

Exponential Stabilization of Relative Equilibria for Mechanical Systems with Symmetries

Francesco Bullo*
Control and Dynamical Systems
Caltech, mail stop 107-81
Pasadena, CA 91125
bullo@cds.caltech.edu

Abstract

This paper proposes a systematic procedure for the exponential stabilization of relative equilibria of mechanical systems. The emphasis is on 1 dimensional symmetries. A key design idea is to distinguish between horizontal forces, which preserve the momentum, and vertical forces that affect it. A proportional, derivative control in the horizontal directions and a first order regulator in the vertical direction lead to exponential stability of the closed loop provided some assumptions hold. In particular, two necessary conditions are that the relative equilibrium be Lyapunov stable and that the system satisfy a certain linear controllability test. Relevant applications to autonomous vehicle control are described.

Keywords: mechanical control systems, relative equilibria, asymptotic stabilization

1 Introduction

Control of underactuated mechanical systems is a challenging research area of increasing interest. On the theoretical side, control problems for mechanical systems benefit from the wealth of geometric mechanics tools available. On the other hand, strong motivation for these problems comes from applications to autonomous vehicles design and control. In this paper, we investigate stabilization techniques for the steady motions called relative equilibria. This family of trajectories is of great interest in theory and applications.

Stabilization of underactuated Hamiltonian systems was originally investigated in [8], see [6] for a standard treatment. Recently, geometric tools have been employed to address a the class of mechanical systems with symmetries. Stability of underwater vehicles is studied in [4] where symmetry breaking potentials were employed to shape the energy of the closed loop system. In [1], a novel and powerful approach is introduced to deal with an even larger class of systems.

In this paper, we build on the work in [4] and focus on the exponential stabilization problem (as opposed to Lyapunov or asymptotic stabilization). The control design is based in ideas from two areas: the theory of Hamiltonian reduction (and the Energy–Momentum method in particular), see [7], and the theory of passive nonlinear systems, see [8]. We divide the control synthesis into two steps: first we design a controller for the reduced system employing only the momentum preserving forces, then we regulate the value of the momentum with the remaining control authority. The main contribution is a set of coordinate independent conditions that ensures the exponential stability for the full (internal variables and momentum) system.

A key feature of our approach is that we focus on 1 dimensional (Abelian) symmetries because applications to control of vehicles require this assumption. This restriction leads to strong results and a simple exposition.

In Jalnapurkar and Marsden [3] the authors obtain stabilizing controllers for underactuated mechanical systems with non-Abelian symmetry. In their treatment the family of input forces is assumed momentum preserving and stability in the reduced space is characterized in terms of certain Poisson brackets.

Acknowledgments The author thanks Sameer M. Jalnapurkar for insightful conversations and acknowledges his suggestion to apply the results in [8] to the present setting.

2 Stabilization of Nonlinear Systems

In this section we review some basic tools in stabilization of nonlinear control systems. Let M be a smooth n dimensional manifold and consider the smooth control system

$$\dot{x} = f(x) + \sum g_i(x)u_i \quad (1)$$

*This research is supported in part by the National Science Foundation under grant CMS-9502224.

Let x_0 be an equilibrium point for f , and assume $V : M \rightarrow \mathbb{R}_+$ is smooth function such that

$$\begin{aligned} V(x) &> 0 & \forall x \neq x_0 \in B_\epsilon(x_0) \\ V(x_0) &= 0 \end{aligned} \quad (2)$$

where $B(x_0)$ is a neighborhood of x_0 . If it holds

$$\begin{aligned} 0 &= \mathcal{L}_f V(x) \\ u_i(x) &= -\mathcal{L}_{g_i} V(x), \end{aligned} \quad (3)$$

the point x_0 is Lyapunov stable. Additionally if it holds

$$\dim\{g_i, \text{ad}_f g_i, \dots, \text{ad}_f^n g_i, \forall i\}(x_0) = n, \quad (4)$$

the solution $x(t)$ will converge asymptotically to x_0 .

