MOTION CONTROL FOR UNDERACTUATED MECHANICAL SYSTEMS ON LIE GROUPS

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Abstract

Control design for underactuated mechanical systems is an active area of research. In this paper we focus on mechanical control systems defined on Lie groups with the Lagrangian equal to kinetic energy. Examples include satellites and underwater vehicles. Under a controllability assumption, we propose two algorithms to compute small amplitude, periodic inputs that achieve arbitrary reconfigurations.

1. Introduction

In this paper we design control laws to change as desired the position and orientation (i.e., the configuration) of an underactuated Lagrangian system. We focus on mechanical systems on Lie groups, with the Lagrangian equal to the kinetic energy and with fewer input forces than degrees of freedom. Mechanical control systems are second-order systems with drift. Accordingly, we address motion control of a class of underactuated systems with drift. This is in contrast to the bulk of previous work on motion control for underactuated drift-free systems.

The motivation for studying mechanical control systems comes from both practice and theory. From the practical point of view, we are motivated by the interest in a variety of autonomous mechanical systems such as underwater and aerospace vehicles. These systems are sometimes underactuated by design or because of a component failure, yet despite there being fewer actuators than degrees of freedom, important tasks like station keeping and short range reconfigurations can still be achieved.

From the theoretical perspective, we are motivated by recent results that exploit geometric structure of mechanical systems and systems on Lie groups to advantage in control. In particular, we refer to the work on configuration controllability in [8] and on motion control for kinematic models in [7] and for dynamic models with dissipation in [6]. The difficulty of the motion planning problem for these systems is indicated for example in [11], where it is shown that flatness is not a generic properties of underactuated mechanical systems. Other relevant results on oscillatory controls and mechanical systems include [1], [12], and [14]. Asymptotically stabilizing control laws for an underactuated satellite are designed in [10] and [15].

In contrast to these previous works, this paper deals with the class of simple mechanical systems on Lie groups. Examples include a satellite on the group of rotations and an underwater vehicle on the group of rotations and translations. After reviewing some nonlinear controllability results, we apply the perturbation method to study the effect of small–size periodic inputs. Our main contributions are two feedback algorithms which allow for arbitrary change in position and orientation. Both theoretical controllability questions as well as constructive problems are answered within a comprehensive theoretical framework.

The paper is organized as follows. In Section 2 we review models and controllability conditions. In Section 3 formulas for approximate solutions are obtained, which are then used to design motion control algorithms in Section 4. Finally, our conclusions are given in Section 5.

2. Models and controllability results

We assume the reader to be familiar with some Lie groups and nonlinear controllability theory. We refer to [7] for a description of the kinematics of systems on Lie groups and to [9] for their dynamics.

2.1. Mechanical systems on Lie groups

A simple mechanical control system on a Lie group is described by the following objects: an n dimensional Lie group G with its algebra $\mathfrak{g}$, an inertia tensor $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$ and a set of body-fixed forces $\{f^1, \dots, f^m\} \subset \mathfrak{g}^*$ (input one-forms). Because the system is underactuated, it holds $m < n$. We denote with Greek letters ξ, η elements in $\mathfrak{g}$ and with $\operatorname{ad}_\xi \eta = [\xi, \eta]$ the Lie bracket on $\mathfrak{g}$.

Let $g \in G$ denote the configuration of the system and

$\xi \in \mathfrak{g}$ the body-fixed velocity. The kinematic and dynamic equations of motions are

$$\dot{g} = g \cdot \xi \tag{1}$$

$$\mathbb{I}\dot{\xi} = \mathrm{ad}^*_\xi \mathbb{I}\xi + \sum_{i=1}^m f^i u_i(t), \tag{2}$$

where ad^*_ξ is the dual of ad_ξ and $\sum f^i u_i(t)$ is the resultant force acting on the system. The dynamic equation (2) is often called the Euler-Poincaré equation.

