y^j_a Y^k_b + Y^k_a Y^j_b\right).$$

For a more geometric definition of symmetric product we refer the reader to (Lewis and Murray, 1997). The adjective "symmetric" comes from the straightforward equality

$$\langle Y_a : Y_b \rangle = \langle Y_b : Y_a \rangle.$$

Finally, let $Y^{\text{lift}}(q,t) = \sum Y_a^{\text{lift}}(q)u_a(t)$ and, at fixed q, denote finite integrals with respect to time t as:

$$\overline{Y}(q,t) \triangleq \int_0^t Y(q,s)ds. \tag{7}$$

We insist that q is fixed during the integration with respect to t. Via an integration by parts and thanks to the symmetry of the symmetric product, one can verify that

$$\int_0^t \int_0^{s_1} [Y^{\text{lift}}(s_2), [Y^{\text{lift}}(s_1), Z_g]]ds_2 ds_1 = -\frac{1}{2}\left\langle \overline{Y}(t) : \overline{Y}(t)\right\rangle^{\text{lift}}, \tag{8}$$

where the $(q,\dot{q})$ dependency is dropped for simplicity.

3.3 *Solutions of ordinary differential equations with polynomial vector fields*

In this section we compute solutions to a few differential equations defined by the polynomial vector fields introduced above. In particular we can make significant simplifications in the following two cases:

(i) Let $Y(q,t)$ be a vector field on $\mathbb{R}^n$, and consider the differential equation on $x = (q,v) \in T\mathbb{R}^n$:

$$\dot{x} = Y^{\text{lift}}(q,t).$$

In coordinates the previous equation reads $\dot{q} = 0$ and $\dot{v} = Y(q,t)$. If $(q_0, v_0) \in T_{q_0}\mathbb{R}^n$ are the initial conditions we compute

$$\Phi_{0,t}^{Y^{\text{lift}}}\left(\begin{bmatrix} q_0 \\ v_0 \end{bmatrix}\right) = \begin{bmatrix} q_0 \\ v_0 + \overline{Y}(q_0,t) \end{bmatrix}.$$

(ii) For $m > 0$, consider the differential equation on $x = (q,v) \in T\mathbb{R}^n$

$$\begin{aligned} \dot{x} &= X_0(q,v,t) + X_m(q,v,t) \\ x(0) &= (q_0, 0), \end{aligned} \tag{9}$$

where the X_0 and X_m belong respectively to $\mathcal{P}_0$ and $\mathcal{P}_m$. We write X_0 as

$$X_0 = \begin{bmatrix} X_{0,1}(q,t) \\ X_{0,2}(q,v,t) \end{bmatrix}$$

where $X_{0,2}$ depends linearly on v. Since every component of the vector field X_m is at lest linear in v and since the initial velocity $v(0)$ is assumed zero, $v(t)$ remains zero for all time t. Accordingly, the differential equation (9) on $T\mathbb{R}^n$ reduces to a differential equation on $\mathbb{R}^n$. In short:

$$\begin{aligned} q(t) &= \Phi_{0,t}^{X_{0,1}}(q_0) \\ v(t) &= 0. \end{aligned}$$

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We would like to describe the evolution of system (6) when starting from rest. We plan to do this by writing the total flow into a flow in the velocity variables and a flow in the configuration variables.

Theorem 4.1. Let $\{\gamma(t), t \in [0,T]\}$ denote the solution to the differential equation

$$\ddot{\gamma}^i + \Gamma^i_{jk}(\gamma)\dot{\gamma}^j\dot{\gamma}^k = Y^i(\gamma,t)$$

with initial conditions $\gamma(0) = q_0$ and $\dot{\gamma}(0) = 0$. Let the functions $\Gamma^i_{jk}(q)$ and the vector field $Y(q,t)$ be uniformly integrable and bounded analytic in a neighborhood of q_0. At fixed q, let $V_1(q,t) = \int_0^t Y(q,s)ds$ and

$$V_{k+1}(q,t) = -\int_0^t \Big\langle V_k(q,s) : \sum_{m=1}^{k-1} V_m(q,s) + \frac{1}{2}V_k(q,s)\Big\rangle ds.$$

There exists a sufficiently small T_c such that the series $\sum_{k=1}^{\infty} V_k(q,t)$ converges absolutely and uniformly in t and q for all $t \in [0,T_c]$ and for all q in a neighborhood of q_0. Over the same interval the solution $\gamma(t)$ satisfies

$$\dot{\gamma}(t) = \sum_{k=1}^{+\infty} V_k(\gamma(t),t). \tag{10}$$

For a detailed proof of convergence of the series we refer to a forthcoming publication. In what follows, we present a formal derivation of the series expansion.