Finally we examine the stability properties of the linearization of the control system (1). Consider a local chart about the point x_0 . With no loss in generality we let $x \in \mathbb{R}^n$ denote a coordinate system about the point $x_0 = 0$. Additionally, f, g_i denote the corresponding quantities in the coordinate system.

Lemma 1. *Consider the nonlinear control system (1) with $x \in \mathbb{R}^n$ and let 0 be an equilibrium point. Assume the conditions in equations (2), (3) and (4). If the second variation of V at $x = 0$ is positive definite, i.e., if*

$$\frac{\partial^2 V}{\partial x \partial x}(0) = P > 0, \quad (5)$$

then the point $x = 0$ is locally exponentially stable.

We refer the reader to [2] for a proof.

3 Mechanical Control Systems

A *simple mechanical control system* is defined by the following objects:

- (i) an n -dimensional configuration manifold Q , with local coordinates $\{q^1, \dots, q^n\}$,
- (ii) a metric M_q on Q (the kinetic energy), alternatively denoted by $\langle \cdot, \cdot \rangle$,
- (iii) a function V on Q describing the potential energy, and
- (iv) an m -dimensional codistribution $\mathcal{F} = \text{span}\{F^1, \dots, F^m\}$ defining the input forces.

Let $q(t) \in Q$ be the configuration of the system and $\dot{q}(t) \in T_q Q$ its velocity. The Lagrangian for the system is

$$L(q, \dot{q}) = \frac{1}{2} M_{ij}(q) \dot{q}^i \dot{q}^j - V(q) = \frac{1}{2} \langle \dot{q}, \dot{q} \rangle - V(q),$$

where the summation convention is enforced here and in what follows. In local coordinates the forced Euler-Lagrange's equations are

$$\ddot{q}^i + \Gamma_{jk}^i \dot{q}^j \dot{q}^k = M^{ij}(q) \left(-\frac{\partial V}{\partial q^j}(q) + F_j^k u_k \right), \quad (6)$$

where $\Gamma_{jk}^i(q)$ are the Christoffel symbols of the metric M_q and where M^{ij} is the inverse tensor to M_{ij} .

A symmetry of the mechanical system is a group action that leaves kinetic and potential energy invariant. We refer to [5] for the most general definition of group action. In what follows, we present a simplified treatment of the the trivial 1-dimensional case.

Abelian Symmetries

We assume that the manifold Q can be written as $Q = R \times G$, where $(G, +)$ is either the torus or the real line. Correspondingly, the configuration is $q = (r, x) \in R \times G$, and the group action is the smooth diffeomorphism of Q defined by $(y, (r, x)) = (r, x + y)$. The kinetic and potential energy are *invariant* under the action of G if $\partial L / \partial x = 0$. The vector field $\partial / \partial x$ is sometime called an infinitesimal isometry or the infinitesimal generator of the group action.

The *momentum map* μ is a one-form, that is a map $TQ \rightarrow \mathbb{R}$ defined by

$$\mu(X_q) \triangleq \langle X_q, \frac{\partial}{\partial x} \rangle \quad X_q \in T_q Q.$$

The value of the momentum map is constant along the solutions to the unforced mechanical system (i.e., equation (6) with $u_k = 0$). In particular one can prove that:

$$\frac{d}{dt} \mu(\dot{q}) = \sum_{k=1}^m \langle F^k, \frac{\partial}{\partial x} \rangle u_k. \quad (7)$$

Given $\mu_0 \in \mathbb{R}$, we define the *locked inertia* $I(r)$ and the *amended potential* $V_{\mu_0}(r)$ as

$$\begin{aligned} I(r) &= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x} \rangle, \\ V_{\mu_0}(r) &= V(r) + \frac{1}{2} I(r)^{-1} \mu_0^2, \end{aligned}$$

where we write $V(q) = V(r)$ thanks to $\partial V / \partial x = 0$. A *relative equilibrium* is a pair (μ_0, r_0) , where $\mu_0 \in \mathbb{R}$ and $r_0 \in R$ is a critical point of V_{μ_0} , i.e., a point such that $dV_{\mu_0}(r_0)$ vanishes.