Example 1. (Satellite in $SO(3)$) Let $R \in SO(3)$ be the attitude matrix and $\Omega \in \mathfrak{so}(3) \approx \mathbb{R}^3$ be the body angular velocity. We write the Euler equations for the rotation of a satellite as

$$\dot{R} = R\Omega$$

$$\mathbb{J}\dot{\Omega} = \mathbb{J}\Omega \times \Omega + \sum_{i=1}^m \tau^i u_i(t), \tag{3}$$

where $\mathbb{J}$ is the inertia matrix, the covectors τ^i describe the external torques and $\times$ is the crossproduct on $\mathbb{R}^3$. •

Example 2. (Underwater vehicle in $SE(3)$) The motion of a rigid body in incompressible, irrotational and inviscid fluid is Hamiltonian with an inertia tensor which includes added masses and inertias, see [5] or the original work of Kirchhoff. Let $(R, b) \in SE(3)$ and $(\Omega, V) \in \mathfrak{se}(3)$. The kinematic equations are $\dot{R} = R\Omega$ and $\dot{b} = RV$. For a neutrally buoyant ellipsoidal body with uniformly distributed mass, the dynamic equations are

$$\begin{aligned} \mathbb{J}\dot{\Omega} &= \mathbb{J}\Omega \times \Omega + \mathbb{M}V \times V && + \textstyle\sum_{i=1}^m \tau^i u_i(t) \\ \mathbb{M}\dot{V} &= \mathbb{M}V \times \Omega && + \textstyle\sum_{i=1}^m f^i u_i(t), \end{aligned}$$

where the covectors $[\tau^i \ f^i]$ describe the body-fixed inputs. The mass and inertia matrices of the body-fluid system are $\mathbb{M} = \mathrm{diag}\{m_1, m_2, m_3\}$ and $\mathbb{J} = \mathrm{diag}\{I_1, I_2, I_3\}$. •

Finally, we define the *symmetric product* of two vectors ξ, η on the Lie algebra $\mathfrak{g}$ as

$$\langle \xi : \eta \rangle \triangleq -\mathbb{I}^{-1}\big(\mathrm{ad}^*_\xi \mathbb{I}\eta + \mathrm{ad}^*_\eta \mathbb{I}\xi\big).$$

For example on $\mathfrak{so}(3) \approx \mathbb{R}^3$ with the inertia tensor $\mathbb{J}$, we have $\langle \xi : \eta \rangle = -\mathbb{J}^{-1}\big(\mathbb{J}\eta \times \xi + \mathbb{J}\xi \times \eta\big)$. Since it will be useful later, we rewrite the dynamic equation (2) as

$$\dot{\xi} = -\frac{1}{2}\langle \xi : \xi \rangle + \sum_{i=1}^m b_i u_i(t),$$

where we define $b_i \triangleq \mathbb{I}^{-1} f^i$ for simplicity.

2.2. Nonlinear Controllability

We briefly describe here some nonlinear controllability properties of mechanical systems on Lie groups. We refer to [13] for the notion of local controllability, to [8] for a treatment focused on mechanical systems and to [3] for the Lie group case. We start with some definitions.

Let $\mathcal{B} = \mathrm{span}\{b_1, \ldots, b_m\} \subset \mathfrak{g}$ be the input subspace. Equivalently, one can think of $\mathcal{B}$ as a left-invariant distribution on G. We define $\overline{\mathrm{Lie}}(\mathcal{B})$ and $\overline{\mathrm{Sym}}(\mathcal{B})$ as the closure of $\mathcal{B}$ under, respectively, Lie brackets and symmetric products. The *order* of an iterated product of factors from $\overline{\mathrm{Sym}}(\mathcal{B})$ is the total number of factors. We shall say that a symmetric product from $\overline{\mathrm{Sym}}(\mathcal{B})$ is *bad* if it contains an even number of each of the vectors in $\mathcal{B}$. Otherwise, it is *good*. Some examples of good/bad products can be found in Example 5.

A configuration g_1 is said to be *reachable starting from g_0 at zero velocity*, if there exist a time $T > 0$ and an input $\{u(t), 0 \leq t \leq T\}$, such that the solution $\big(g, \xi\big)(t)$ to the system (1) and (2) with initial conditions $\big(g_0, 0\big)$ satisfies $g(T) = g_1$. The point (g_1, ξ_1) is said to be *reachable starting from g_0 at zero velocity*, if, under the same assumptions as before, $g(T) = g_1$ and $\xi(T) = \xi_1$.

Next we present two notions of controllability and some algebraic tests for them. Note that these two definitions are weakenings of "full state" accessibility/controllability. The key idea is to simplify the controllability question by focusing on problems where the system is initially at rest.

