TRAJECTORY TRACKING FOR FULLY ACTUATED MECHANICAL SYSTEMS

CIT-CDS CANDIDACY REPORT

FRANCESCO BULLO

April 23, 1996

ABSTRACT. This paper presents a general framework for the control of mechanical systems with as many control inputs as degrees of freedom. An intrinsic interpretation of proportional derivative feedback and feed-forward control is provided through a Riemannian geometry framework. An error function and a transport map are the crucial design ingredients. The proposed approach includes various results on joint and workspace control of manipulators, rigid bodies and pointing devices.

Keywords: nonlinear control, mechanical control systems, robotics.

1. Introduction

This report is a contribution in the area of nonlinear control theory for mechanical systems (Murray 1995) and it is the completion of our previous works, (Bullo and Murray 1995) and (Bullo, Murray and Sarti 1995). Given a fully actuated control system, that is a system wth as many inputs as degrees of freedom, our control objective is to track a trajectory with exponential convergence rates, as to guarantee performance and robustness.

Exponential tracking for a robotic manipulator was first achieved by Wen and Bayard (1988) using a Lyapunov technique and is now standard in textbooks (Murray, Li and Sastry 1994). Since then, similar techniques have been applied to other fields in robotics, like position and attitude stabilization and for satellites and autonomous underwater vehicles. Additionally, relevant work on global stability issues was performed by Koditschek (1989).

Here we consider general mechanical systems defined on Riemannian manifolds and extend Koditschek's (1989) approach to the exponential tracking problem. The solution is the sum of a proportional derivative feedback and a feedforward control, which are designed using two key ingredients: an error function and a transport map. The gradient of the error function gives the proportional action. The transport map allows for a coordinate independent definition of velocity error and of derivative action. Furthermore, the feedforward control is the sum of two terms, depending on the reference input, the transport map and its covariant derivative.

The control design is therefore reduced to the design of these basic objects. More precisely we require a compatibility condition on how the transport map acts on the gradient of the error function. Provided some relevant operator are bounded, this condition is sufficient to prove exponential convergence. The main result is then applied to the various examples: the Euclidean space $\mathbb{R}^n$, the Lie group $SO(3)$ and the two sphere $\mathbb{S}^2$. Generally, there is a great freedom in designing error function and transport map and this is reflected by the numerous available works on the subject. However, our compatibility condition represents a precise test with instructive implications even in the basic examples. Additionally, instructive design trade-offs become clear and are outlined in the various examples.

Regarding the feedforward terms, the mechanical structure of the problem reveals itself clearly within our solution: the feedback part does not depend on the mechanical structure, while the feedforward control does! Additionally, the feedforward term is expressed in terms of intrinsic objects, which constitute a novel contribution. Indeed, following the discussion on "two degree of freedom system design" in (Murray 1995), we expect these terms to be important also in underactuated situations.

The report is organized as follows. In Section 2 we review the necessary tools from Riemannian geometry and fix some notation. In Section 3 we describe the design process, introducing the error function and the transport map. Also we compute the covariant derivative of the latter. Section 4 contains the main theorem with some useful comments. In the following sections we apply the general theory to the various cases: $\mathbb{R}^n$, $SO(3)$ and $\mathbb{S}^2$. Some final comments are reported in Section 8.

2. Mechanical systems on Riemannian manifolds

In this section we review some standard material, however some adhoc notation is introduced in Subsection 2.2. See in particular equations (2.5) and (2.6).

2.1. Elements of Riemannian geometry.

We refer to (Kobayashi and Nomizu 1963) and (Do Carmo 1992) for an introduction to Riemannian geometry. Here we simply review some notation.

Let M be a Riemannian manifold, denote with $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ or with g its metric tensor and with the symbols $^\flat : TM \to T^*M$ and $^\sharp : T^*M \to TM$ the musical isomorphisms. An *affine connection* on M is a map that assigns to each pair of smooth vector fields X, Y a smooth vector field $\nabla_X Y$ such that

(i) $\nabla_{fX} Y = f \nabla_X Y$ and

(ii) $\nabla_X fY = f\nabla_X Y + (\mathcal{L}_X f)\, Y$ for all $f \in C^\infty(M)$.

In a local chart with coordinates (x^i) we define the *Christoffel symbols* by

$$\nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} = \Gamma^i_{jk} \frac{\partial}{\partial x^i}.$$

Given any three vector fields X, Y, Z on M, we say that the affine connection ∇ on M is *torsion-free* if $[X, Y] = \nabla_X Y - \nabla_Y X$ and is *compatible* with the metric $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$ if

$$\mathcal{L}_X \langle\!\langle Y\,,\,Z \rangle\!\rangle = \langle\!\langle \nabla_X Y\,,\,Z \rangle\!\rangle + \langle\!\langle Y\,,\,\nabla_X Z \rangle\!\rangle. \tag{2.1}$$

By the Levi-Civita theorem, there exists a unique torsion-free affine connection ∇ on M compatible with the metric. We call this ∇ the Riemannian (or Levi-Civita) connection on $(M, \langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle)$. Its Christoffel symbols are computed as

$$\Gamma^m_{ij} = \frac{1}{2} g^{km} \left(\frac{\partial g_{kj}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right).$$

Given a smooth real valued function f on M, define its *gradient* as

$$\langle\!\langle \nabla f\,,\,X \rangle\!\rangle \triangleq \mathcal{L}_X f.$$

Next, let K be a $(1,1)$ type tensor field on M and let $X, Y \in \mathfrak{X}(M)$. It holds

$$\nabla_X (K(Y)) = (\nabla K)(Y; X) + K(\nabla_X Y)$$

where ∇K is a type $(1,2)$ tensor field on M such that

$$(\nabla K)^i_{jk} = K^i_{j;k} \;=\; (K^i_j)_{,k} + \Gamma^i_{k\ell} K^\ell_j - \Gamma^m_{kj} K^i_m. \tag{2.2}$$

Similarly, let ω be a $(0,1)$ type tensor field (hence a one form). Its covariant derivative is a $(0,2)$ type tensor which satisfies

$$(\nabla \omega)_{jk} = \omega_{j;k} \;=\; \omega_{j,k} - \Gamma^m_{kj} \omega_m. \tag{2.3}$$

For a complete treatment on the covariant derivative of a tensor, we refer to (Kobayashi and Nomizu 1963, page 122 and page 146). The same authors, in Proposition 7.10 at page 147, study the difference between two Riemannian connections ∇' and ∇. For any pair $X, Y \in \mathfrak{X}(M)$, it holds

$$\nabla'_X Y = \nabla_X Y + S(X, Y) \tag{2.4}$$

where S is a symmetric tensor field on M, whose coordinates satisfy $S^i_{jk} = {\Gamma'}^i_{jk} - \Gamma^i_{jk}$. Conversely, given any Riemannian connection ∇ and any symmetric tensor field S, equation (2.4) defines a new Riemannian connection ∇'.

2.2. **Mechanical control systems on Riemannian manifolds.** We now turn to mechanical systems. The following definitions are standard:

Definition 1. A *simple mechanical control system* is defined by a Riemannian metric on a configuration manifold Q (defining the kinetic energy), a function V on Q (defining the potential energy), and m one-forms, $F^1, \ldots, F^m$, on Q (defining the inputs).

A simple mechanical system is said to be *fully actuated*, if for all $q \in Q$ it holds $\operatorname{span}\{F^1(q), \ldots, F^m(q)\} = T_q^*Q$, i.e. if there exists an independent input one form corresponding to each degree of freedom. □

In the following we will assume that the configuration manifold Q comes provided with a *natural* Riemannian metric $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$. We denote with

$$\nabla \qquad \text{the Levi-Civita connection on} \qquad (Q\,, \langle\!\langle \cdot\,, \cdot \rangle\!\rangle). \tag{2.5}$$

Equivalently, tangent and cotangent bundle are naturally identified:

$$T_q^*Q \;=\; T_qQ. \tag{A1}$$

Remark 1. This basic assumption is emphasized with the label (A1). Beside simplifying notation, this identification holds in every example and is usually exploited in the design of a dissipation and error function, see next sections. □

With respect to this natural structure, the kinetic energy of the mechanical system can be expressed in terms of a positive definite symmetric tensor field $M_q : T_qQ \to T_qQ$. We denote with $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle_M = \langle\!\langle M \cdot\,, \cdot \rangle\!\rangle$ the inner product associated to the kinetic energy (*mechanical metric*) and with ${}_M\nabla$ the Riemannian connection associated to the simple mechanical system. Hence we have

$${}_M\nabla \qquad \text{the Levi-Civita connection on} \qquad (Q\,, \langle\!\langle \cdot\,, \cdot \rangle\!\rangle_M). \tag{2.6}$$

Let us denote with $q(t) \in Q$ the configuration of the system and with $\dot{q}(t) \in T_qQ$ its velocity. Using the formalism introduced in the previous section, the Euler-Lagrange equations for a simple mechanical control system can be written as

$${}_M\nabla_{\dot{q}}\dot{q} \;=\; {}_M\nabla V(q) + M_q^{-1} f_a(q) u^a, \tag{2.7}$$

where the input vector fields $f_a(q) \in T_qQ$ are identified with the input one forms $F^a(q) \in T_q^*Q$ through (A1).