If (μ_0, r_0) is a relative equilibrium, then any curve of the form $\{(r(t), x(t)) \in Q \mid r(t) = r_0, \dot{x}(t) = \mu_0 / I(r_0), t \in \mathbb{R}\}$ is a solution to the unforced mechanical system (i.e., equation (6) with $u_k = 0$). In other words, a relative equilibrium is an ‘‘equilibrium solution’’ in the reduced space $R \times \mathbb{R}$ that correspond to a family of solution in the full phase space.

Finally, we introduce the notion of momentum preserving forces. We define the *horizontal codistribution* hor_q as the annihilator of the distribution spanned by $\partial / \partial x$, that is:

$$\text{hor}_q = \text{Ker span}\left\{ \frac{\partial}{\partial x} \right\}.$$

According to equation (7), a force F is horizontal, i.e., it takes values in hor_q , if and only if it preserves the momentum μ . Next, we let $\text{hor}_q \mathcal{F}$ denote the largest horizontal subspace (subbundle) of \mathcal{F} . The dimension of this subspace is either $m-1$ or m , depending on whether the external forces affect on the momentum or not.

4 Exponential Stabilization of Relative Equilibria

In what follows we let (μ_0, r_0) be a relative equilibrium for the mechanical system with Abelian symmetry (6).

Problem 1. *Design a feedback law $u = u(q, \dot{q})$ such that $(\mu(\dot{q}(t)), r(t))$ converge exponentially fast to (μ_0, r_0) .*

We devise our control strategy on the base of the following assumptions. First, we assume that the horizontal input codistribution is $m-1$ dimensional and integrable, i.e., we assume that there exist $m-1$ functions $\phi_i : R \rightarrow \mathbb{R}$ such that

$$\text{hor}_q \mathcal{F} = \text{span}\{\mathbf{d}\phi_1, \dots, \mathbf{d}\phi_{m-1}\}. \quad (\text{A1})$$

This also implies that we can arbitrarily pick a covector F_{vert} to complete \mathcal{F} . In other words, we have decomposed the set of inputs \mathcal{F} into horizontal forces (momentum preserving) and a complementary force. Accordingly, we re-parameterize the input force as

$$\sum_{k=1}^m F^k u_k = F_{\text{vert}} u_{\text{vert}} + \sum_{i=1}^{m-1} \mathbf{d}\phi_i u_{\text{horiz},i}.$$

Notice that this is a different requirement than asking for \mathcal{F} itself to be integrable. Condition (A1) implies that the set of allowable inputs \mathcal{F} can exert a force of the form $\mathbf{d}V_{\text{fdbk}}$, for any function $V_{\text{fdbk}} = V_{\text{fdbk}}(\phi_1, \dots, \phi_{m-1})$.

Next, we require the second variation of the amended potential $V_{\mu_0}(r)$ to be positive definite over the “uncontrolled” subspace $\text{Ker}\{\text{hor}_q \mathcal{F}\} \subset T_r R$. More precisely, we require that the matrix

$$\left(\frac{\partial^2 V_{\mu_0}}{\partial r^i \partial r^j} (r_0) \right) \text{ be positive definite restricted to } \text{Ker span}\{\mathbf{d}\phi_1, \dots, \mathbf{d}\phi_{m-1}\}(r_0) \subset T_{r_0} R. \quad (\text{A2})$$

This requirement is closely related to certain conditions given in [7] and [8, Proposition 2.3].

