TRAJECTORY TRACKING FOR FULLY ACTUATED MECHANICAL SYSTEMS

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Abstract

We present a general framework for the control of Lagrangian systems with as many inputs as degrees of freedom. Relying on the geometry of mechanical systems on manifolds, we propose a design algorithm for the tracking problem. The notions of error function and transport map lead to a proper definition of configuration and velocity error. These are the crucial ingredients in designing a proportional derivative feedback and feedforward controller. The proposed approach includes as special cases a variety of results on control of manipulators, pointing devices and autonomous vehicles.

1. Introduction

Mechanical control systems provide an important and challenging research area that falls between the study of classical mechanics and modern nonlinear control theory. From a theoretical viewpoint, the geometric structure of mechanical systems gives way to stronger control algorithms than those obtained for generic nonlinear systems. Some results in this area are surveyed for example in [12] and [10]. The driving applications are motion control problems in underwater and aerospace environments, where relevant Lagrangian models are available and a nonlinear analysis can successfully exploit this structure.

This paper deals with the trajectory tracking problem for a class of Lagrangian systems. The control objective is to track a trajectory with exponential convergence rates in order to guarantee performance and robustness. The mechanical systems we consider have Lagrangian equal to the kinetic energy and are fully actuated, that is, they have as many independent input forces as degrees of freedom. A wide variety of aerospace and underwater vehicles, as well as robot manipulators, fulfill these assumptions. The main emphasis in this paper is on the fact that the configuration space of these systems is a generic manifold Q.

The tracking problem for robot manipulators has received much attention in the literature. Examples are the contributions in [16], [13] and [17], where asymptotic and exponential tracking are achieved via a nonlinear analysis. These results are now standard in textbooks on control [12] and robotics [11]. Since then, similar techniques have been applied to the attitude control problem for satellites [18], and likewise to the attitude and position control for underwater vehicles [5, Section 4.5.4]. A common feature in all these works is the preliminary choice of a parametrization, that is a choice of coordinates for the configuration manifold. The synthesis of both control law and relative Lyapunov function is performed in this specific chart.

In this paper we propose a general framework that relies on the geometry of mechanical systems on manifolds. In the spirit of [6], our approach avoids the parametrization step and focuses directly on how to define a configuration and a velocity error on a manifold. The notions of "error function" and "transport map" yield to a coordinate-free definition of differences (errors) between configurations and between velocities. Additionally, an intrinsic definition for the feedforward control is obtained using the theory of Riemannian connections. These ideas lead to a coordinate independent system design. The resulting control law, even though expressed in a specific set of coordinates, is not biased by that choice. Global stability is ensured as long as error function and transport map satisfy a compatibility condition. The resulting design algorithm can then be applied to a variety of examples. We treat here only the case of a robot manipulator on $\mathbb{R}^n$ and of a satellite on the group of rotations $SO(3)$. We refer to the report [3] for an integral version of this work, including the instructive case of an underwater vehicle on the group of rotations and translations $SE(3)$.

The paper is organized as follows. Section 2 reviews some concepts from Riemannian geometry and from mechanical systems. In Section 3 we define error function and transport map. The main theorem is presented in Section 4 and two examples are discussed in Section 5.

2. Mathematical Preliminaries

In this section we introduce the mathematical machinery needed for the remainder of the paper. For an introduction to Riemannian geometry we refer to [2] and [4]. For an introduction to mechanics we refer to [8].

2.1. Elements of Riemannian geometry

A *Riemannian metric* on a manifold Q is a smooth map that associates to each tangent space T_qQ an inner product $\langle\langle \cdot\,,\,\cdot \rangle\rangle_q$. An *affine connection* on Q is a smooth map that assigns to each pair of smooth vector fields X, Y a smooth vector field $\nabla_X Y$ such that for all smooth functions f on Q