We conclude the section with a boundedness condition. We shall say that the mechanical system (2.7) has bounded inertia, if there exist $m_1 \geq m_2 > 0$ such that

$$m_1 \geq \sup_{q \in Q} \|M_q\| \geq \inf_{q \in Q} \|M_q\| \geq m_2, \tag{A2}$$

where $\|\cdot\|$ is the operator norm on the normed linear space $(T_qQ\,, \langle\!\langle \cdot\,, \cdot \rangle\!\rangle)$.

3. The design process

In this section we introduce the notion of error function φ and transport map τ. An error function provides us with a notion of state error and generalizes the notion of proportional action (Koditschek 1989). A transport map provides us with a notion of velocity error and, coupled with a dissipation function, generalizes the notion of derivative action. Additionally the transport map is crucial in the computation of the feedforward controls. As in the previous section and for the rest of the paper, we assume (A1). Also, we label with (An) the main assumptions in the design construction.

3.1. **The error and dissipation functions.** Let φ be a smooth real valued function on $Q \times Q$. We shall call φ an *error function* if it is *symmetric*, i.e. $\varphi(q,r) = \varphi(r,q)$, and if it is *positive definite*, i.e. $\varphi(q,r) = 0$ if and only if $q = r$. Denote with $\nabla\varphi$ the gradient of $\varphi(q,r)$ with respect to its first argument. We shall say that the error function φ is *quadratic at* r if there exists a neighborhood N_r of r and $b_1 \geq b_2 > 0$ such that

$$b_1 \|\nabla\varphi(q,r)\|^2 \geq 2\varphi(q,r) \geq b_2 \|\nabla\varphi(q,r)\|^2. \tag{A3}$$

for all $q \in N_r$. Furthermore, φ is said to be quadratic if it is quadratic at each r with the same constants b_1, b_2.

Remark 2. The quadratic assumption on the error function is needed to prove exponential convergence rates. We could impose weaker conditions, as for example a class $\mathcal{K}$ requirement, and obtain only asymptotic type results. But exponential convergence helps in proving robustness properties and in designing adaptive schemes. Additionally, in the examples we are always able to find such φ's and indeed we later show a standard procedure to design such error functions. □

Define a *(Rayleigh) dissipation function* by introducing a symmetric positive definite tensor field $(K_d)_q : T_qQ \to T_qQ$. We assume K_d admits upper and lower bounds over Q, that is there exist $d_2 \geq d_1 > 0$ such that

$$d_2 \geq \sup_{q\in Q} \|K_d\| \geq \inf_{q \in Q} \|K_d\| \geq d_1, \tag{A4}$$

where $\|\cdot\|$ is the operator norm on the normed linear space $(T_qQ\,,\, \langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle)$. □

3.2. **The transport map.** Given two points $q, r \in Q$, the linear map $\tau_{(q,r)} : T_rQ \to T_qQ$ is said to be a *transport map* if it is *compatible with the natural inner product* (A1)

$$\langle\!\langle X_r\,,\, Y_r \rangle\!\rangle_r = \langle\!\langle \tau_{(q,r)} Y_r\,,\, \tau_{(q,r)} X_r \rangle\!\rangle_q \tag{A5}$$

and it is *compatible with the error function*

$$\tau_{(q,r)}\, \nabla\varphi(r,q) = -\nabla\varphi(q,r). \tag{A6}$$

We say that the trasport map τ is *smooth*, if for all $r \in Q$ and $Y_r \in T_rQ$ there exists a neighborhood N_r of r, such that $\tau_{(q,r)} Y_r$ is a smooth vector field for all $q \in N_r$. In the next lemma, we motivate the introduction of a transport map with such properties as (A5) and (A6).

Lemma 1 (Time derivative of error function). *Let $r(t)$ and $q(t)$ be two curves in Q, such that $\nabla\varphi(q(t),r(t))$ is well-defined for all t. Assume* (A1), (A5) *and* (A6). *Then*

$$\frac{d}{dt}\varphi(q(t),r(t)) = \langle\!\langle \nabla\varphi(q,r)\,,\, \dot{q} - \tau_{(q,r)}\dot{r} \rangle\!\rangle.$$

Proof. Using the symmetry property and the transport map on the second addendum:

$$\begin{aligned}
\frac{d}{dt}\varphi(q(t),r(t)) &= \langle\!\langle \nabla\varphi(q,r)\,,\, \dot{q} \rangle\!\rangle_q + \langle\!\langle \nabla\varphi(r,q)\,,\, \dot{r} \rangle\!\rangle_r \\
&= \langle\!\langle \nabla\varphi(q,r)\,,\, \dot{q} \rangle\!\rangle_q + \langle\!\langle \tau_{(q,r)}\nabla\varphi(r,q)\,,\, \tau_{(q,r)}\dot{r} \rangle\!\rangle_q && \text{by (A5)} \\
&= \langle\!\langle \nabla\varphi(q,r)\,,\, \dot{q} \rangle\!\rangle_q - \langle\!\langle \nabla\varphi(q,r)\,,\, \tau_{(q,r)}\dot{r} \rangle\!\rangle_q && \text{by (A6)} \\
&= \langle\!\langle \nabla\varphi(q,r)\,,\, \dot{q} - \tau_{(q,r)}\dot{r} \rangle\!\rangle_q.
\end{aligned}$$

■

The result can equivalently be stated as follows. Recall that $\varphi : Q \times Q \to \mathbb{R}$ and $(q, r) \in Q \times Q$. The time derivative of φ on the full space reduces to a derivative only with respect to the first argument

$$\mathcal{L}_{(\dot q,\, \dot r)}\varphi \;=\; \mathcal{L}_{(\dot q - \tau_{(q,r)}\dot r,\, 0)}\varphi, \tag{3.1}$$

where $(X,\ Y)$ denotes a vector field on $Q \times Q$. In the following, we shall call the quantity

$$\dot e \;\triangleq\; \dot q - \tau_{(q,r)}\dot r \qquad \in T_qQ$$

the *velocity error*. Note the slight abuse of terminology, given that no corresponding "position error" exists.

Remark 3 (Standard design using Riemannian tools)**.** On a generic manifold, a choice of error function and transport map is always given by distance function and parallel transport map associated to a Riemannian metric. A detailed description of these ideas is contained in Appendix A. □

3.3. **Covariant derivative of the transport map.** In the computation of the feedforward control, the "time derivative" of the transport map τ will be required. In this subsection, we show how one can covariantly differentiate τ after turning it into a tensor field. This is implemented through the following tedious, but straightforward construction.

We start with some notation. Let $\overline{Q} = Q \times Q$ and denote with (q, r) a generic point on it. In the natural trivialization of $T_{(q,r)}\overline{Q}$, write $(X_q, Y_r) \in T_qQ \times T_rQ$. Next, define $\pi_1 : T_{(q,r)}\overline{Q} \to T_qQ$ and $\pi_2 : T_{(q,r)}\overline{Q} \to T_rQ$ to be the first and second projection, so that

$$\pi_1(X_q, Y_r) = X_q \qquad \text{and} \qquad \pi_2(X_q, Y_r) = Y_r.$$

Also, denote with π_1^{-1} and π_2^{-1} the generalized inverse maps (lifts) such that

$$(\pi_1^{-1})_{(q,r)}X_q = (X_q, 0_r) \qquad \text{and} \qquad (\pi_2^{-1})_{(q,r)}Y_r = (0_q, Y_r).$$

Next, we introduce a Riemannian metric and the relative Levi-Civita connection on $T_{(q,r)}\overline{Q}$. For any pair $U, V \in T\overline{Q}$ let

$$\langle\!\langle U\,,\, V\rangle\!\rangle_{\overline{Q}} \;\triangleq\; \langle\!\langle \pi_1 U\,,\, \pi_1 V\rangle\!\rangle_M + \langle\!\langle \pi_2 U\,,\, \pi_2 V\rangle\!\rangle,$$

Note the different inner products on the two subspaces: we employ the mechanical metric $\langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle_M$ on the T_qQ subspace and the natural metric $\langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle$ on the T_rQ subspace (A1). Denote with

$$\overline{\nabla} \qquad \text{the Levi-Civita connection on} \qquad (\overline{Q}\,,\, \langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle_{\overline{Q}}). \tag{3.2}$$