PROOF. Let k be a positive integer, $x_k \in \mathbb{R}^{2n}$, and let X_k and Y_k be time varying vector fields on $\mathbb{R}^n$. Consider the differential equation

$$\begin{aligned} \dot{x}_k &= \left(Z_g + [X_k^{\text{lift}}, Z_g] + Y_k^{\text{lift}}\right)(x_k,t) \\ x_k(0) &= \begin{bmatrix} q_0 \\ 0 \end{bmatrix}. \end{aligned} \tag{11}$$

We recover the mechanical system in equation (6) by setting $k = 1$, $X_1 = 0$, $Y_1 = Y(q,t)$, and accordingly $x(t) = x_1(t)$.

Using the variation of constants formula in equations (2.1) and (3), we set

$$x_k(t) = \Phi_{0,t}^{Y_k^{\text{lift}}}(x_{k+1}(t)) \tag{12}$$

and

$$\begin{aligned} \dot{x}_{k+1} &= \left(\left(\Phi_{0,t}^{Y_k^{\text{lift}}}\right)^* (Z_g + [X_k^{\text{lift}}, Z_g])\right)(x_{k+1}) \\ x_{k+1}(0) &= \begin{bmatrix} q_0 \\ 0 \end{bmatrix}. \end{aligned} \tag{13}$$

where we compute the pull-back along the flow by means of the infinite series in equation (5). Remarkably, this series reduces to a finite sum. From the discussion in Section 3.1 on the Lie algebraic structure of the various vector fields, we have

$$\begin{aligned}\mathrm{ad}^m_{Y_k^{\mathrm{lift}}} Z_g &= 0\\ \mathrm{ad}^m_{Y_k^{\mathrm{lift}}}[X_k^{\mathrm{lift}}, Z_g] &= 0,\end{aligned}$$

for all $m \geq 3$. With a little book-keeping we can exploit these equalities and compute

$$\begin{aligned}&\left(\Phi_{0,t}^{Y_k^{\mathrm{lift}}}\right)^* (Z_g+[X_k^{\mathrm{lift}}, Z_g])\\ &= Z_g+[X_k^{\mathrm{lift}}, Z_g]+\int_0^t [Y_k^{\mathrm{lift}}(s), \left(Z_g + [X_k^{\mathrm{lift}}, Z_g]\right)]ds\\ &\quad + \int_0^t \int_0^{s_1} [Y_k^{\mathrm{lift}}(s_2), [Y_k^{\mathrm{lift}}(s_1), Z_g]]ds_2 ds_1\\ &= Z_g + [X_k^{\mathrm{lift}} + \overline{Y}_k^{\mathrm{lift}}, Z_g] + [\overline{Y}_k^{\mathrm{lift}}(s), [X_k^{\mathrm{lift}}, Z_g]]\\ &\quad + \int_0^t \int_0^{s_1} [Y_k^{\mathrm{lift}}(s_2), [Y_k^{\mathrm{lift}}(s_1), Z_g]]ds_2 ds_1\\ &= Z_g + [X_k^{\mathrm{lift}} + \overline{Y}_k^{\mathrm{lift}}, Z_g] - \left\langle \overline{Y}_k^{\mathrm{lift}} : X_k^{\mathrm{lift}} \right\rangle\\ &\qquad - \frac{1}{2}\left\langle \overline{Y}_k^{\mathrm{lift}} : \overline{Y}_k^{\mathrm{lift}} \right\rangle,\end{aligned}$$

where we have used equation (8). Therefore, the differential equation describing the evolution of $x_{k+1}(t)$ is of the same form as equation (11), where

$$\begin{aligned}X_{k+1} &= X_k + \overline{Y}_k\\ Y_{k+1} &= -\left\langle \overline{Y}_k \ : \ X_k + \tfrac{1}{2}\overline{Y}_k \right\rangle.\end{aligned}$$

We easily compute

$$X_k = \sum_{m=1}^{k-1} \overline{Y}_m$$

and set

$$Y_{k+1} = -\left\langle \overline{Y}_k \ : \ \sum_{m=1}^{k-1} \overline{Y}_m + \tfrac{1}{2}\overline{Y}_k \right\rangle.$$

According to the the iteration in the statement of the theorem, we have proven that $V_k = \overline{Y}_k$.