Finally, we require the following controllability condition. Set $u_{\text{vert}} = 0$ (i.e. only horizontal forces are allowed) and let (A, B_{horiz}) denote the linearization about the point $(r, \dot{r}, \mu) = (r_0, 0, \mu_0)$ of the mechanical system in equation (6). We say that the horizontal forces have full linear controllability rank if

$$\text{rank} \begin{bmatrix} B_{\text{horiz}} & AB_{\text{horiz}} & \cdots & A^{2(n-1)} B_{\text{horiz}} \end{bmatrix} = 2(n-1). \quad (\text{A3})$$

This condition allows us to prove asymptotic and exponential stability. A similar statement can be expressed in terms of an involutivity condition of certain Poisson brackets, see [8]. We finally state the main result.

Theorem 1 (Exponential Stabilization). *Consider the simple mechanical control system in equation (6) and let (μ_0, r_0) be a relative equilibrium. Let assumptions (A1), (A2) and (A3) hold. Then there exist $(m-1)$ positive constants $k_{p1}, \dots, k_{p(m-1)}$ such that*

$$V_{\mu_0}(r) + \sum_{i=1}^{m-1} k_{p_i} \phi_i(r)^2 > 0$$

for all $r \neq r_0$ in a neighborhood of r_0 . Let k_{vert} and $k_{d1}, \dots, k_{d(m-1)}$ be positive constants and define

$$\begin{aligned} u_{\text{vert}} &= -k_{\text{vert}}(\mu(\dot{q}) - \mu_0) \\ u_{\text{horiz},i} &= -k_{p_i} \phi_i(r) + k_{d_i} \dot{\phi}_i(r, \dot{r}) \quad \forall i = 1, \dots, m-1. \end{aligned}$$

Then $(\mu, r)(t)$ locally exponentially converges to (μ_0, r_0) .

We refer the reader to [2] for a proof.

There are two advantages of the controller described in the previous theorem over a standard linear controller based on linearization: First, by investigating the amended potential V_{μ_0} we can establish the region of attraction of our controller. Second, our design is independent from the exact knowledge of the inertia coefficients, leading to robustness to parameter uncertainty.

The assumption that the amended potential be positive definite over the “uncontrolled subspace” is quite strong. Should this condition fail, one can employ the techniques introduced in [1].

5 Applications to Vehicle Control

We consider the model of a planar body moving in an idealized fluid, see [4] for more details. Let (θ, x, y) denote the configuration of the planar body and $\xi = (\omega, v_x, v_y)$ its velocity expressed in a body-fixed reference frame. The Lagrangian is $L = \frac{1}{2} J \omega^2 + \frac{1}{2} m_x v_x^2 + \frac{1}{2} m_y v_y^2$ where $J > 0$ and $m_x > m_y > 0$. The two control inputs consist of forces $\{f_1, f_2\}$ applied at a distance h from the center of mass. The equations of motion are:

$$\begin{aligned} \dot{\theta} &= \omega & J\dot{\omega} &= (m_x - m_y)v_x v_y - h u_2 \\ \dot{x} &= \cos(\theta)v_x - \sin(\theta)v_y, & m_x \dot{v}_x &= m_y \omega v_y + u_1 \\ \dot{y} &= \sin(\theta)v_x + \cos(\theta)v_y & m_y \dot{v}_y &= -m_x \omega v_x + u_2. \end{aligned}$$

Even though the planar body has a full SE(2) symmetry, we focus on the Abelian group action $(\chi, (\theta, x, y)) \mapsto (\theta, x + \chi, y)$. According to the definitions, we compute

$$\begin{aligned} \mu &= (m_x \cos^2 \theta + m_y \sin^2 \theta) \dot{x} + (m_x - m_y) \sin \theta \cos \theta \dot{y}, \\ I(r) &= m_x \cos^2 \theta + m_y \sin^2 \theta, \end{aligned}$$

and $V_{\mu_0}(r) = \frac{1}{2} \mu_0^2 / I(r)$. The control goal is to stabilize the relative equilibrium described by $r = (\theta, y) = (0, 0)$ and $\mu = \mu_0 = m_x \dot{x}_0$. Assumption (A1) holds because