The connection $\overline{\nabla}$ is determined by the two connections ∇ on $(Q, \langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle)$ (defined in (2.5)) and ${}_M\nabla$ on $(Q, \langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle_M)$ (defined in (2.6)). Indeed, let $\left\{\frac{\partial}{\partial q^a}\right\}$ be a basis for T_qQ and $\left\{\frac{\partial}{\partial r^\alpha}\right\}$ for T_rQ. One can verify that

$$\overline{\nabla}_{\frac{\partial}{\partial q^a}}\frac{\partial}{\partial q^b} = \left({}_M\nabla_{\frac{\partial}{\partial q^a}}\frac{\partial}{\partial q^b}\,,\, 0\right) \qquad \text{and} \qquad \overline{\nabla}_{\frac{\partial}{\partial r^\alpha}}\frac{\partial}{\partial r^\beta} = \left(0\,,\, \nabla_{\frac{\partial}{\partial r^\alpha}}\frac{\partial}{\partial r^\beta}\right),$$

while the cross covariant derivatives between $\partial/\partial q^a$ and $\partial/\partial r^\alpha$ vanish. In other words, if ${}_M\Gamma$ are the Christoffel symbols for ${}_M\nabla$ and Γ are the ones for ∇, the only non vanishing Christoffel symbols of $\overline{\nabla}$ are $\overline{\Gamma}^c_{ab} = {}_M\Gamma^c_{ab}$ and $\overline{\Gamma}^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}$. □

We now ready to study the transport map τ. Define the $(1,1)$ type tensor field $\overline{\tau}$ on $\overline{Q}$ as

$$\overline{\tau}_{(q,r)}(X_q,\ Y_r) = (\tau_{(q,r)}Y_r, 0),$$

or, more precisely, $\overline{\tau}_{(q,r)} \triangleq \pi_1^{-1} \cdot \tau_{(q,r)} \cdot \pi_2 r \quad : \quad T_{(q,r)}\overline{Q} \to T_{(q,r)}\overline{Q}$. Next, we compute coordinate expressions for τ, $\overline{\tau}$ and $\overline{\nabla}\overline{\tau}$. We have

$$(\tau\dot{r})^a = \tau^a_\alpha \dot{r}^\alpha, \qquad \text{and} \qquad (\overline{\tau}(0,\dot{r}))^a = \tau^a_\alpha \dot{r}^\alpha.$$

(This equality in coordinates reveals once again how the construction above is indeed trivial.) Additionally note thay $\overline{\tau}^\alpha_k = \overline{\tau}^k_a = 0$ for all k, since the tensor $\overline{\tau}$ has the T_qQ subspace as its kernel and image. To compute the $(1,2)$ type tensor $\overline{\nabla}\overline{\tau}$ we use equation (2.2):

$$\begin{aligned}
(\overline{\nabla}\overline{\tau})^a_{\alpha k} = \overline{\tau}^a_{\alpha;k} &= \tau^a_{\alpha,k} + \overline{\Gamma}^a_{k\ell}\tau^\ell_\alpha - \overline{\Gamma}^m_{k\alpha}\tau^a_m \\
&= \tau^a_{\alpha,k} + {}_M\Gamma^a_{kb}\tau^b_\alpha - \Gamma^\beta_{k\alpha}\tau^a_\beta \\
&= \begin{cases} \tau^a_{\alpha,c} + {}_M\Gamma^a_{cb}\tau^b_\alpha & \text{if } k = c, \\ \tau^a_{\alpha,\gamma} - \Gamma^\beta_{\gamma\alpha}\tau^a_\beta & \text{if } k = \gamma. \end{cases}
\end{aligned} \tag{3.3}$$

Next, note that for any $X \in T_{(q,r)}\overline{Q}$, the map $\overline{\nabla}_X\overline{\tau}$ has the same kernel and image as $\overline{\tau}$ and therefore it drops to a function from $T_rQ \to T_qQ$. In particular we define

$$\overline{\nabla}_X\tau \triangleq \pi_1 \cdot \overline{\nabla}_X\overline{\tau}_{(q,r)} \cdot \pi_2^{-1}. \tag{3.4}$$

Remark 4. In the examples of the later sections, we will need to compute the quantity $\overline{\nabla}_{(\dot{q},\,\dot{r})}\big(\overline{\tau}_{(q,r)}(0,\dot{r})\big)$. The previous coordinate expressions give of course a correct (sometimes tedious) answer. However, simpler procedure are often available, as for example in Section 6. □

We conclude the section by formalizing two boundedness conditions. We say that the transport map has bounded covariant derivative if

$$\sup_{(q,r)\in Q\times Q} \|\overline{\nabla}\overline{\tau}\| < \infty, \tag{A7}$$

where $\|\cdot\|$ is the operator norm for $(1,2)$ type tensors on $Q \times Q$. Also, we say that the error function φ has bounded second covariant derivative if

$$\sup_{(q,r)\in Q\times Q} \|{}_M\nabla\, d\varphi\| < \infty, \tag{A8}$$

where $\|\cdot\|$ is the operator norm for $(0,2)$ type tensors on $Q \times Q$.

Given equation (2.3) and (3.3), a sufficient condition for both the previous bounds to hold, is given by bounds on the quantities $\tau^a_{\alpha,k}$, $\varphi_{,ab}$, $\Gamma^\alpha_{\beta\gamma}$ and ${}_M\Gamma^a_{bc}$.

4. Tracking on Riemannian manifolds

In this section we introduce and solve the exponential tracking problem for a fully actuated mechanical system. We start by introducing two additional definitions which will help state the control objective precisely.

First, since no coordinate independent "state error" can be defined on a generic manifold, we resort to the error function φ to define the notion of exponential convergence. In particular, we shall say that $q(t)$ converges φ-exponentially to $r(t)$ if $\varphi(q(t), r(t))$ converges exponentially fast to zero for all initial conditions $(q(0), r(0))$, with $q(0)$ in an appropriate neighborhood of $r(0)$.

Second, since the quantity $\ddot{r}$ is generally not well-defined, [1] we resort to the Riemannian connection ∇ in equation (2.5) to have a notion of higher order derivative of $r(t)$. In particular the final control law will depend on $\nabla_{\dot{r}}\dot{r}$. (Note that we would have the design freedom to choose ${}_M\nabla$ instead of ∇. However, this latter choice is consistent with our treatment in the previous section and simplifies some computations.)

Control objective: Consider the mechanical system in equation (2.7) and let $r(t) \in Q$ be a reference trajectory with bounded time derivative

$$\sup_t \|\dot{r}\| < \dot{r}_{\max}. \tag{A9}$$

Design a feedback control law $f = f(q, \dot{q};\ r, \dot{r}, \nabla_{\dot{r}}\dot{r})$ that achieves φ-exponential convergence.

Let Q be a configuration manifold with a natural metric structure (A1). Let the mechanical system in equation (2.7) have bounded inertia (A2) and let the reference trajectory $(r(t), \dot{r}(t), \nabla_{\dot{r}}\dot{r})$ have bounded velocity (A9).

Let φ be a quadratic error function (A3) with bounded second covariant derivative (A8). Let τ be a compatible transport map (A5), (A6) with bounded covariant derivative (A7). Let K_d be a bounded dissipation function (A4). Define $\dot{e} = \dot{q} - \tau_{(q,r)}\dot{r}$ to be the velocity error.

Theorem 4.1 (Exponential tracking on Riemannian manifolds). *Assume (A1) through (A9) and define the proportional plus derivative feedback f_{PD} and feedforward term f_{FF} as*

$$\begin{aligned} f_{PD}(q, \dot{q};\ r, \dot{r}) &= -\nabla\varphi(q, r) - K_d\dot{e} \\ M_q^{-1} f_{FF}(q, \dot{q};\ r, \dot{r}, \nabla_{\dot{r}}\dot{r}) &= \left(\overline{\nabla}_{(\dot{q},\,\dot{r})}\tau_{(q,r)}\right)\dot{r} + \tau_{(q,r)}\,\nabla_{\dot{r}}\dot{r}, \end{aligned}$$

where the map $\overline{\nabla}_{(\dot{q},\dot{r})}\,\tau_{(q,r)} : T_rQ \to T_qQ$ is defined in equation (3.4).

Then the feedback control law $f = f_{PD} + f_{FF}$ exponentially stabilizes φ and $\dot{e}$, for all initial conditions such that the quantity $\left(\varphi(q,r) + \|\dot{e}\|^2\right)|_{t=0}$ is small enough.

We refer to Appendix B for the detailed proof. Some comments follow.

Remark 5 (Global/local properties of the closed loop)**.** The convergence properties of the closed loop are global in the reference position $r(t)$ but only local in the "error" $\varphi(q,r) + \|\dot{e}\|^2$. Thus the expression "global/local." This is due to the nature of the design procedure and of the boundedness assumptions, all of which are intrinsically local (in the error). However, example by example, we will be able to specify precisely the size of this local neighborhood.