In summary, the iteration procedure proves that, for any $k \geq 2$, the solution to the original mechanical system $x(t) = x_1(t)$ satisfies

$$x(t) = \left(\Phi_{0,t}^{Y_1^{\mathrm{lift}}} \circ \Phi_{0,t}^{Y_2^{\mathrm{lift}}} \circ \ldots \circ \Phi_{0,t}^{Y_{k-1}^{\mathrm{lift}}}\right)(x_k(t)),$$

where $\{x_k(t), t \in [0,T]\}$ is the solution to equation (11). We immediately simplify this result as follows. Since the Lie bracket

$$[Y_i^{\mathrm{lift}}(x, s_1), Y_j^{\mathrm{lift}}(x, s_2)] = 0$$

for all i, j and for all s_1, s_2, the flows of Y_i^{lift} and Y_j^{lift} commute. Accordingly, we write

$$x(t) = \Phi_{0,t}^{\sum_{m=1}^{k-1} Y_m^{\mathrm{lift}}}(x_k(t)).$$

Since the previous equality holds for all k, we next investigate the quantities $\sum_{m=1}^{k} Y_m(q,t)$ and $\{x_k(t), t \in [0,T]\}$ in the limit $k \to +\infty$.

Since the input $Y_1(q,t) = Y(q,t)$ is assumed analytic and bounded, one can prove the existence of a constant M and positive values c_k such that

$$\|\overline{Y}_k(q,t)\| \leq c_k t^{2k-1} \quad \text{and} \quad c_{k+1} \leq M c_k,$$

where $\|\cdot\|$ is an appropriate semi-norm. In the interest of brevity, we refer to a forthcoming publication for the proof of these statements.

In short, the series $\sum_{m=1}^{\infty} Y_m(q,t)$ converges absolutely and uniformly in q and t, for q in a neighborhood of q_0 and for sufficiently small t. We denote

$$Y_\infty(q,t) = \sum_{m=1}^{\infty} Y_m(q,t).$$

Since the vector field Y_∞^{lift} belongs to $\mathcal{P}_{-1}$ we can explicitly integrate its flow, see Section 3.3. From the initial condition $[q_0^T, 0^T]^T \in T_{q_0}\mathbb{R}^n$, we have

$$\Phi_{0,t}^{Y_\infty^{\mathrm{lift}}}\left(\begin{bmatrix} q_0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} q_0 \\ \overline{Y}_\infty(q_0, t) \end{bmatrix}. \tag{14}$$

Next, we investigate the limit as $k \to \infty$ of the solution $\{x_k(t), t \in [0,T]\}$ to the differential equation (11). The previous arguments lead to uniform convergence in q for small time t of the limits:

$$\lim_{k\to\infty} Y_k(q,t) = 0 \qquad \text{and} \qquad \lim_{k\to\infty} X_k(q,t) = \overline{Y}_\infty.$$

The limit $x_\infty(t) = \lim_{k\to\infty} x_k(t)$ satisfies

$$\begin{aligned}\dot{x}_\infty &= \left(Z_g + [\overline{Y}_\infty^{\mathrm{lift}}, Z_g]\right)(x_\infty)\\ x_\infty(0) &= \begin{bmatrix} q_0 \\ 0 \end{bmatrix}.\end{aligned}$$

This differential equation can be integrated as discussed in Section 3.3, since the initial velocity is zero and the vector field belongs to $\mathcal{P}_1 + \mathcal{P}_0$. From the initial condition $[q_0^T, 0^T]^T$, we have

$$x_\infty(t) = \begin{bmatrix} q_\infty(t) \\ \dot{q}_\infty(t) \end{bmatrix} = \begin{bmatrix} \Phi_{0,t}^{\overline{Y}_\infty}(q_0) \\ 0 \end{bmatrix}. \tag{15}$$

The final result in the theorem follows from combining the two flows in equations (14) and (15).

Notice that equation (10) is well-defined, since at fixed q, the integration is performed with respect to the time variable. Using the abbreviated notation introduced in equation (7), a few terms of the iteration are computed as

$$V_1 = \overline{Y}$$
$$V_2 = -\frac{1}{2}\overline{\langle \overline{Y} : \overline{Y} \rangle}$$
$$V_3 = \frac{1}{2}\overline{\left\langle \overline{\langle \overline{Y} : \overline{Y} \rangle} : \overline{Y} \right\rangle} - \frac{1}{8}\overline{\left\langle \overline{\langle \overline{Y} : \overline{Y} \rangle} : \overline{\langle \overline{Y} : \overline{Y} \rangle} \right\rangle},$$

so that we can write

$$\dot{\gamma}(t) = \overline{Y}(\gamma, t) - \frac{1}{2}\overline{\langle \overline{Y} : \overline{Y} \rangle}(\gamma, t) + \frac{1}{2}\overline{\left\langle \overline{\langle \overline{Y} : \overline{Y} \rangle} : \overline{Y} \right\rangle}(\gamma, t) + O(t^7).$$

4.1 *Expansions under small amplitude forcing*

The series in equation (10) converges under the assumption of small final time T and bounded total acceleration $Y(q,t)$. Here we derive a series that converges under the assumption of bounded final time T and small acceleration $Y(q,t)$.