1. $\nabla_{fX} Y = f \nabla_X Y$, and
2. $\nabla_X fY = f \nabla_X Y + (\mathfrak{L}_X f)\, Y$

where $\mathfrak{L}_X f$ denotes the Lie derivative of f with respect to X. In system of local coordinates $(q^1, \ldots, q^n)$ we define the *Christoffel symbols* by

$$\nabla_{\frac{\partial}{\partial q^i}} \left(\frac{\partial}{\partial q^j} \right) = \Gamma^k_{ij} \frac{\partial}{\partial q^k}.$$

where the summation convention is enforced here and in what follows. Given any three vector fields X, Y, Z on Q, we say that the affine connection ∇ on Q is *torsion-free* if

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

and is *compatible* with the metric $\langle\langle \cdot\,,\,\cdot \rangle\rangle$ if

$$\mathfrak{L}_X \langle\langle Y\,,\, Z \rangle\rangle = \langle\langle \nabla_X Y\,,\, Z \rangle\rangle + \langle\langle Y\,,\, \nabla_X Z \rangle\rangle.$$

The Levi-Civita theorem states that on the Riemannian manifold Q there exists a unique affine connection ∇, which is torsion-free and compatible with the metric. We call this ∇ the Riemannian connection on Q.

We conclude with two useful definitions. Given a real valued function f on Q, the *gradient* of f is the vector field ∇f such that

$$\langle\langle \nabla f\,,\, X \rangle\rangle \triangleq \mathfrak{L}_X f.$$

Given a one form ω and a vector field X, the covariant derivative of ω with respect to X is the one form $\nabla_X \omega$ such that

$$(\nabla_X \omega) \cdot Y = \mathfrak{L}_X (\omega \cdot Y) - \omega \cdot \nabla_X Y,$$

for all vector fields Y.

2.2. Computing covariant derivatives

Loosely speaking, covariant derivatives are directional derivatives of quantities defined on manifolds. Equation (1) relates them to the notion of Lie differentiation, whereas equation (1) plays the role of the Leibniz rule. In the following we present some useful approaches on how to compute covariant derivatives.

A first instructive case is when the manifold Q is a submanifold of $\mathbb{R}^n$ and the Riemannian metric on Q is the one induced by the Euclidean metric on $\mathbb{R}^n$. Then we can denote with π_q the orthogonal projection from $\mathbb{R}^n$ onto the tangent bundle T_qQ. Given any two vector fields X, Y on Q, it holds that

$$(\nabla_X Y)(q_0) = \pi_{q_0} \left(\frac{d}{dt}\Big|_{t=0} Y(q(t)) \right),$$

where $\{q(t), t \in \mathbb{R}\}$ is any curve on Q with $q(0) = q_0$ and $\dot{q}(0) = X(q_0)$. We refer to [2, Chapter VII] for more details on this description of covariant differentiation.

In the general case, e.g. whenever the previous assumptions are not satisfied, we can express covariant derivatives in a system of local coordinates. The Christoffel symbols Γ^k_{ij} of a Riemannian connection can be computed as follows. Denoting with $M_{ij} = \langle\langle \frac{\partial}{\partial q^i}\,,\, \frac{\partial}{\partial q^j} \rangle\rangle$, we have

$$\Gamma^k_{ij} = \tfrac{1}{2} M^{mk} \left(\frac{\partial M_{mj}}{\partial q^i} + \frac{\partial M_{mi}}{\partial q^j} - \frac{\partial M_{ij}}{\partial q^m} \right),$$

where $\{M^{ij}\}$ is the inverse of the tensor M. The covariant derivative of a vector field is then written as

$$\nabla_X Y = \left(\frac{\partial Y^i}{\partial q^j} X^j + \Gamma^i_{jk} X^j Y^k \right) \frac{\partial}{\partial q^i}.$$

2.3. Mechanical systems in a Riemannian context

Here we describe a mechanical system and its equations of motion in a coordinate free fashion. Key ideas are regarding the system's kinetic energy as a Riemannian metric and writing the Euler-Lagrange's equations in terms of the associated Riemannian connection. For a more complete treatment, see for example [7].

We start with some definitions. A *simple mechanical control system* is defined by a Riemannian metric on a configuration manifold Q (defining the kinetic energy), a function V on Q (defining the potential energy), and m one-forms, $F^1, \ldots, F^m$, on Q (defining the inputs).