Additionally we refer to (Koditschek 1989) for precise statements on global limit behavior of mechanical systems controlled by proportional derivative feedback.

Remark 6 (Feedback transformations)**.** A global feedback transformation that preserves the second order nature of the mechanical system can only be a "change of connection", as defined by equation (2.4). With the notation of the previous sections, there exists a tensor field S_M on Q such that

$${}_M\nabla_{\dot{q}}\dot{q} = \nabla_{\dot{q}}\dot{q} + S_M(\dot{q}, \dot{q}).$$

[1] We could employ the theory of jet bundles, but we will not follow this approach here.

Hence, recalling the definitions in Subsection 2.2, a (coordinate independent) feedback transformation of

$$ {}_M\nabla_{\dot q}\dot q = M(q)^{-1} f \qquad \text{into} \qquad \nabla_{\dot q}\dot q = u $$

is of the form

$$ f = M(q)\big(u - S_M(\dot q, \dot q)\big). \tag{4.1} $$

An example of this procedure is the so-called "computed torque" strategy in $\mathbb{R}^n$ and our treatment of the sphere case in Section 7. □

Remark 7 (General Lagrangian systems)**.** While the theorem is stated for the case of mechanical systems with Lagrangian equal to kinetic energy, it can be generalized to systems with potentials functions, viscous and gyroscopic forces by pre-compensating for the extra terms. In other words the simplest approach is a straightforward cancellation, using a feedback transformation. An alternative solution is discussed in the next remark. □

Remark 8 (Approximate feedforward and gravity compensation)**.** Practical reasons might suggest the implementation of a feedforward control different from the one presented above. For example Wen and Bayard (1988) describe various possible feedforward compensations, which might be computationally simpler.

The idea underlying these alternative formulations is that they all agree up to higher order terms in $\dot e$ or perpendicular to $\dot e$. Indeed the proof in Appendix B can be modified to account for such changes. Similar remarks hold when compensating for gravity or gyroscopic effects. □

Remark 9 (Design trade-offs on (φ, τ))**.** Important design issues appear when understanding the design dependencies between the feedback and feedforward terms. The design of

$$ f_{\mathrm{PD}}(q, \dot q;\ r, \dot r) = -\nabla\varphi(q, r) - K_d(\dot q - \tau_{(q,r)}\dot r) $$

does not depend on the mechanical metric M_q. The design of

$$ M_q^{-1} f_{\mathrm{FF}}(q, \dot q;\ r, \dot r, \nabla_{\dot r}\dot r) = \big(\overline{\nabla}_{(\dot q,\ \dot r)}\tau_{(q,r)}\big)\, \dot r + \tau_{(q,r)}\, \nabla_{\dot r}\dot r, $$

does not depend directly on the error function. Therefore, modulo the compatibility assumption (A6), the design of the proportional action (more precisely of φ) is independent from the design of the derivative action and of the feedforward term (more precisely of τ).

(i) The simpler φ we choose, the simpler τ's will be compatible and the simpler control laws we will design. On the other hand, Koditschek (1989) shows that the global properties of the closed loop system depend critically on φ, being for example a Morse function. An instructive example of this tradeoff is the attitude tracking problem for a rigid body with external torques. See Section 6.

(ii) Assume (φ, τ) is a compatible design, and s is an appropriate scaling function s. Then also $(s \circ \varphi, \tau)$ is compatible. Indeed scaling φ preserves the direction of $\nabla\varphi$ and the assumption (A6) on τ keeps holding. An instructive example of this design freedom is exploited in Remark 12 of Section 6 to rederive various different controllers proposed in the literature. □

Remark 10 (Two degree of freedom system design). As expected, the final control law is sum of a feedback and a feedforward term. This is in agreement with the ideas exposed in (Murray 1995) on "two degree of freedom system design". In particular it holds true that the feedforward control achieves exact tracking, that is, when starting with zero error, the closed-loop system maintans zero error. □

5. On flat manifolds: the $\mathbb{R}^n$ case

This section considers the tracking problem for a robotic manipulator with n joints. We denote the configuration variables with $q \in \mathbb{R}^n$ and the inertia matrix of the manipulator with $M(q)$. In the following, we shall recover the standard results on control of manipulators contained in (Murray et al. 1994) by applying Theorem 4.1. We refer to (Koditschek 1989) for a treatment of problems related to obstacle avoidance and kinematic singularites.

Start by noting that $\mathbb{R}^n$ has a natural metric structure (A1). Then, neglect kinematic singularities and assume $M(q)$ to have bounded magnitude over $\mathbb{R}^n$ (A2). Finally let the reference trajectory $r(t)$ have bounded time derivative (A7). The design described in Section 3 is performed as follows:

Error function: The natural choice is

$$\varphi(q,r) = \frac{1}{2}(q-r)^T K_p (q-r).$$

Transport map: Since $T_q\mathbb{R}^n = T_r\mathbb{R}^n$, let $\mathcal{T}_{(q,r)} = \mathrm{id}$, the identity map. This is possible every time the configuration manifold is a vector space.

For these choices, we check (A3) to (A5).

(A3): φ is positive definite, symmetric and quadratic.

(A4): Since $\mathrm{id} \in SO(n)$, the transport map preserves the inner product.

(A5): We compute

$$\nabla\varphi(q,r) = K_p(q-r) \qquad \text{and} \qquad \nabla\varphi(r,q) = K_p(r-q) = -\nabla\varphi(q,r),$$

so that id is compatible with φ. □

From Subsection (3.3) recall the definition of the tensor $\overline{\mathcal{T}}$ and of affine connection $\overline{\nabla}$ on $\overline{Q} = Q \times Q$. To design the feedforward action as well as to check for (A6), we compute the covariant derivative of $\overline{\mathcal{T}}$. From equation (3.3) we have

$$\begin{aligned}(\overline{\nabla}\,\overline{\mathrm{id}})^a_{\alpha;k} &= \mathrm{id}^a_{\alpha,k} + {}_M\Gamma^a_{kb}\mathrm{id}^b_\alpha - \Gamma^\beta_{k\alpha}\mathrm{id}^a_\beta \\ &= {}_M\Gamma^a_{kb}\mathrm{id}^b_\alpha \;=\; {}_M\Gamma^a_{k\alpha},\end{aligned}$$

since the Christoffel symbols Γ^k_{ij} of the natural metric of $\mathbb{R}^n$ are zero. Therefore, in contrast to a naive guess, the covariant derivative of the identity map is different from zero!

Next, we impose the boundedness assumption (A7) by requiring $\sup_q \|{}_M\Gamma^a_{k\alpha}(q)\| < \infty$. Additionally we compute

$$\begin{aligned}\overline{\nabla}_{(\dot q,\,\dot r)}(\overline{\mathrm{id}}\,(0,\dot r)) &= \;(\overline{\nabla}\,\overline{\mathrm{id}})\big((0,\dot r);(\dot q,\dot r)\big) \;+\; \mathrm{id}\,\overline{\nabla}_{(\dot q,\dot r)}(0,\dot r) \\ &= \;(\overline{\nabla}\,\overline{\mathrm{id}})^a_{\alpha;d}\dot r^\alpha \dot q^d \frac{\partial}{\partial q^a} \;+\; \nabla_{\dot r}\dot r \\ &= \;{}_M\Gamma^a_{db}\dot r^b \dot q^d \frac{\partial}{\partial q^a} \;+\; \ddot r,\end{aligned}$$

where $\left\{\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}\right\}$ is the standard base in $\mathbb{R}^n$.

Finally, after the design and the computation, we state the control law as follows:

$$f_{\mathrm{PD}} = -K_p(q-r) - K_d(\dot q - \dot r)$$

$$f_{\mathrm{FF}} = M\ddot r + M^c_a{}_M\Gamma^a_{bd}\dot r^b \dot q^d \frac{\partial}{\partial q^c} \;=\; M(q)\ddot r + C(q,\dot q)\dot r \tag{5.1}$$

where the *Coriolis matrix* $C(\cdot,\cdot)$ is defined as usual. The control law $f = f_{\mathrm{PD}} + f_{\mathrm{FF}}$ agrees with the one presented in (Murray et al. 1994, page 195) under the name of "augmented PD control".

Note that our approach is able to provide an intrinsic meaning to the second term in equation (5.1). Indeed this sort of cross term represent the covariant derivative of the identity transport map.