Definition 3. *The system* (1) *and* (2) *is* locally accessible from zero velocity at g_0 *if the set of reachable points (g, ξ) starting from g_0 at zero velocity contains an open non-empty set of $G \times \mathfrak{g}$. The system is* locally controllable from zero velocity at g_0 *if $(g_0, 0)$ belongs to the interior of this set. If this latter property holds for each $g_0 \in G$, then the system is called* locally controllable from zero velocity.

The system (1) *and* (2) *is* locally configuration accessible at g_0 *if the set of reachable configurations g starting from g_0 at zero velocity contains an open non-empty set of G. The system is* locally configuration controllable at g_0 *if g_0 belongs to the interior of this set. If this latter property holds for each $g_0 \in G$, then the system is called* locally configuration controllable. •

Proposition 4. *Consider the system* (1) *and* (2) *on G and let $\mathcal{B} = \mathrm{span}\{b_1, \ldots, b_m\}$ be the input subspace.*

1. *The system is locally accessible from zero velocity if and only if* rank $\overline{\mathrm{Sym}}(\mathcal{B}) = n$. *It is locally controllable from zero velocity if in addition any bad product is a linear combination of lower order good products.*

2. *The system is locally configuration accessible if and only if* rank $\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathcal{B})) = n$. *It is locally configuration controllable if in addition any bad product is a linear combination of lower order good products.* •

The proof is based on the results in [8] and [13]. As emphasized in [3], the previous proposition characterizes the controllability properties of the nonlinear system (1) and (2) by means of simple algebraic operations. Note that single-input systems $(n > m = 1)$ always fail the sufficient condition for both controllability notions: if only

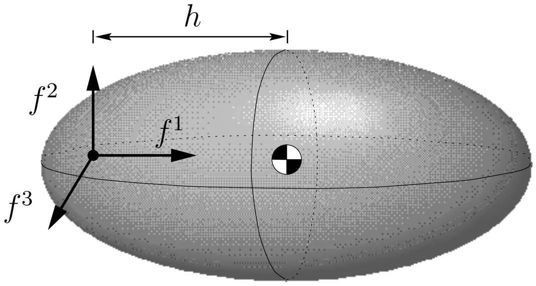

Fig. 1: Rigid body in $SE(3)$ with three forces applied at a point a distance h from the center of mass.

one input vector is available, the only possible nontrivial second order product is bad.

Example 5. (Ex. 2 continued: controllability) Consider the underwater vehicle described in Example 2, and assume there are three body-fixed forces applied at a point a distance h from the center of mass, see Figure 1. The matrix with columns defined by the input vectors is

$$[b_1\ b_2\ b_3] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{h}{I_2} \\ 0 & \frac{-h}{I_3} & 0 \\ \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix}.$$

The system is generically[1] locally controllable from zero velocity considering only second-order symmetric products. Indeed, the good second-order products are

$$[\langle b_i : b_j \rangle] = \begin{bmatrix} 0 & 0 & \frac{\frac{h^2}{I_3} - \frac{h^2}{I_2} - \frac{1}{m_3} + \frac{1}{m_2}}{I_1} \\ 0 & \frac{m_1 - m_3}{I_2 m_1 m_3} & 0 \\ \frac{m_2 - m_1}{I_3 m_1 m_2} & 0 & 0 \\ 0 & 0 & 0 \\ \frac{-h}{I_3 m_2} & 0 & 0 \\ 0 & \frac{-h}{I_2 m_3} & 0 \end{bmatrix}$$

where $(i,j) = (1,2), (1,3), (2,3)$. Additionally all the bad second-order products $\langle b_k : b_k \rangle$ for $k = 1,2,3$, are proportional to b_1 and hence are spanned by good lower order products (b_1 is a good product of order 1). •

Motivated by the previous example, we introduce an additional definition. A system is *locally controllable from zero velocity with second-order symmetric products* if it satisfies the following property:

(A1) The subspace $\mathrm{span}\{b_i, \langle b_j : b_k \rangle\}_{i,j,k=1,\dots,m}$ has full rank and any bad product $\langle b_i : b_i \rangle$ is a linear combination of the vectors $\{b_1, \dots, b_m\}$.