A simple mechanical system is said to be *fully actuated* if for all $q \in Q$, the family of vectors $\{F^1(q), \ldots, F^m(q)\}$ spans the whole vector space T^*_qQ, that is if there exists an independent input one form corresponding to each degree of freedom.

Let $M_q : T_qQ \to T^*_qQ$ denote the metric tensor associated to the kinetic energy and ∇ the corresponding Riemannian connection. Let $q(t) \in Q$ be the configuration of the system and $\dot{q}(t) \in T_qQ$ its velocity. Using the formalism introduced in the previous section, the forced Euler-Lagrange equations can be written as

$$\nabla_{\dot{q}} \dot{q} \;=\; M_q^{-1} \big(\mathrm{d}V(q) + F(t, q, \dot{q}) \big), \tag{1}$$

where $\mathrm{d}V(q)$ is the differential of the potential function V and where the resultant force $F(q,t) = \sum F^a(q) u_a(t)$ is the input. In a system of local coordinates $(q^1, \ldots, q^n)$ the previous equations read

$$\ddot{q}^i + \Gamma^i_{jk} \dot{q}^j \dot{q}^k = M^{ij} \left(\frac{\partial V}{\partial q^j} + F_j \right).$$

Note that the Euler-Lagrange's equations are coordinate independent (intrinsic), in the sense that they are satisfied in every system of local coordinates.

We shall say that the mechanical system (1) has bounded inertia if there exist $m_1 \geq m_2 > 0$ such that for all $q \in Q$ it holds that

$$m_1 \geq \sup_{q\in Q} \|M_q^* M_q\|_{M_q} \geq \inf_{q\in Q} \|M_q^* M_q\|_{M_q} \geq m_2, \tag{B1}$$

where $\|\cdot\|_{M_q}$ is the operator norm on the inner product space $(T_qQ,\, M_q)$. Here and in the following, the tag (Bn) denotes some boundedness assumptions which will play a crucial role in later sections.

3. Geometric Description of Configuration and Velocity Error

In this section we study the geometric objects involved in the controller design. To measure the distance between reference and actual configuration, we introduce the notion of error function. To measure the distance between reference and actual velocity, we introduce the notion of transport map. A design on two sphere manifold provides an example of our definitions. Finally we study the time derivative of the transport map. Together with a dissipation function, these ingredients are crucial in designing a tracking control.

3.1. Error function and configuration error

Let φ be a smooth real valued function on $Q \times Q$. We shall call φ an *error function* if it is *positive definite*, that is $\varphi(q,r) \geq 0$ for all q and r, and $\varphi(q,r) = 0$ if and only if $q = r$. We shall say that the error function φ is *symmetric*, if $\varphi(q,r) = \varphi(r,q)$ for all q and r.

Let $\mathrm{d}_1\varphi$ and $\mathrm{d}_2\varphi$ denote the gradient of $\varphi(q,r)$ with respect to its first and second argument. We shall say that the error function φ is *(uniformly) quadratic with constant* L if for all $\epsilon > 0$ there exist two constants $b_1 \geq b_2 > 0$ such that $\varphi(q,r) < L - \epsilon$ implies

$$b_1 \|\,\mathrm{d}_1\varphi(q,r)\|^2_{M_q} \;\geq\; \varphi(q,r) \;\geq\; b_2 \|\,\mathrm{d}_1\varphi(q,r)\|^2_{M_q}. \tag{A1}$$

Here and in the following, the tag (An) denotes some design assumptions which will play a crucial role in later sections.

When q and r are actual and reference configuration, we will sometimes call the quantity $\varphi(q,r)$ configuration error. As mentioned above, the error function φ will be instrumental in designing the proportional action.