$$\begin{aligned} \mathcal{F} &= \text{span}\{\cos\theta \mathbf{d}x + \sin\theta \mathbf{d}y, \cos\theta \mathbf{d}y - \sin\theta \mathbf{d}x - h \sin\theta \mathbf{d}\theta\} \\ &= \text{span}\{\mathbf{d}y - h \cos\theta \mathbf{d}\theta\} + \text{span}\{\mathbf{d}x + h \cos\theta \mathbf{d}\theta\}, \end{aligned}$$

where one can verify that $\text{hor}_q \mathcal{F} = \text{span}\{\mathbf{d}(y - h \sin\theta)\}$ and where we have chosen $\mathbf{d}x + h \cos\theta \mathbf{d}\theta$ to complete the input codistribution. Regarding (A2), we compute the second variation of V_{μ_0} as

$$\left(\frac{\partial^2 V_{\mu_0}}{\partial \theta \partial y}(0, 0) \right) = \frac{\mu_0^2}{m_x^2} \begin{bmatrix} m_x - m_y & 0 \\ 0 & 0 \end{bmatrix},$$

which is positive definite for $m_x > m_y$ when restricted to the subspace $\text{Ker span}(\mathbf{d}y - h \cos\theta \mathbf{d}\theta) = \text{span}\{\frac{\partial}{\partial y} + h \cos\theta \frac{\partial}{\partial \theta}\}$. The controllability assumption (A3) can be easily verified. If we let $c_\theta = \cos\theta$ and $s_\theta = \sin\theta$, local exponential stability is obtained by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix} \begin{bmatrix} -k_{\text{vert}}(\mu - m_x v_0) \\ -k_p(y - h s_\theta) - k_d(\dot{y} - h \omega c_\theta) \end{bmatrix}$$

Next, we consider the classic model of a satellite with two thrusters, see for example [5]. Let $\text{SO}(3)$ denote the group of rotations in the Euclidean space \mathbb{R}^3 . The attitude and the body fixed velocity are (R, Ω) , the inertia matrix is \mathbb{J} so that the kinetic energy is $\frac{1}{2} \Omega^T \mathbb{J} \Omega$. The two inputs consist of two torques about the first and second axes. The equations of motion are:

$$\begin{aligned} \dot{R} &= R \widehat{\Omega} \\ \mathbb{J} \dot{\Omega} &= \mathbb{J} \Omega \times \Omega + \mathbf{e}_1 u_1(t) + \mathbf{e}_2 u_2(t), \end{aligned} \quad (8)$$

where $\mathbf{e}_1 = (1, 0, 0)$ and $\mathbf{e}_2 = (0, 1, 0)$. We assume $\mathbb{J} = \text{diag}\{J_1, J_2, J_3\}$ with $J_1 > J_2 > J_3$.

Even though the satellite has a full $\text{SO}(3)$ symmetry, we focus on the Abelian group action $(\chi, R) \mapsto \exp(\chi \widehat{\mathbf{e}}_1) R$ where the exponential map $\exp : \mathfrak{so}(3) \rightarrow \text{SO}(3)$ maps an angle-axis of rotation to the corresponding rotation matrix. A convenient parameterization of $\text{SO}(3)$ is as follows. We write R as

$$R(\alpha, \beta, \gamma) = \exp(\alpha \widehat{\mathbf{e}}_1) \exp(\beta \widehat{\mathbf{e}}_2) \exp(\gamma \widehat{\mathbf{e}}_3),$$

that is, we parameterize $\text{SO}(3)$ by a set of Euler angles (α, β, γ) , that is singular at $\beta = \pm\pi/2$. The unusual order of rotation is well-suited to the symmetry we consider and to the set of input vector fields. We refer to [2] for the expression of the Jacobian relating Euler angles rates and body fixed velocity and of the inertia matrix as a function of (α, β, γ) . In terms of the (α, β, γ) chart we compute