[1] It is locally controllable from zero velocity using only second-order symmetric products for all values of the parameters, except when $h^2 m_1 m_2 + I_3(m_1 - m_2) = 0$ or when $h^2 m_1 m_3 + I_2(m_1 - m_3) = 0$ or when $h^2(1/I_3 - 1/I_2) = 1/m_3 - 1/m_2$.

3. Approximate solutions

In this section we compute approximations for the solutions of equations (1) and (2) under small amplitude periodic forcing. The key tool is the standard perturbation method as described in [4]. Before stating the result, we review some standard material. We start by making the following assumption:

(A2) The group G is the Cartesian product of an arbitrary number of copies of $SE(3)$ and its proper subgroups.

Under (A2), it can be show that the exponential map $\exp : \mathfrak{g} \to G$, defined for example in [9], is a local diffeomorphism. Denoting with log its inverse in a neighborhood of the identity $Id \in G$, we call *exponential coordinates* of g the vector $\log(g)$. For example, if $R \in SO(3)$ is such that $\mathrm{tr}(R) \neq -1$, then

$$\log(R) = \frac{\phi}{2\sin\phi}(R - R^T) \quad \in \mathfrak{so}(3),$$

where ϕ satisfies $2\cos\phi = \mathrm{tr}(R) - 1$ and $|\phi| < \pi$. In other words, $\log(R)$ is the product of the axis and angle of rotation of R.

Also, we introduce the following notation. Given a (vector-valued) function $h(t)$ for $t \in [0, 2\pi]$, define its first integral function $\overline{h}$ as

$$\overline{h}(t) \ \triangleq\ \int_0^t h(\tau) d\tau.$$

Higher order integrals, as for example $\overline{\overline{h}}(t)$, are defined recursively. Given a positive constant $\epsilon \ll 1$, we decompose the input $\sum_i b_i u_i(t,\epsilon)$ into the sum of two terms of different order in ϵ:

$$\begin{aligned} \sum_{i=1}^m b_i u_i(t,\epsilon) &= \sum_{i=1}^m b_i \left(\epsilon u_i^1(t) + \epsilon^2 u_i^2(t)\right) \\ &= \epsilon\, b^1(t) + \epsilon^2\, b^2(t). \end{aligned} \tag{4}$$

The following proposition describes the system's behavior when forced by small (ϵ) amplitude inputs. As predicted from the controllability computations, both symmetric products and Lie brackets show up in the Taylor expansions.

Proposition 6. *Let $(g(t), \xi(t))$ be the solutions of equations (1) and (2). Let $\epsilon \ll 1$ and define the inputs as in equation (4). Let $x(t)$ be the exponential coordinates of $g(t)$ about the initial condition $g(0) = Id$. If $\xi(0) = 0$, then for $t \in [0, 2\pi]$ it holds that*

$$\begin{aligned} \xi(t) = {}& \epsilon\overline{b^1}(t) + \epsilon^2 \left(\overline{b^2} - \tfrac{1}{2}\overline{\left\langle \overline{b^1} : \overline{b^1}\right\rangle}\right)(t) \\ &+ \epsilon^3 \left(\tfrac{1}{2}\overline{\left\langle \overline{b^1} : \overline{\left\langle \overline{b^1} : \overline{b^1}\right\rangle}\right\rangle} - \overline{\left\langle \overline{b^1} : \overline{b^2}\right\rangle}\right)(t) + O(\epsilon^4) \end{aligned}$$

$$\begin{aligned} x(t) = {}& \epsilon\overline{\overline{b^1}}(t) \ +\ \epsilon^2 \left(\overline{\overline{b^2}} - \tfrac{1}{2}\overline{\overline{\left\langle \overline{b^1} : \overline{b^1}\right\rangle}} + \tfrac{1}{2}\overline{\left[\overline{\overline{b^1}}, \overline{b^1}\right]}\right)(t) \\ &+\ O(\epsilon^3). \end{aligned}$$

If $\xi(0) = \epsilon\xi^1 + \epsilon^2\xi^2$, then for $t \in [0, 2\pi]$ it holds that

$$\xi(t) = \epsilon\left(\xi^1 + \overline{b^1}(t)\right) + \epsilon^2\left(\xi^2 - \tfrac{1}{2}t\left\langle\xi^1 : \xi^1\right\rangle\right) + \epsilon^2\left(\overline{b^2} - \left\langle\xi^1 : \overline{\overline{b^1}}\right\rangle - \tfrac{1}{2}\overline{\left\langle\overline{b^1} : \overline{b^1}\right\rangle}\right)(t) + O(\epsilon^3)$$

$$x(t) = \epsilon\left(\overline{\xi^1} + \overline{\overline{b^1}}\right)(t) + O(\epsilon^2).$$

•

The proof is based on the standard perturbation method as described in [4]. Since the unperturbed system is not exponentially stable, the approximations hold only over a finite period of time.