3.2. Transport map and velocity error

Given two points $q, r \in Q$, we shall call a linear map $\mathcal{T}_{(q,r)} : T_rQ \to T_qQ$ a *transport map* if it is *compatible with the error function*, that is if

$$\mathrm{d}_2\varphi(q,r) = -\mathcal{T}^*_{(q,r)}\, \mathrm{d}_1\varphi(q,r), \tag{A2}$$

where $\mathcal{T}^*_{(q,r)} : T_q^*Q \to T_r^*Q$ is the dual map of $\mathcal{T}_{(q,r)}$. The transport map $\mathcal{T}$ is also required to be *smooth*, i.e., for all point r in Q and tangent vectors Y_r in T_rQ, the vector field $\mathcal{T}_{(q,r)}Y_r$ is smooth.

Using a transport map, velocities belonging to different tangent bundles can be compared. In the following, we shall call *velocity error* the quantity

$$\dot e \;\triangleq\; \dot q - \mathcal{T}_{(q,r)}\dot r \quad \in T_qQ.$$

Note the slight abuse of terminology, given that the velocity error is not the time derivative of a position error. Also note that since the definition of $\mathcal{T}$ and $\dot e$ are equivalent, we will sometimes talk about compatibility between configuration and velocity errors. The next lemma provides some insight into the meaning of the velocity error and of condition (A2).

Lemma 1. (Time derivative of an error function) *Let* $\{q(t), t \in \mathbb{R}_+\}$ *and* $\{r(t), t \in \mathbb{R}_+\}$ *be two smooth curves in* Q. *Let* φ *be an error function and* $\mathcal{T}$ *a compatible transport map. Then*

$$\frac{d}{dt}\varphi\big(q(t), r(t)\big) = \mathrm{d}_1\varphi\big(q(t), r(t)\big)\cdot \dot e(t), \qquad \forall t \in \mathbb{R}_+.$$

•

The result can be restated as follows. As both q and r are functions of time, the time derivative of $\varphi : Q \times Q \to \mathbb{R}$ reduces to a derivative only with respect to the first argument

$$\mathcal{L}_{(\dot q,\, \dot r)}\varphi \;=\; \mathcal{L}_{(\dot e,\, 0)}\varphi,$$

where $(X,\, Y)$ denotes a vector field on the product manifold $Q \times Q$.

Last, we introduce the notion of dissipation function, which will be useful in defining a derivative action. We define a *(linear Rayleigh) dissipation function* as a smooth, self-adjoint, positive definite tensor field $(K_d)(q) : T_qQ \to T_q^*Q$. We shall say that K_d is bounded if there exist $d_2 \geq d_1 > 0$ such that

$$d_2 \geq \sup_{q\in Q} \|K_d\|_{M_q} \geq \inf_{q\in Q} \|K_d\|_{M_q} \geq d_1, \tag{B2}$$

where $\|\cdot\|_M$ is the operator norm for $(1,1)$ type tensors on T_qQ induced by the metric M_q on T_qQ.

3.3. Derivatives of the transport map and boundedness assumptions

So far we have introduced configuration and velocity errors that will be key ingredients in designing a proportional and derivative feedback in the next section. We now study how the transported reference velocity $(\mathcal{T}_{(q,r)}\dot r)$ varies as a function of both $q(t)$ and $(r, \dot r)(t)$. This will be useful in designing the feedforward action. We denote the total derivative of $(\mathcal{T}_{(q,r)}\dot r)$ with

$$\frac{D\,(\mathcal{T}\dot r)}{dt} = \nabla_{\dot q}\,(\mathcal{T}\dot r) + \frac{d}{dt}\Big|_{q \text{ fixed}} (\mathcal{T}\dot r)\,,$$

where the two terms are described as follows:

1. At $(r, \dot r)$ fixed, $\mathcal{T}_{(q,r)}\dot r$ is a vector field on Q and therefore its covariant derivative $\nabla_{\dot q}(\mathcal{T}\dot r)$ is well-defined on Q. We call *covariant derivative of the transport map* the map $\nabla\mathcal{T} : T_qQ \times T_rQ \to T_qQ$ defined as

$$(\nabla_X\mathcal{T})Y_r \;\triangleq\; \nabla_X(\mathcal{T}Y_r),$$

for all tangent vectors $X \in T_qQ$ and $Y_r \in T_rQ$.