Remark 11 (Computed torque)**.** The other famous approach to control of robotic manipulators is the so-called computed torque approach. Within our framework, this approach is described by equation (4.1), where

$$(S_M)^k_{ij} = {}_M\Gamma^k_{ij} - \Gamma^k_{ij} = {}_M\Gamma^k_{ij}.$$

Hence the feedback transform is $f = M(q)u - C(q,\dot q)\dot q$, and the tracking control law for the transformed system is $u = -K_p(q-r) - K_d(\dot q - \dot r) + \ddot r$. □

6. On the Lie group $SO(3)$

In this section we perform the design process and apply the main theorem to the Lie group $SO(3)$. Regarding the error function design, we follow the approach in (Koditschek 1989) and our previous work (Bullo and Murray 1995).

Recalling that the set of matrices $\mathfrak{gl}(3)$ is equipped with the natural inner product

$$\langle\!\langle A\,,\,B\rangle\!\rangle_{\mathfrak{gl}(3)} = \frac{1}{2}\operatorname{tr}(AB^t),$$

which induces the orthogonal decomposition $\mathfrak{gl}(3) = \mathrm{sym}(3)\oplus\mathrm{skew}(3)$ into symmetric and skew matrices. The Lie algebra $\mathfrak{so}(3) = \mathrm{skew}(3)$ comes therefore equipped with a natural inner product, [2] which induces a Riemannian metric on $SO(3)$ by left translation. Assumption (A1) is thus satisfied.

As far as notation is concerned, let $q, r \in SO(3)$ denote actual and reference configuration, let I_3 denote the identity matrix, and, for $g \in SO(3)$, denote left and right translation by L_g and R_g. The exponential map, its inverse the logarithm, adjoint action and matrix commutator are defined as usual, see (Murray et al. 1994).

Error function: Using the group operation, let $e \triangleq r^t q \in SO(3)$ be the configuration error and define the error function as $\varphi(q,r) \triangleq \phi(e)$. Given be a 3×3 positive definite matrix K_p, define

$$\phi(e) \triangleq -\operatorname{tr}(K_p e) + \operatorname{tr}(K_p),$$

where the second addendum is the constant needed to have $\phi(I_3) = 0$. Using Rodrigues' formula and some properties of the trace function, one can verify that this error function is symmetric, positive definite and quadratic. Next, we compute the gradient of ϕ. Defining $\mathrm{skew}(A) = (A - A^t)/2$, we have

$$\dot\phi = -\operatorname{tr}(K_p\dot e) = 2\langle\!\langle K_p e\,,\, e^t\dot e\rangle\!\rangle_{\mathfrak{gl}(3)} = 2\langle\!\langle \mathrm{skew}(K_p e)\,,\, e^t\dot e\rangle\!\rangle_{\mathfrak{so}(3)},$$

[2] Indeed this inner product coincides modulo a constant with the Killing form.

and therefore

$$\nabla\phi(e) \;=\; 2\,TL_e \cdot \operatorname{skew}(K_p e) = TL_e \cdot (K_p e - e^t K_p).$$

For the scalar gain case $K_p = k_p I_3$, we compute $\phi(e) = 2 - 2\cos\theta(e)$, where $\theta(e)$ is the angle of rotation associated with e. Some additional insight is gained by noting that $\theta(e)$ is equal to the Riemannian distance of e from I_3. Using the geometric tools discussed in Appendix A or computing by hand, one can verify that $\nabla\phi(e)$ is parallel to

$$\nabla\theta(e) = -TL_e \cdot \log(e).$$

□

Transport map: The most general transport map $\tau_{(q,r)} : T_r SO(3) \to T_q SO(3)$ can be written as

$$\tau = TL_q \cdot A \cdot TL_{r^t},$$

for some linear function $A : \mathfrak{so}(3) \to \mathfrak{so}(3)$. Since the adjoint action of $SO(3)$ is an orthogonal representation on $\mathfrak{so}(3)$, the assumption (A5) implies $A = \operatorname{Ad}_x$ for some $x \in SO(3)$. Also, since

$$\nabla\varphi(q,r) = TL_r \cdot \nabla\phi(e),$$

assumption (A6) (on the compatibility with the error function) is equivalent to

$$\operatorname{Ad}_g(K_p e^t - e K_p) = e^t K_p - K_p e, \qquad \text{for some} \quad g \in SO(3).$$

For general K_p's, the previous equation is satisfied by $g = e^t$. For $K_p = k_p I_3$, an even simpler solution is $g = I_3$. Corresponding to these two choices, we define

$$\tau_R \triangleq TR_{r^t q} \qquad \text{and} \qquad \tau_L \triangleq TL_{q r^t}.$$

Note that other possibilities would also be feasible.

Next, we write coordinate expressions for these transport maps. Let $\dot q = TL_q \cdot \omega$ and $\dot r = TL_r \cdot \omega_r$. Then the two velocity error are

$$\dot q - \tau_L \dot r = TL_q \cdot \big(\omega - \omega_r\big)$$
$$\dot q - \tau_R \dot r = TL_q \cdot \big(\omega - \operatorname{Ad}_{e^t} \omega_r\big).$$

Note that τ_L is a somehow "simpler" choice, but it does not allow for matrix gains (it requires $K_p = k_p I_3$). On the other hand, the transport map τ_R preserves the "second order property"

$$\dot e = e\big(\omega - \operatorname{Ad}_{e^t} \omega_r\big) = TL_r \cdot (\dot q - \tau_R \dot r).$$

□

Before computing the covariant derivatives of the transport maps, we can already state the proportional and derivative feedbacks. Depending on the choice of τ, Theorem 4.1 leads to

$$TL_{q^t} \cdot f_{\mathrm{PD}}(q, \dot q;\; r, \dot r) = -2\operatorname{skew}(K_p e) - K_d(\omega - \operatorname{Ad}_{e^t} \omega_r),$$

or

$$TL_{q^t} \cdot f_{\mathrm{PD}}(q, \dot q;\; r, \dot r) = -2k_p \operatorname{skew}(e) - K_d(\omega - \omega_r),$$

where K_d is a positive definite gain on $\mathfrak{so}(3)$.

Remark 12 (Variations of the error function)**.** Recalling the discussion in Remark 9, we emphasise the importance of the direction of the gradient of φ, rather than its magnitude. This latter quantity can indeed be changed with a scaling function, without affecting the choice of transport map.

As an example we apply this scaling procedure to the trace function introduced above (the scalar case $K_p = k_p I$). Let

$$\varphi(g,r) = \frac{1}{2}k^2(\theta(e)),$$

where $k : \mathbb{R} \to \mathbb{R}$ is an increasing function, that is first order in x for $x \to 0$. For different choices of k, we recover previous results obtained in the literature by introducing a parametrization on $SO(3)$:

$k(x) = k_p x$	Meyer (1971), exponential coordinates,
$= k_p \tan(x/2)$	Slotine and Benedetto (1990), Gibbs vector,
$= k_p \sin x$	Wen and Kreutz-Delgado (1991), quaternion,
$= k_p \sin(2x)$	Wen and Kreutz-Delgado (1991), vector quaternion,

where the standing assumption is $k_p > 0$.

For all of these designs, both transport maps τ_L and τ_R can be used, since the gradient of φ will always have the same direction (parallel to $TL_e \cdot \log(e)$). □

6.1. **Feedforward action.** The feedforward term depends on the metric of the mechanical system, hence we need to specify this latter. In this subsection, we restrict our attention to the case of a rigid body with external torques.

We denote with $\mathbb{I} : \mathfrak{so}(3) \to \mathfrak{so}(3)^*$ the body inertia tensor and with ${}_{\mathbb{I}}\nabla$ the Riemannian connection associated to the $SO(3)$ metric $M_q = T^*L_{q^t} \cdot \mathbb{I} \cdot TL_{q^t}$. As one can verify, see for example (Helgason 1978, Chapter II), the connection ${}_{\mathbb{I}}\nabla$ satisfies

$$TL_g \cdot ({}_{\mathbb{I}}\nabla_X Y)_h = ({}_{\mathbb{I}}\nabla_X Y)_{gh}$$

for any pair of left invariant vector fields X, Y and $g, h \in SO(3)$. Therefore, restricting the connection to left invariant vector fields, one obtains a map from $\mathfrak{so}(3) \times \mathfrak{so}(3)$ to $\mathfrak{so}(3)$, which we denote with the same symbol ${}_{\mathbb{I}}\nabla$. For all $\xi, \eta \in \mathfrak{so}(3)$, this latter map is computed as

$$2\,{}_{\mathbb{I}}\nabla_\xi \eta = [\xi, \eta] - \mathbb{I}^{-1}\big(\operatorname{ad}^*_\xi \mathbb{I}\eta + \operatorname{ad}^*_\eta \mathbb{I}\xi\big).$$

From Theorem 4.1 we need to compute

$$M_q^{-1} f_{\mathrm{FF}} \;=\; \overline{\nabla}_{(\dot q,\, \dot r)} \left(\overline{\tau}_{(q,r)}(0,\ \dot r)\right) \qquad \in T_q SO(3).$$

We examine the two cases $\tau = \tau_L$ and $\tau = \tau_R$ separately. Let $\{\xi_a\}$ be a base for $\mathfrak{so}(3)$ and recall the definitions $\omega = TL_{q^t} \cdot \dot q$ and $\omega_r = TL_{r^t} \cdot \dot r$.