Example 7. (In-phase inputs) To give some insight into the previous result, we apply it to the Euler equation (3). The latter reads in coordinates as:

$$I_1\dot{\Omega}_1 = (I_2 - I_3)\Omega_2\Omega_3 \ + \ u_1$$
$$I_2\dot{\Omega}_2 = (I_3 - I_1)\Omega_1\Omega_3 \ + \ u_2$$
$$I_3\dot{\Omega}_3 = (I_1 - I_2)\Omega_1\Omega_2.$$

If $\Omega_i(0) = 0$, and if $u_1 = u_2 = \epsilon\sin(t)$, then by the expansions above it holds that

$$\Omega_3(2\pi) = \epsilon^2\,\frac{3\pi(I_1 - I_2)}{2I_1I_2I_3} + O(\epsilon^3)$$

and that $\Omega_1(2\pi) = \Omega_2(2\pi) = O(\epsilon^3)$. Note that the motion in the symmetric product direction is generated with signals *in-phase*, rather than out of phase (which is a typical feature of the algorithms in [7]). •

We now simplify the expansions in Proposition 6 by an appropriate choice of inputs. Let N be an integer and for $a = 1, \dots, N$, consider the scalar functions

$$\psi_a(t) = \tfrac{1}{\sqrt{2\pi}}\Big(a\sin at - (a+N+1)\sin(a+N+1)t\Big),$$

defined on the interval $t \in [0, 2\pi]$. These functions satisfy a few useful properties, as for example $\overline{\psi_a}(2\pi) = \overline{\overline{\psi_a}}(2\pi) = 0$ and $\overline{\overline{\psi_a}\,\overline{\psi_b}}(2\pi) = \delta_{ab}$ (Kronecker delta). In what follows we impose the following structure to the input functions:

(A3) (Given the notations in equation (4)) Let the functions $u_i^1(t)$ be linear combination of the $\psi_a(t)$ and let the functions $u_i^2(t)$ be constants. Equivalently, let

$$b^1(t) = \sum_{i,a} c_{ia}^1\psi_a(t)b_i \quad \text{and} \quad b^2 = \sum_i c_i^2 b_i,$$

for some coefficients c_{ia}^1 and c_i^2.

Proposition 8. (Approximate evolution) *Let the inputs (b^1, b^2) be as in* (A3). *Under the assumptions of Proposition 6 and if $\xi(0) = 0$, we have*

$$\xi(2\pi) = \epsilon^2\left(\overline{b^2} - \tfrac{1}{2}\overline{\left\langle\overline{b^1} : \overline{b^1}\right\rangle}\right)(2\pi) + O(\epsilon^4) \tag{5}$$

$$x(2\pi) = \epsilon^2\left(\overline{\overline{b^2}} - \tfrac{1}{2}\overline{\overline{\left\langle\overline{b^1} : \overline{b^1}\right\rangle}} + \tfrac{1}{2}\overline{\left[\overline{\overline{b^1}}, \overline{b^1}\right]}\right)(2\pi) + O(\epsilon^3). \tag{6}$$

Under the assumptions of Proposition 6 and if $\xi(0) = \epsilon\xi^1 + \epsilon^2\xi^2$, we have

$$\xi(2\pi) = \epsilon\,\xi^1 + \epsilon^2\Big(\xi^2 - \pi\left\langle\xi^1 : \xi^1\right\rangle\Big) + \epsilon^2\Big(\overline{b^2} - \tfrac{1}{2}\overline{\left\langle\overline{b^1} : \overline{b^1}\right\rangle}\Big)(2\pi) + O(\epsilon^3) \tag{7}$$

$$x(2\pi) = \epsilon\,2\pi\xi^1 + O(\epsilon^2). \tag{8}$$

•

We interpret equation (5) as follows: up to an higher order error, the final value of ξ is determined by certain symmetric products and the final value of x is determined by certain symmetric products and Lie brackets.