2. At q fixed, $\mathcal{T}_{(q,r)}\dot r$ is a vector on the vector space T_qQ and therefore its time derivative is well-defined. We denote it with the symbol:

$$\frac{d}{dt}\Big|_{q \text{ fixed}}(\mathcal{T}\dot r) \quad \in T_qQ.$$

We conclude the section with some boundedness assumptions. We shall say that the transport map $\mathcal{T}$ has bounded covariant derivative and that the error function φ has bounded second covariant derivative if

$$\sup_{(q,r)\in Q\times Q} \|\nabla\mathcal{T}_{(q,r)}\|_M < \infty, \tag{B3}$$

and

$$\sup_{(q,r)\in Q\times Q} \|\nabla\, \mathrm{d}_1\varphi\,(q,r)\|_M < \infty, \tag{B4}$$

where $\|\cdot\|_M$ is the induced operator norm on the inner product space $(T_qQ,\, M_q)$. We shall say that the twice differentiable curve $\{r(t), t \in \mathbb{R}_+\} \subset Q$ is a reference trajectory with bounded time derivative if

$$\sup_{t\in\mathbb{R}} \|\dot r\|_{M_r} < \infty. \tag{B5}$$

Notice that a sufficient condition for the bounds (B3) and (B4) to hold, is that the quantities $\partial\mathcal{T}^i_\alpha/\partial q^k$, $\partial^2\varphi/(\partial q^i\partial q^j)$ and $\Gamma^k_{ij}(q)$ are bounded.

4. Tracking on Manifolds

In this section we state and solve the exponential tracking problem. Since we are dealing with second order systems on a manifold, special care is needed in defining Lyapunov and exponential stability. We express the latter notions in terms of a total energy function, defined as the sum of a generalized potential (the configuration error) and kinetic energy (the norm of the velocity error):

$$W_{\text{total}}(q,\dot q;\; r,\dot r) \;\triangleq\; \varphi(q,r) + \tfrac{1}{2}\|\dot e\|^2_{M_q}.$$

The control goal is to drive the total energy to zero, since at $W_{\text{total}} = 0$ the state $q(t)$ tracks the trajectory $r(t)$ exactly. This is stated as follows:

Problem 2. (Control objective) Given a reference trajectory $\{r(t), t \in \mathbb{R}_+\}$, design a control law $F = F(q,\dot q;\; r,\dot r)$ such that the total energy function W_{total} is exponentially convergent to zero.

Recall that by exponential convergence for W_{total} we mean the existence of two positive constants k and λ such that

$$W_{\text{total}}(t) \;\le\; k\, W_{\text{total}}(0)\, \mathrm{e}^{-\lambda t},$$

where we write $W_{\text{total}}(t)$ for $W_{\text{total}}(q(t), \dot q(t);\; r(t), \dot r(t))$. We are now ready to state the main result.

Theorem 3. (Exponential tracking) *Consider the mechanical system*

$$\nabla_{\dot q}\dot q \;=\; M_q^{-1}F, \qquad\qquad q \in Q$$

and let the twice differentiable curve $\{r(t), t \in \mathbb{R}_+\}$ *be a reference trajectory. Let* φ *be an error function,* $\mathcal{T}$ *be a transport map satisfying the compatibility condition* (A2) *and* K_d *be a dissipation function.*

If the control input is defined as $F = F_{\text{PD}} + F_{\text{FF}}$ *with*

$$F_{\text{PD}}(q,\dot q;\; r,\dot r) \;=\; -\,\mathrm{d}_1\varphi(q,r) - K_d\,\dot e$$

$$F_{\text{FF}}(q,\dot q;\; r,\dot r) \;=\; M_q\left((\nabla_{\dot q}\mathcal{T}_{(q,r)})\dot r \;+\; \frac{d}{dt}\Big|_{q \text{ fixed}}(\mathcal{T}_{(q,r)}\dot r)\right),$$

then the curve $q(t) = r(t)$ *is Lyapunov stable, in the sense that* $W_{\text{total}}(t) \le W_{\text{total}}(0)$ *from all initial conditions* $\big(q(0), \dot q(0)\big)$.