Left translation: Setting $\tau = \tau_L$ we have

$$\begin{aligned}
\overline{\nabla}_{(\dot q,\, \dot r)} \overline{\tau}_L(0, \dot r) &= \overline{\nabla}_{(\dot q,\, \dot r)}(TL_q \cdot \omega_r, 0) \\
&= TL_q \cdot \dot\omega_r + (\omega_r)^a \overline{\nabla}_{(\dot q,\, \dot r)}(TL_q \cdot \xi_a, 0) \\
&\overset{*}{=} TL_q \cdot \dot\omega_r + (\omega_r)^a \omega^b\, {}_{\mathbb{I}}\nabla_{(TL_q \cdot \xi_b)}(TL_q \cdot \xi_a) \\
&= TL_q \cdot \big(\dot\omega_r + {}_{\mathbb{I}}\nabla_\omega \omega_r\big),
\end{aligned}$$

where in equality $(*)$ the covariant derivative drops from $\overline{Q}$ to Q. Hence we have

$$T^*L_{q^t}\cdot f_{\mathrm{FF}} = \mathbb{I}\dot{\omega}_r + \mathbb{I}\,{}_{\mathbb{I}}\nabla_{\omega}\omega_r.$$

Right translation: Similarly for $\tau = \tau_R$

$$\begin{aligned}\overline{\nabla}_{(\dot q,\,\dot r)}\overline{\tau}_L(0,\dot r) &= \overline{\nabla}_{(\dot q,\,\dot r)}(TL_q\cdot(\mathrm{Ad}_{e^t}\,\omega_r),0)\\ &= TL_q\cdot\Big(\frac{d}{dt}(\mathrm{Ad}_{e^t}\,\omega_r) + {}_{\mathbb{I}}\nabla_{\omega}(\mathrm{Ad}_{e^t}\,\omega_r)\Big),\end{aligned}$$

and

$$\begin{aligned}T^*L_{q^t}\cdot f_{\mathrm{FF}} &= \mathbb{I}\Big(\mathrm{Ad}_{e^t}\,\dot\omega_r + [\mathrm{Ad}_{e^t}\,\omega_r\,,\,\omega] + {}_{\mathbb{I}}\nabla_{\omega}(\mathrm{Ad}_{e^t}\,\omega_r)\Big)\\ &= \mathbb{I}\Big(\mathrm{Ad}_{e^t}\,\dot\omega_r + {}_{\mathbb{I}}\nabla_{(\mathrm{Ad}_{e^t}\,\omega_r)}\omega\Big).\end{aligned}$$

Remark 13. The expressions we obtained are different from the standard one described in (Wen and Kreutz-Delgado 1991) and (Bullo et al. 1995). We recover these previous results through the equality

$$\langle\!\langle \xi-\eta\,,\,\mathbb{I}\,{}_{\mathbb{I}}\nabla_{\xi}\eta\rangle\!\rangle = \langle\!\langle \xi-\eta\,,\,\mathrm{ad}^*_{\xi}\,\mathbb{I}\eta\rangle\!\rangle$$

and the discussion in Remark 8.

7. On the two sphere $\mathbb{S}^2$

Our approach to the tracking problem on the manifold $\mathbb{S}^2$ relies on some of the geometric tools described in Appendix A. This section is however self-contained. Main reference is our previous work (Bullo et al. 1995).

Denote the natural inner and outer products on $\mathbb{R}^3$ by $\langle\!\langle\cdot\,,\,\cdot\rangle\!\rangle$ and $[\cdot,\cdot]$. At any $q\in\mathbb{S}^2\subset\mathbb{R}^3$, every tangent vector $X_q\in T_q\mathbb{S}^2$ can be uniquely represented as a vector $X_q\in\mathbb{R}^3$ such that $X_q\perp q$ and more generally $T_q\mathbb{S}^2=\mathrm{span}\{q\}^{\perp}$. The natural inner product on $\mathbb{R}^3$ induces a Riemannian metric on $\mathbb{S}^2$ in the natural way:

$$\langle\!\langle X_q\,,\,Y_q\rangle\!\rangle_{T_q\mathbb{S}^2} \triangleq \langle\!\langle X_q\,,\,Y_q\rangle\!\rangle \qquad \forall X_q,Y_q\in T_q\mathbb{S}^2\subset\mathbb{R}^3.$$

We denote with ∇ the Riemannian connection associated with this inner product.

Error function: The geodesics of $\mathbb{S}^2$ are great circles and the distance between any two points is the angle between them. We define

$$\varphi(q,r) \triangleq \frac{1}{2}dist(q,r)^2 = \frac{1}{2}\arccos^2\langle\!\langle p\,,\,r\rangle\!\rangle,$$

with arccos taking values in $[0,\pi]$. Provided q and r are neither equal nor opposite, there exists a unit vector $V_{(q,r)}\in T_q\mathbb{S}^2$, called the *geodesic versor*, which points from q toward r. It holds

$$V_{(q,r)} = [\mathrm{vers}\big([q,r]\big),q] \qquad \in T_q\mathbb{S}^2$$

where $\mathrm{vers}(x)=x/\|x\|$. Following the approach in Appendix A, or computing by hand, one can see that

$$\nabla\varphi(q,r) = dist(q,r)V_{(q,r)}.$$

Transport map: Define $\tau_{(q,r)} \in SO(3)$ as the rotation about $[q,r]$ which maps r to q. This can be expressed by the two conditions $\tau_{(q,r)} \cdot r = q$ and $\tau_{(q,r)} \cdot [r,q] = [r,q]$, or by the equation

$$\tau_{(q,r)} \triangleq I_3 + ([r,q]\times) + \frac{1-\cos(dist(q,r))}{\sin(dist(q,r))^2}([q,r]\times)^2,$$

where $(a\times)b = [a,b]$. Also, define $\omega \in \mathbb{R}^3$ such that $\dot{\tau}_{(q,r)} = (\omega\times)\tau_{(q,r)}$. As we show in (Bullo et al. 1995), it holds

$$\langle\!\langle \omega\, ,\, q \rangle\!\rangle = \langle\!\langle \tan\big(\tfrac{1}{2} dist(q,r)\big) V_{(q,r)} - q\ ,\ \ [\dot{q}, r] + [q, \dot{r}] \rangle\!\rangle.$$

We consider the tracking problem for the system

$$\nabla_{\dot{q}}\dot{q} = f. \tag{7.1}$$

This assumes that either the mechanical system has a trivial inertia, or that a feedback transformation has already been performed as described in Remark 6. Let $r(t) \in \mathbb{S}^2$ be the reference trajectory, $\dot{r}, \nabla_{\dot{r}}\dot{r} \in T_r\mathbb{S}^2$ be the reference velocity and acceleration, and assume $\sup_t \|\dot{r}\|$ bounded.

Lemma 2 (Tracking on the two sphere). *Consider the system in equation* (7.1). *Given a positive k_p and positive definite K_d, define*

$$\begin{aligned} f_{PD} &= k_p dist(q,r) V_{(q,r)} - K_d(\dot{q} - \tau_{(q,r)}\dot{r}) \\ f_{FF} &= \tau_{(q,r)} \nabla_{\dot{r}}\dot{r} + \langle\!\langle \omega\, ,\, q\rangle\!\rangle [q, \dot{q}]. \end{aligned}$$

Then the control law $f = f_{PD} + f_{FF}$ exponentially stabilizes $\{dist(q,r)\, ,\, \dot{q} - \tau_{(q,r)}\dot{r}\}$ to zero from any initial condition $q(0) \neq -r(0)$ and for all k_p, $\dot{q}(0)$ and $\dot{r}(0)$ such that

$$k_p > \frac{\|\dot{q}(0) - \tau_{(q,r)}\dot{r}(0)\|^2}{\pi^2 - dist(q(0), r(0))^2}.$$

The cross term in the feedforward control has not been computed by covariantly differentiating τ. Instead the Lyapunov function used in the proof has been differentiated by hand. We refer to (Bullo et al. 1995) for the missing details.

8. Conclusions and future directions

In this report, we have described the geometry underlying the tracking problem for fully actuated mechanical systems. A general theoretical framework allows us to recover and understand standard results (as in the $\mathbb{R}^n$ and $SO(3)$ cases) and design new controllers (as in the $\mathbb{S}^2$ case). Future relevant extensions will deal with the $SE(3)$ case.