Let ϵ be a small positive constant. Motivated by the treatment in (Bullo *et al.*, 1997), consider a total acceleration dependent on ϵ and of the form:

$$Y(q, t, \epsilon) = \epsilon X(q, t).$$

Equation (10) is equivalent to

$$\dot{\gamma}(t) = \epsilon \overline{X}(\gamma, t) - \frac{\epsilon^2}{2}\overline{\langle \overline{X} : \overline{X} \rangle}(\gamma, t) + \frac{\epsilon^3}{2}\overline{\left\langle \overline{\langle \overline{X} : \overline{X} \rangle} : \overline{X} \right\rangle}(\gamma, t) + O(\epsilon^4). \quad (16)$$

This expression generalizes the results presented in Proposition 4.1 in (Bullo *et al.*, 1997), since in that context the treatment was restricted to manifold with Lie group structure and invariant vector fields. Notice that this series now converges under two different set of assumptions: either ϵ is bounded and t goes to zero (see main theorem above), or t is bounded and ϵ goes to zero (see statement in (Bullo *et al.*, 1997)).

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5. REFERENCES

Agračhev, A. A. and R. V. Gamkrelidze (1978). The exponential representation of flows and the chronological calculus. *Math. USSR Sbornik* **35**(6), 727–785.

Bullo, F. and N. E. Leonard (1997). Motion control for underactuated mechanical systems on Lie groups. In: *European Control Conference.* Brussels, Belgium.

Bullo, F., N. E. Leonard and A. D. Lewis (1997). Controllability and motion algorithms for underactuated Lagrangian systems on Lie groups. To appear, *IEEE Transactions on Automatic Control.* Also Technical Report CDS 97–013, California Institute of Technology.

Kawski, M. (1995). Geometric homogeneity and applications to stabilization. In: *IFAC World Conference.* San Francisco, CA.

Kawski, M. and H. J. Sussmann (1997). Noncommutative power series and formal Lie-algebraic techniques in nonlinear control theory. In: *Operators, Systems, and Linear Algebra* (U. Helmke, D. Pratzel-Wolters and E. Zerz, Eds.). pp. 111–128. Teubner. Stuttgart, Germany.

Kolmanovsky, I. and N. H. McClamroch (1996). Stabilizing feedback laws for internally actuated multibody systems in-space. *Nonlinear Analysis, Theory, Methods & Applications* **26**(9), 1461–1479.

Lafferriere, G. and H. J. Sussmann (1991). Motion planning for controllable systems without drift. In: *IEEE Conf. on Robotics and Automation.* Sacramento, CA. pp. 1148–1153.

Leonard, N. E. and P. S. Krishnaprasad (1995). Motion control of drift-free, left-invariant systems on Lie groups. *IEEE Transactions on Automatic Control* **40**(9), 1539–1554.

Lewis, A. D. (1997). Simple mechanical control systems with constraints. Submitted to the *IEEE Transactions on Automatic Control.*

Lewis, A. D. (1998). Affine connections and distributions with applications to nonholonomic mechanics. *Reports on Mathematical Physics* **42**(1/2), 135–164.

Lewis, A. D. and R. M. Murray (1997). Controllability of simple mechanical control systems. *SIAM Journal of Control and Optimization* **35**(3), 766–790.

Murray, R. M. and S. S. Sastry (1993). Nonholonomic motion planning: Steering using sinusoids. *IEEE Transactions on Automatic Control* **5**(38), 700–726.

Sontag, E. D. and H. J. Sussmann (1986). Time-optimal control of manipulators. In: *IEEE Conf. on Robotics and Automation.* San Francisco, CA. pp. 1692–1697.

Sussmann, H. J. (1986). A product expansion of the Chen series. In: *Theory and Applications of Nonlinear Control Systems* (C. I. Byrnes and A. Lindquist, Eds.). pp. 323–335. Elsevier. Oxford, UK.