In addition, if the error function φ *satisfies the quadratic assumption* (A1) *with a constant* L*, and if the boundedness assumptions* (B1–B5) *hold, then* $W_{\text{total}}(t)$ *converges exponentially to zero, from all initial conditions* $(q(0), \dot q(0))$ *such that*

$$\varphi\big(q(0), r(0)\big) + \tfrac{1}{2}\|\dot e(0)\|^2_M < L. \qquad \bullet$$

While similar results are quite common and well-established in the robotics literature, note that the contribution of the previous theorem lies in a coordinate free design performed on a generic manifold. We refer to the report [3] for the proof, but we include a few remarks in the following.

The design process and the theorem's results are global in the reference position $r(t)$ but only local in the configuration q (the error function $\varphi(q,r)$ must remain smaller than the parameter L). This cannot be avoided because of (possible) topological properties of the manifold Q. For additional details we refer to [6], where the global aspects of the point stabilization problem are discussed.

By constraining the choices of admissible couples $(\varphi, \mathcal{T})$, the compatibility condition (A2) affects important design aspects of F_{PD} and F_{FF}. For example, one particular transport map might generate a "simple" velocity error and a "simple" feedforward control, but it might also require a "complicated" error function. The next section contains some examples of this tradeoff.

As expected, the final control law is sum of a feedback and a feedforward term. This is in agreement with the ideas exposed in [10] on "two degree of freedom system design" for mechanical systems. While the feedforward term depends on the geometry of both the manifold and the mechanical system, the feedback term is designed knowing only the configuration manifold Q. We expect the ideas of configuration and velocity error to be relevant for more general second order nonlinear systems on manifolds.

5. Applications and Extensions

We present only two applications of Theorem 3 and we refer to the report [3] for additional examples and details.

5.1. A robot manipulator on $\mathbb{R}^n$

In this section, we shall recover the standard results on tracking control of manipulators contained in [11]. Let $q \in \mathbb{R}^n$ be the joint variables and $M(q)$ be the inertia matrix of the manipulator. The design described in Section 3 is performed as follows.

Let $\varphi(q,r) = \frac{1}{2}(q-r)^T K_p (q-r)$ be the quadratic error function and as $T_q\mathbb{R}^n = T_r\mathbb{R}^n$, let the transport map be equal to the identity matrix: $\mathcal{T}_{(q,r)} = I_n$. Assumptions (A1) and (A2) are easily verified. To design the feedforward action, we compute the covariant derivative of $\mathcal{T}$. Let $\left\{\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\right\}$ be the standard basis in $\mathbb{R}^n$, let $\{i,j,k,\ldots\}$ be indices over q and $\{\alpha,\beta,\ldots\}$ be indices over r. Then one can show that

$$(\nabla I_n)^i_{\alpha j} = \frac{\partial (I_n)^i_\alpha}{\partial q^j} + \Gamma^i_{jk}(I_n)^k_\alpha = \Gamma^i_{j\alpha},$$

Therefore, in contrast to a naive guess, the covariant derivative of the identity map is different from zero. The control law is

$$\begin{aligned} F_{\mathrm{PD}} &= -K_p(q-r) - K_d(\dot q - \dot r) \\ F_{\mathrm{FF}} &= M(q)\Big((\nabla_{\dot q} I_n)\dot r + \frac{d}{dt}\Big|_{q \text{ fixed}} \dot r \Big) \\ &= M(q)\big(\Gamma^i_{j\alpha}\dot q^j \dot r^\alpha \tfrac{\partial}{\partial q^i} + \ddot r\big) \;\equiv\; M(q)\ddot r + C(q,\dot q)\dot r, \end{aligned}$$

where $C(\cdot,\cdot)$ is the *Coriolis matrix* typically encountered in robotics. The control law $F = F_{\mathrm{PD}} + F_{\mathrm{FF}}$ agrees with the one presented in [11, Chapter 4, Section 5.3] under the name of "augmented PD control". The assumptions (B1–B5) can be written in terms of M_q, Γ^k_{ij} and $\dot r$ being bounded over $t \in \mathbb{R}$ and $q \in \mathbb{R}^n$.