Algorithm 1. (Inversion) Given an $\eta \in \mathfrak{g}$ and assuming (A1), we compute $\left(b^1, b^2\right)$ such that $\left(\overline{b^2} - \frac{1}{2}\overline{\langle\overline{b^1} : \overline{b^1}\rangle}\right)(2\pi) = \eta$.

1. Set $N = m(m-1)/2$ and let $P = \{(j,k) \mid 1 \le j < k \le m\}$. Number the elements in P with the set of integers $1, \dots, N$, and let $a(j,k)$ be the integer associated with the pair (j,k).
2. Compute $(m+N)$ real numbers z_i and z_{jk} such that $\eta = \sum_i z_i b_i + \sum_{j<k} z_{jk}\left\langle b_j : b_k\right\rangle$.
   This is possible thanks to the assumption (A1).
3. Set
   $b^1(t) = \sum_{j<k}\sqrt{|z_{jk}|}\Big(b_j - \operatorname{sign}(z_{jk})b_k\Big)\psi_{a(j,k)}(t),$
   $b^2 = \frac{1}{2\pi}\sum_i z_i b_i + \frac{1}{4\pi}\sum_{j<k}|z_{jk}|\Big(\left\langle b_j : b_j\right\rangle + \left\langle b_k : b_k\right\rangle\Big).$

Motion along the good product $\langle b_j : b_k\rangle$ (with $j \neq k$) is generated with the periodic function $\psi_{a(j,k)}$ within the definition of b^1. The second order input b^2 compensates for the motion excited along bad product directions. •

We conclude this section with a remark on scaling and frequencies. The extent of applicability of our approximations is better appreciated when parameter and time scaling are introduced. Indeed, for the approximations to be numerically sound, one needs to scale the inertia parameters to about unity. Additionally, if high frequency inputs are possible, one can obtain motion on a faster time-scale.

4. Motion control algorithms

Here we present two motion control algorithms based on Proposition 8 and Algorithm 1 for systems that satisfy assumptions (A1–3). The goal of these algorithms is to reconfigure the system, i.e. change its position and orientation, starting and ending at zero velocity. In contrast to the work in [6], the algorithms we introduce here rely on the use of symmetric product rather than Lie bracket. Note that Examples 1 and 2 motivate the assumption (A1). The latter could be relaxed allowing for higher order symmetric products and Lie brackets.

We start by fixing some notation. Without loss of generality, we assume $(g(0), \xi(0)) = (Id, 0) \in G \times \mathfrak{g}$ and we let $(g_d, 0) \in G$ denote the desired final configuration. We assume that $\log(g_d)$ is well defined; for example in $SE(3)$ this means that the change in attitude is less that π. Also, we write the input as in equation (4) and we introduce a positive constant $\sigma \ll 1$.

Algorithm 2. (Constant speed) Consider 3 steps. In Step 1, we apply a control to generate an appropriate velocity over the first time interval. In Step 2, maintain the velocity close to this constant value for an appropriate number of periods. In Step 3, stop the system when at the desired destination. The details are as follows:

Initialization Let $t = 0$ and compute $N \in \mathbb{N}$, $\xi_d \in \mathfrak{g}$ such that $\log(g_d) = 2\pi\sigma N \xi_d$ and $\|\xi_d\| \approx 1$. The desired velocity is $\sigma\xi_d$.

Step 1 Let $\epsilon = \sqrt{\sigma}$, let $\eta = \xi_d$ and apply Algorithm 1. After one period, i.e. at $t = 2\pi$, it holds that $\xi(2\pi) = \sigma\xi_d + O(\sigma^2)$ by the approximation in equation (5).

Step 2 Let $\epsilon = \sigma$ and repeat for $k = 1, \ldots, N-1$: measure $\xi(2k\pi)$ and compute $\xi_{err} = O(1)$ such that $\xi(2k\pi) = \sigma\xi_d + \sigma^2\xi_{err}$. Set $\eta = \pi \langle \xi_d : \xi_d \rangle - \xi_{err}$ and apply Algorithm 1. After one period, $\xi(2(k+1)\pi)$ is again $\sigma\xi_d + O(\sigma^2)$ by the approximation in equation (7).