$$\begin{aligned} f^1 &= (\cos\beta \cos\gamma) \mathbf{d}\alpha + (\sin\gamma) \mathbf{d}\beta \\ f^2 &= (-\cos\beta \sin\gamma) \mathbf{d}\alpha + (\cos\gamma) \mathbf{d}\beta \\ \mu &= (J_3 \sin(\beta)^2 + \cos(\beta)^2 (J_1 \cos(\gamma)^2 + J_2 \sin(\gamma)^2)) \dot{\alpha} \\ &\quad + \frac{1}{2} (J_1 - J_2) \cos(\beta) \sin(2\gamma) \dot{\beta} + J_3 \sin(\beta) \dot{\gamma}, \\ I(r) &= J_1 \cos^2 \beta \cos^2 \gamma + J_3 \sin^2 \beta + J_2 \cos^2 \beta \sin^2 \gamma. \end{aligned}$$

The control goal is to stabilize the relative equilibrium described by $r = (\beta, \gamma) = (0, 0)$ and $\mu = J_1 \dot{\alpha}_0$. This problem is often referred to as ‘‘spin axis stabilization.’’ Assumption (A1) holds because

$$\begin{aligned} \mathcal{F} &= \text{span}\{(\cos\beta \cos\gamma) \mathbf{d}\alpha + (\sin\gamma) \mathbf{d}\beta, \\ &\quad (\cos\gamma) \mathbf{d}\beta - (\cos\beta \sin\gamma) \mathbf{d}\alpha\} \\ &= \text{span}\{\mathbf{d}\beta\} + \text{span}\{\cos\beta \mathbf{d}\alpha\}, \end{aligned}$$

where one can verify that $\text{hor}_q \mathcal{F} = \text{span}\{\mathbf{d}\beta\}$ and where we have chosen $\cos\beta \mathbf{d}\alpha$ to complete the input codistribution. Regarding (A2), we compute the second variation of V_{μ_0} as

$$\left(\frac{\partial^2 V_{\mu_0}}{\partial \beta \partial \gamma}(0, 0) \right) = \frac{\mu_0^2}{J_1^2} \begin{bmatrix} J_1 - J_3 & 0 \\ 0 & J_1 - J_2 \end{bmatrix},$$

which is positive definite for $J_1 > J_2$ when restricted to the subspace $\text{Ker span}(\mathbf{d}\beta) = \text{span}\{\frac{\partial}{\partial \gamma}\}$. The controllability assumption (A3) can be easily verified. Summarizing, local exponential stability is obtained by

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix} \begin{bmatrix} -k_{\text{vert}}(\mu - J_1 \omega_0) \\ -k_p \beta - k_d \dot{\beta} \end{bmatrix}.$$

References

- [1] A. M. Bloch, N. E. Leonard, and J. E. Marsden. Stabilization of mechanical systems using controlled Lagrangians. In *CDC*, San Diego, CA, December 1997.
- [2] F. Bullo. *Nonlinear Control of Mechanical Systems*. PhD thesis, California Institute of Technology, 1999.
- [3] S. M. Jalnapurkar and J. E. Marsden. Stabilization of relative equilibria. Technical Report CDS 98-007, California Institute of Technology, July 1998.
- [4] N. E. Leonard. Stabilization of underwater vehicle dynamics with symmetry-breaking potentials. *Systems & Control Letters*, 32:35–42, October 1997.
- [5] J. E. Marsden and T. S. Ratiu. *Introduction to Mechanics and Symmetry*. Springer Verlag, New York, 1994.
- [6] H. Nijmeijer and A. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer Verlag, New York, 1990.
- [7] J. C. Simo, D. R. Lewis, and J. E. Marsden. Stability of relative equilibria I: The reduced energy momentum method. *Archive for Rational Mechanics and Analysis*, 115:15–59, 1991.
- [8] A. J. van der Schaft. Stabilization of Hamiltonian systems. *Nonl. Analysis, Theory, Methods & App.*, 10(10):1021–1035, 1986.