From a theoretical viewpoint, we feel that this contribution is basic and instrumental in understanding design techniques for mechanical control systems. Novel contributions are the concept of transport map, of compatibility with the error function and the interpretation of the feedforward control. From a pratical viewpoint, the contribution lies in showing what the intrinsic objects are and how to tweak them while preserving convergence properties. For example, in tracking problems on Lie groups, it is important to push along the gradient of a well-defined error function, rather along an arbitrary choice of local coordinate.

Finally, we emphasize that our framework relies on the same tools used in other recent papers on mechanical control systems. For example Lewis and Murray (1996) and Rathinam and Murray (96) characterize controllability and dynamic feedback linearization in terms of Riemannian geometry concepts. Hopefully these same tools will help us construct a general "control theory for mechanical systems" and in particular a design technique for underactuated systems.

A Research Proposal is included with a discussion on these last comments.

References

Bullo, F. and Murray, R. M. (1995). Proportional derivative (PD) control on the Euclidean group, *Tech. Rep. CIT/CDS 95–011*, Caltech. Short version appeared in the Proc. of ECC'95. Available electronically via `http://avalon.caltech.edu/cds`.

Bullo, F., Murray, R. M. and Sarti, A. (1995). Control on the sphere and reduced attitude stabilization, *Tech. Rep. CIT/CDS 95–005*, Caltech. Short version appeared in Proc. of Nolcos'95. Available electronically via `http://avalon.caltech.edu/cds`.

Do Carmo, M. P. (1992). *Riemannian Geometry*, Birkhäuser, Boston.

Helgason, S. (1978). *Differential Geometry, Lie Groups and Symmetric Spaces*, Academic Press, New York.

Kobayashi, S. and Nomizu, K. (1963). *Foundations of Differential Geometry*, Vol. I, Interscience Publishers, New York.

Koditschek, D. E. (1989). The application of total energy as a Lyapunov function for mechanical control systems, *in* P. S. K. J. E. Marsden and J. C. Simo (eds), *Dynamics and Control of Multibody Systems*, Vol. 97, AMS, pp. 131–157.

Lewis, A. D. and Murray, R. M. (1996). Controllability of simple mechanical control systems, *SIAM Journal of Control and Optimization.* To appear.

Meyer, G. (1971). Design and global analysis of spacecraft attitude control systems, *Technical report*, NASA. Tech. Report R-361.

Murray, R. M. (1995). Nonlinear control of mechanical systems: a Lagrangian perspective, *Nolcos*, Lake Tahoe, CA.

Murray, R. M., Li, Z. X. and Sastry, S. S. (1994). *A Mathematical Introduction to Robotic Manipulation*, CRC Press, Boca Raton, Florida.

Rathinam, M. and Murray, R. M. (96). Configuration flatness of Lagrangian systems underactuated by one control, *CDC*, Kobe, Japan.

Slotine, J.-J. E. and Benedetto, M. D. D. (1990). Hamiltonian adaptive control of spacecraft, *T. Automatic Control* **AC-35**: 848–852.

Wen, J. T.-Y. and Bayard, D. S. (1988). A new class of control laws for robotic manipulators. Part I: Non–adaptive case, *Int. J. Control* **47**(5): 1361–1385.

Wen, J. T.-Y. and Kreutz-Delgado, K. (1991). The attitude control problem, *T. Automatic Control* **AC-36**: 1148–1162.

APPENDIX A. The design process using Riemannian tools

This Appendix describes a general method to obtain quadratic error functions and compatible transport maps using geometric tools.

Let assumption (A1) hold and assume that the Riemannian manifold $(Q, \langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle)$ is complete (see (Do Carmo 1992)). Then a choice of error function is given by squared distance function:

$$\phi(q,r) = \frac{1}{2} dist(q,r)^2 .$$

Such error function is symmetric and positive definite. To compute the gradient and prove ϕ is quadratic, we prove the following lemma:

Lemma 3 (Extension of Gauss's Lemma). *Let $(Q, \langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle)$ be a Riemannian manifold and let $r \in Q$. Let $U \subset Q$ denote a neighborhood of r, such that $\forall q \in U$ there exists a unique (unit speed) geodesic curve connecting q to r and contained in U. Denote this curve with $\gamma_q^r(s)$, where $s \in [0, dist(q,r)]$.*

Given a curve $q : [a,b] \to U$, then

$$\frac{d}{dt} dist(q(t), r) = -\langle\!\langle \frac{d}{ds}\Big|_{s=0} \gamma_q^r\,,\, \dot{q} \rangle\!\rangle, \tag{A.1}$$

where $\frac{d}{ds}\Big|_{s=0} \gamma_q^r$, the initial velocity vector of the unit speed geodesic connecting q to r, is called geodesic versor.

Proof. If $\dot{q} \perp \frac{d}{ds}|_{s=0}\gamma_q^r$, then we have $\frac{d}{dt} d(q,r) = 0$ by Gauss's Lemma (Do Carmo 1992). Hence consider only the component of $\dot{q}$ parallel to $\frac{d}{ds}|_{s=0}\gamma_q^r$. Equation (A.1) simply states that the distance between q and r increases as the velocity of the geodesic with velocity equal to the parallel component of $\dot{q}$. But this holds true, since the distance function is the integral of the velocity of the geodesic curve. ■

Since equation (A.1) holds for all $\dot{q}$, we compute $\nabla\varphi(q,r) = dist(q,r) \frac{d}{ds}|_{s=0}\gamma_q^r$. Thus ϕ is quadratic.

Next, set the transport map τ to be the parallel transport map along the geodesic γ_q^r connecting q to r, which we denote with P_q^r. We remark that distance and parallel transport are computed with respect to the natural metric $\langle\!\langle \cdot\,,\,\cdot \rangle\!\rangle$. We now check that assumption (A3) through (A6) hold.

(A3): holds with $b_1 = b_2 = 1$, since $\|\nabla\varphi(q,r)\| = dist(q,r)$.

(A5): holds, since the parallel transport preserves the inner product.

(A6): holds, since P_q^r maps geodesic curves to geodesic curves .

Hence, this standard construction applies to every Riemannian manifold Q. Indeed, performing this construction, is an instructive step in most of the examples.

Appendix B. Proof of main theorem

Proof. [*Lyapunov stability from total energy*] Consider the candidate Lyapunov function

$$W_{\text{total}}(q,\dot q;\; r,\dot r) = \varphi(q,r) + \frac{1}{2}\langle\langle \dot e\,,\,\dot e\rangle\rangle_M,$$

which is positive definite in $\varphi(q,r)$ and $\dot e = \dot q - \tau_{(q,r)}\dot r$. Its time derivative is

$$\begin{aligned}\frac{d}{dt}W_{\text{total}} &= \frac{d}{dt}\varphi(q,r) \;+\; \frac{1}{2}\frac{d}{dt}\langle\langle(\dot e,0)\,,\,(\dot e,0)\rangle\rangle_{\overline{Q}}\\ &= \langle\langle\nabla\varphi(q,r)\,,\,\dot e\rangle\rangle + \langle\langle(\dot e,0)\,,\,\overline{\nabla}_{(\dot q,\,\dot r)}(\dot e,0)\rangle\rangle_{\overline{Q}},\end{aligned}$$

where we have used Lemma 3 and equation (2.1) on the full space $\overline{Q} = Q\times Q$. Using the linearity properties of $\overline{\nabla}$, we compute

$$\begin{aligned}\overline{\nabla}_{(\dot q,\,\dot r)}(\dot e,0) &= \overline{\nabla}_{(\dot q,\,\dot r)}\big((\dot q,0) - \overline{\tau}_{q,r}(0,\dot r)\big)\\ &= \overline{\nabla}_{(\dot q,\,\dot r)}(\dot q,0) - \big(\overline{\nabla}_{(\dot q,\,\dot r)}\overline{\tau}_{q,r}\big)(0,\dot r) - \overline{\tau}_{q,r}\overline{\nabla}_{(\dot q,\,\dot r)}(0,\dot r)\\ &= ({}_M\nabla_{\dot q}\dot q,0) - \big(\overline{\nabla}\overline{\tau}_{(q,r)}\big)\Big((0,\dot r);(\dot q,\dot r)\Big) - (\tau_{(q,r)}\nabla_{\dot r}\dot r,0),\end{aligned}$$

and substituting the feedforward part of f, we have exactly

$$\overline{\nabla}_{(\dot q,\,\dot r)}(\dot e,0) = (M_q^{-1}f_{\text{PD}},0). \tag{B.1}$$