Step 3 At time $t = 2N\pi$, let $\epsilon = \sqrt{\sigma}$. Measure $\xi(2N\pi)$, set $\eta = -\xi(2N\pi)/\sigma$ and apply Algorithm 1, so that $\xi(2(N+1)\pi)$ is $O(\sigma^2)$ by the approximation in equation (7).

The final position can be computed with an $O(\sigma)$ error: neglect the first and last periods (Step 1 and Step 3), since $\xi = O(\sigma)$ can give only a $O(\sigma)$ contribution over a finite time. During Step 2, which lasts[2] $2(N-1)\pi = O(1/\sigma)$, it holds $\xi(t) = \sigma\big(\xi_d + \overline{b^1}\big) + O(\sigma^2)$. Therefore, iterating $(N{-}1)$ times the approximation in equation (8), we have

$$\begin{aligned} \log(g(2(N+1)\pi)) &= 2(N{-}1)\pi\Big(\sigma\xi_d + O(\sigma^2)\Big) + O(\sigma) \\ &= 2(N{-}1)\pi\sigma\xi_d + O(\sigma). \end{aligned} \tag{9}$$

Since G is the Cartesian product of copies of $SE(3)$ (assumption (A2)), there exist a metric on G such that equality (9) implies $g = g_d$ up to order σ. We refer to [7] for a precise treatment of this point. •

Algorithm 3. (Constant acceleration) Consider 2 steps. In Step 1, for N intervals, keep increasing ξ at a constant rate $\epsilon\xi_d/(2\pi)$. In Step 2, decelerate the velocity to zero over the same number of intervals.

Initialization Let $t = 0$ and compute $N \in \mathbb{N}$, $\xi_d \in \mathfrak{g}$ such that $\log(g_d) = 2\pi\sigma^2 N^2 \xi_d$ and $\|\xi_d\| \approx 1$.

[2] Step 2 lasts $(N-1)$ periods, as Step 1 and 3 give each a 1/2 period contribution to the final reconfiguration.

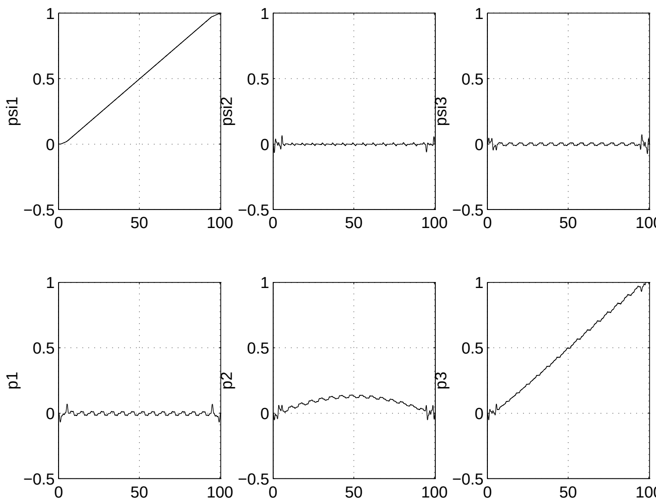

Fig. 2: Simulation of Algorithm 2. The configuration (R, p) in $SE(3)$ is depicted with the variables $(\log(R), p) \in \mathbb{R}^6$. The final error is of order σ, where $\sigma = .1125$.

Step 1 For $k = 0, \ldots, N-1$, during the period $2k\pi \le t < 2(k+1)\pi$: measure $\xi(2k\pi)$ and compute $\alpha = O(1)$ and $\xi_{err} = O(1)$ such that $\xi(2k\pi) = \epsilon\alpha\xi_d + \epsilon^2\xi_{err}$ (this is possible by the approximation in equation (7)). Set $\eta = \xi_d - \xi_{err} + \pi\alpha^2 \langle \xi_d : \xi_d \rangle$ and apply Algorithm 1.