5.2. A satellite on the rotation group $SO(3)$

In the next two sections we design tracking controllers for mechanical systems defined on the group of rotations $SO(3)$ and on the group of rigid motions $SE(3)$. We focus on rigid bodies with body-fixed forces and invariant kinetic energy, as satellites and underwater vehicles. Nevertheless our treatment is relevant also for workspace control of robot manipulators. This section presents the attitude control problem for a satellite.

The configuration of the satellite (rigid body) is the rotation matrix R representing the position of a frame fixed with the rigid body with respect to an inertially fixed frame. A rotation matrix on $\mathbb{R}^3$ is an element on the special orthogonal group $SO(3) = \{R \in \mathbb{R}^{3\times3} | \, RR^T = I_3, \ \det(R) = +1\}$. The kinematic equation describing the evolution of $R(t)$ is

$$\dot R = R\widehat{\Omega}$$

where $\Omega \in \mathbb{R}^3$ is the body angular velocity expressed in the body frame. Recall that the matrix $\widehat{\Omega}$ is defined such that $\widehat{\Omega}x = \Omega \times x$ for all $x \in \mathbb{R}^3$ and it belongs to the space of skew symmetric matrices $\mathfrak{so}(3) = \{S \in \mathbb{R}^{3\times3} | \, S^T = -S\}$. We refer to [11] for additional details.

The kinetic energy of the rigid body is $\frac{1}{2}\Omega^T \mathbb{J}\Omega$, where the inertia matrix $\mathbb{J}$ is symmetric and positive definite. The Euler equations describing the time evolution of Ω are

$$\mathbb{J}\dot\Omega = \mathbb{J}\Omega \times \Omega + f, \tag{2}$$

where $f \in \mathbb{R}^{3*}$ is the resultant torque acting on the body.

Error functions Let $\{R_d(t), t \in \mathbb{R}_+\}$ denote the reference attitude trajectory corresponding to a desired or reference frame and let $\Omega_d = R_d^T \dot R_d$ denote the reference velocity in the reference frame. Using the group operation, we define right and left attitude errors as

$$R_{e,\mathrm{r}} \triangleq R_d^T R \qquad \text{and} \qquad R_{e,\mathrm{l}} \triangleq R R_d^T. \tag{3}$$

The matrix $R_{e,\mathrm{r}}$ is the relative rotation from the body frame to the reference frame. Two error functions are then defined as $\varphi_\mathrm{r}(R,R_d) \triangleq \phi(R_{e,\mathrm{r}})$ and $\varphi_\mathrm{l}(R,R_d) \triangleq \phi(R_{e,\mathrm{l}})$, where $\phi : \ SO(3) \to \mathbb{R}_+$ is defined as [6]

$$\phi(R_e) \triangleq \tfrac{1}{2}\operatorname{tr}\big(K_p(I_3 - R_e)\big).$$

If the eigenvalues $\{k_1,k_2,k_3\}$ of the symmetric matrix K_p satisfy $k_i + k_j > 0$ for $i \neq j$, then both error functions φ_l and φ_R are symmetric, positive definite and quadratic with constant $L = \min_{i\neq j}(k_i + k_j)$. Locally near the identity the function ϕ assigns a weight $k_2 + k_3$ to a rotation error about the first axis (and similarly for the other axes).

Velocity errors To define compatible velocity errors, we compute the time derivative of the two error functions. Let the matrix skew(A) denote $\frac{1}{2}(A - A^T)$ and let $\cdot^\vee$ denote the inverse operator to $\widehat{\cdot}: \mathbb{R}^3 \to \mathfrak{so}(3)$. We have

$$\frac{d}{dt}\varphi_\mathrm{r} = \left(\operatorname{skew}(K_p R_{e,\mathrm{r}})^\vee\right)^T \Omega_{e,\mathrm{r}} \tag{4}$$

$$\frac{d}{dt}\varphi_\mathrm{l} = \left(\operatorname{skew}(K_p R_{e,\mathrm{l}})^\vee\right)^T R_d\, \Omega_{e,\mathrm{l}}, \tag{5}$$

where we define right and left velocity errors in the body frame as

$$\Omega_{e,\mathrm{r}} \triangleq \Omega - R_{e,\mathrm{r}}^T \Omega_d \qquad \text{and} \qquad \Omega_{e,\mathrm{l}} \triangleq \Omega - \Omega_d.$$