Step 2 For $k = N, \ldots, 2N-1$, during the period $2k\pi \le t < 2(k+1)\pi$: measure $\xi(2k\pi)$ and compute $\alpha = O(1)$ and $\xi_{err} = O(1)$ such that $\xi(2k\pi) = \epsilon\alpha\xi_d + \epsilon^2\xi_{err}$ (this is possible by the approximation in equation (7)). Set $\eta = -\xi_d - \xi_{err} + \pi\alpha^2 \langle \xi_d : \xi_d \rangle$ and apply Algorithm 1.

The final position can be computed with an $O(\sigma)$ error: during both steps, it holds $\xi(t) = \sigma\big(\alpha\xi_d + \overline{b^1}\big) + O(\sigma^2)$. Therefore, iterating $2N$ times the approximation in equation (8), we have

$$\begin{aligned} \log(g(2N\pi)) &= 2\pi\sigma \sum_k \alpha(2k\pi)\xi_d + 2NO(\sigma^2) \\ &= 2\pi\sigma^2 N^2 \xi_d + O(\sigma) \end{aligned}$$

since $N = O(1/\sigma)$ and one can show $\sum_k \alpha(2k\pi) = \sigma N^2$. Hence, as in the Algorithm 2, $g = g_d$ up to order σ. •

Some comments are now appropriate. First, the algorithms rely on a discrete-time discontinuous feedback. For example in Step 2 of Algorithm 2 the velocity ξ is measured every 2π periods of time to set η to the correct value. Another common feature of the algorithms is that they both fail to completely stop the system at the end. Indeed, the final velocity is of order σ^2.

Regarding the type of motion that the system performs, these are constant body-velocity motions, that is screw motions. Additionally, if the trajectory connecting initial to final position is a relative equilibrium for the mechanical system, then it can be shown that the final position error is of order σ^2 (rather than σ).

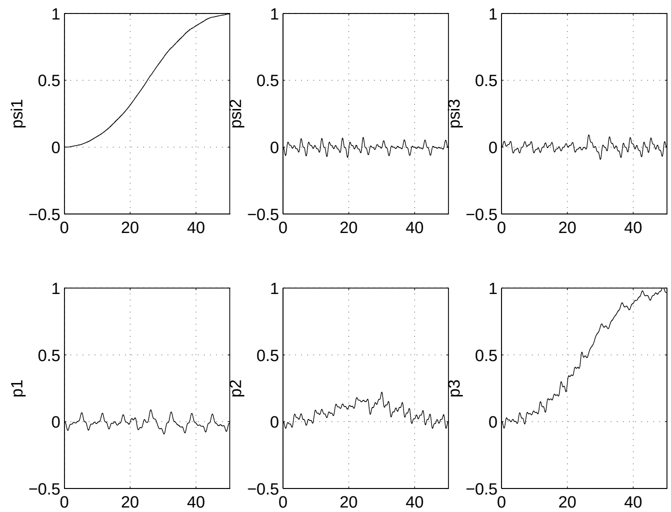

Fig. 3: Simulation of Algorithm 3, Constant acceleration. The variables are displayed are as in the previous Figure.

Example 9. (Ex. 2 continued: simulations) The two algorithms have been implemented and tested numerically on the underwater vehicle model described in Examples 2 and 5. The parameter values were chosen as in [6], and the quantities displayed are in normalized units. We report only the configuration variables for both algorithms applied to the same problem: perform a rotation of 1 unit about the first axis and a translation of 1 unit along the third axis. In both cases the final configuration is achieved with an error proportional to σ (at most).

Despite $\sigma = .1125$ for both simulations, a few differences can be noted. The first algorithm runs over a longer period, is more precise, and the configuration variables evolves linearly in time (hence at constant speed), see Figure 2. The second figure takes less time, achieves a less precise reorientation and the configuration variables depend quadratically on time (constant acceleration, then deceleration), see Figure 3. •

5. Conclusions

This work brings together a number of ideas and techniques in controllability theory, mechanical control systems and averaging theory. For driftless kinematic systems on Lie groups controllability conditions were first due to Brockett [2], while constructive averaging techniques were designed by Leonard and Krishnaprasad in [7]. With the current and previous work [3], we provide the same complete answer to the full dynamic case, where a second-order dynamic and a drift term are present.

Therefore, this work completes the picture for mechanical systems on Lie groups. Future research avenues will focus on how to apply the proposed algorithms to more general mechanical systems and how to include dissipative effects, see [6] for some results in this direction.

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