Note the slightly improper wording, since a velocity error $\dot e = \dot R - \mathcal{T}\dot R_d$ lives on the tangent bundle $T_R SO(3)$. A precise statement is

$$\begin{aligned} \dot e_\mathrm{l} &= R\Omega_{e,\mathrm{l}} \equiv \dot R - \left(RR_d^T\right)\dot R_d \\ \dot e_\mathrm{r} &= R\Omega_{e,\mathrm{r}} \equiv \dot R - \dot R_d \left(R_d^T R\right). \end{aligned}$$

These equalities also motivate the names "left" and "right". A left (right) velocity error is obtained by left (right) translation of the velocity $\dot{R}_d$.

Next we describe compatible couples of configuration and velocity errors. Equation (4) suggests that a right attitude error $R_d^T R$ and a right velocity error $\Omega - R^T R_d \Omega_d$ are compatible. This couple is the most common choice in the literature, see for example [9], [6], and [18].

Left attitude and velocity error appear less frequently. With this choice both the velocity error and, as we show below, the feedforward control have a simple expression. Remarkably, when the gain K_p is a scalar multiple of the identity $k_p I_3$, the left and right error functions are equal and it is possible to use $\varphi_{e,\mathrm{r}}$ with $\Omega_{e,\mathrm{l}}$.

Finally we summarize the design process.

Lemma 4. *Consider the system in equation* (2). *Let* $\{R_d(t), t \in \mathbb{R}_+\}$ *denote the reference trajectory and let* $\Omega_d = R_d^T \dot{R}_d$ *denote its body-fixed velocity. Corresponding to the two choices of attitude error, we define*

$$f_\mathrm{r} = -\operatorname{skew}(K_p R_{e,\mathrm{r}})^\vee - K_d \Omega_{e,\mathrm{r}} + \Omega \times \mathbb{J}(R_{e,\mathrm{r}}^T \Omega_d) + \mathbb{J}(R_{e,\mathrm{r}}^T \dot{\Omega}_d)$$
$$f_\mathrm{l} = -R_d^T \operatorname{skew}(K_p R_{e,\mathrm{l}})^\vee - K_d \Omega_{e,\mathrm{l}} + \Omega_d \times \mathbb{J}\Omega + \mathbb{J}\dot{\Omega}_d$$

where K_d *is a positive definite matrix and* K_p *is a symmetric matrix with eigenvalues* $\{k_1, k_2, k_3\}$ *such that* $k_i + k_j > 0$ *for* $i \neq j$.

Then, for both choices of attitude error, the total energy $\phi(R_e) + \frac{1}{2}\|\Omega_e\|_{\mathbb{J}}^2$ *converges exponentially to zero from all initial conditions* $(R(0), \Omega(0))$ *such that*

$$\phi(R_e(0)) + \tfrac{1}{2}\|\Omega_e(0)\|_{\mathbb{J}}^2 < \min_{i \neq j}(k_i + k_j). \quad \bullet$$

6. Summary and Conclusions

This work unveils the geometry and the mechanics of the tracking problem for fully actuated Lagrangian systems. The design process in Section 3 allows us to characterize in an intrinsic way a tracking controller. The basic answered questions concern how to define configuration and velocity errors and how to compute the feedforward control. Almost global stability and local exponential convergence are proven in full generality. Our framework successfully unifies a variety of examples. We have here presented only two of them and we refer to the report [3] for an integral version of this work, including proofs and additional detailed examples.

This work provides coordinate free design techniques for nonlinear mechanical systems. Other recent papers on modeling [1], controllability [7] and dynamic feedback linearization [14] share the same geometric tools. A parallel avenue of research relies on the Hamiltonian formulation of mechanical systems, see for example [12] and [15]. These geometric techniques are a promising starting point in the design of control policies for underactuated systems.

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