Plugging in

$$\begin{aligned}\frac{d}{dt}W_{\text{total}} &= \langle\langle\nabla\varphi\,,\,\dot e\rangle\rangle + \langle\langle(\dot e,0)\,,\,(M_q^{-1}f_{\text{PD}},0)\rangle\rangle_{\overline{Q}}\\ &= \langle\langle\nabla\varphi\,,\,\dot e\rangle\rangle + \langle\langle\dot e\,,\,f_{\text{PD}}\rangle\rangle\\ &= -\langle\langle\dot e\,,\,K_d\dot e\rangle\rangle\end{aligned}$$

so that $\frac{d}{dt}W_{\text{total}}$ is (only) negative semidefinite. ▼

[*Definition of cross term*] To construct a strict Lyapunov function (i.e. a function with a time derivative strictly definite), we add an ϵ-size cross term to W_{total}. Let

$$W_{\text{cross}}(q,\dot q;\; r,\dot r) \;=\; \frac{d}{dt}\varphi(q,r) \;=\; \langle\langle\nabla\varphi\,,\,\dot e\rangle\rangle$$

and consider the candidate Lyapunov function

$$W \;=\; W_{\text{total}} + \;\epsilon W_{\text{cross}}.$$

We need to show that $\exists\epsilon$ small enough, such that W is positive definite in φ and $\dot e$. Lower bounding

$$\begin{aligned}W &\geq \varphi + \frac{1}{2}\inf_q\|M\|\cdot\|\dot e\|^2 - \epsilon|\nabla\varphi|\cdot\|\dot e\|\\ &\geq \varphi + \frac{1}{2}\inf_q\|M\|\cdot\|\dot e\|^2 - \epsilon(2/\sqrt{b_2})\sqrt{\varphi}\|\dot e\|,\end{aligned}$$

where we have employed the "quadratic" assumption on the error function (A3). Hence we have

$$W > \begin{bmatrix} \sqrt{\varphi} \\ \|\dot{e}\| \end{bmatrix}^t \begin{bmatrix} 1 & -\epsilon/\sqrt{b_2} \\ -\epsilon/\sqrt{b_2} & \inf_q \|M_q\| \end{bmatrix} \begin{bmatrix} \sqrt{\varphi} \\ \|\dot{e}\| \end{bmatrix}$$
$$\triangleq \begin{bmatrix} \sqrt{\varphi} \\ \|\dot{e}\| \end{bmatrix}^t \mathcal{P} \begin{bmatrix} \sqrt{\varphi} \\ \|\dot{e}\| \end{bmatrix},$$

and by choosing $\epsilon < b_2\sqrt{\inf_q \|M_q\|}$, the matrix $\mathcal{P}$ and the function W are positive definite. ▼

[*Time derivative of cross term*] Next, we compute the time derivative of W_{cross}. We do this in a local chart using the coordinate expressions for covariant derivatives described in Subsection 2.1. Let $\{q^a\}$ be a coordinate chart about q and $\{r^\alpha\}$ about r. For convenience, denote partial derivatives with commas: $f_{,a} = \partial f/\partial q^a$. By Lemma 1, we have

$$\dot{\varphi} = \varphi_{,a}\dot{q}^a + \varphi_{,\alpha}\dot{r}^\alpha = \varphi_{,b}\dot{e}^b.$$

Since

$$\frac{d}{dt}\varphi_{,b} = (\dot{\varphi})_{,b} = \varphi_{,ab}\dot{e}^a - \varphi_{,a}(\dot{e}^a)_{,b}$$

and $(\dot{e}^a)_{,b} = -\tau^a_{\alpha,b}\dot{r}^\alpha$, we have

$$\ddot{\varphi} = \varphi_{,ab}\dot{e}^a\dot{e}^b - \varphi_{,a}\tau^a_{\alpha,c}\dot{r}^\alpha\dot{e}^c + \varphi_{,c}\ddot{e}^c,$$

In this equation, some quantities are not intrinsic, that is coordinate independent. Indeed we have

$$\tau^a_{\alpha,c} = \overline{\tau}^a_{\alpha;c} - {}_M\Gamma^a_{bc}\tau^b_\alpha.$$

and

$$\left(\overline{\nabla}_{(\dot{q};\dot{r})}\dot{e}\right)^c = \ddot{e}^c + \overline{\Gamma}^c_{ab}\dot{e}^b\dot{q}^a + \overline{\Gamma}^c_{a\alpha}\dot{e}^a\dot{r}^\alpha = \ddot{e}^c + {}_M\Gamma^c_{ab}\dot{e}^b\dot{q}^a.$$

Substituting

$$\begin{aligned}
\ddot{\varphi} &= \varphi_{,ab}\dot{e}^a\dot{e}^b - \varphi_{,a}\tau^a_{\alpha,c}\dot{r}^\alpha\dot{e}^c + \varphi_{,c}\ddot{e}^c \\
&= \varphi_{,ab}\dot{e}^a\dot{e}^b - \varphi_{,a}\dot{r}^\alpha\dot{e}^c\overline{\tau}^a_{\alpha;c} + \varphi_{,a}\dot{r}^\alpha\dot{e}^c{}_M\Gamma^a_{bc}\tau^b_\alpha + \varphi_{,c}\left(\overline{\nabla}_{(\dot{q};\dot{r})}\dot{e}\right)^c - \varphi_{,c}{}_M\Gamma^c_{ab}\dot{e}^a\dot{q}^b \\
&= \langle\!\langle \nabla\varphi\,,\, \pi_1\cdot\left(\overline{\nabla}_{(\dot{q};\dot{r})}(\dot{e},0)\right) - \left(\overline{\nabla}_{(\dot{e},0)}\tau\right)\rangle\!\rangle + \varphi_{,ab}\dot{e}^a\dot{e}^b - {}_M\Gamma^c_{ab}\dot{e}^a\dot{e}^b\varphi_{,c} \\
&= \langle\!\langle \nabla\varphi\,,\, \pi_1\cdot\left(\overline{\nabla}_{(\dot{q};\dot{r})}(\dot{e},0)\right) - \left(\overline{\nabla}_{(\dot{e},0)}\tau\right)\rangle\!\rangle + {}_M\nabla d\varphi(\dot{e};\dot{e}),
\end{aligned}$$

where ${}_M\nabla d\varphi$ is also an intrinsic quantity. Next, plugging in (B.1), we have

$$\ddot{\varphi} = \langle\!\langle \nabla\varphi\,,\, M_q^{-1}f_{\text{PD}} - \left(\overline{\nabla}_{(\dot{e},0)}\tau\right)\dot{r}\rangle\!\rangle + {}_M\nabla d\varphi(\dot{e};\dot{e})$$

and in the closed loop, setting $f_{\text{PD}} = -\nabla\varphi - K_d\dot{e}$,

$$\ddot{\varphi} = -\langle\!\langle \nabla\varphi\,,\, M_q^{-1}\nabla\varphi\rangle\!\rangle - \langle\!\langle \nabla\varphi\,,\, M_q^{-1}K_d\dot{e} + \left(\overline{\nabla}_{(\dot{e},0)}\tau\right)\dot{r}\rangle\!\rangle + {}_M\nabla d\varphi(\dot{e};\dot{e}).$$

Thanks to the quadratic assumption on φ, we can upper-bound $\dot{W}_{\text{cross}}$ as a function of φ and $\|\dot{e}\|$. We obtain

$$\dot{W}_{\text{cross}} \;=\; \ddot{\varphi} \;\leq\; -\begin{bmatrix} \sqrt{\varphi} \\ \|\dot{e}\| \end{bmatrix}^t \mathcal{Q}_{\text{cross}} \begin{bmatrix} \sqrt{\varphi} \\ \|\dot{e}\| \end{bmatrix},$$

where the symmetric matrix $\mathcal{Q}_{\text{cross}}$ has the following entries:

$$\begin{aligned}(\mathcal{Q}_{\text{cross}})_{1,1} &= 2/\Big(b_1 \sup_{q\in Q} \|M_q\|\Big),\\ (\mathcal{Q}_{\text{cross}})_{1,2} \;=\; (\mathcal{Q}_{\text{cross}})_{2,1} &= -\Big(\sup_{q\in Q}\|M_q^{-1}K_d\| \;+\; \sup_t \|\dot r\| \cdot \sup_{(q,r)\in\overline{Q}} \|\overline{\nabla}\,\overline{\tau}\|\Big)/\sqrt{2b_1}\\ (\mathcal{Q}_{\text{cross}})_{2,2} &= -\sup_{(q,r)\in\overline{Q}} \|{}_M\nabla d\varphi(q,r)\|.\end{aligned}$$

Next, we need to ensure that the operators in $\mathcal{Q}_{\text{cross}}$ are bounded:
$(\mathcal{Q}_{\text{cross}})_{1,1}$ is bounded away from zero thanks to (A2),
$(\mathcal{Q}_{\text{cross}})_{2,2}$ is upper bounded thanks to (A8),
$(\mathcal{Q}_{\text{cross}})_{1,2}$ is upper bounded thanks to assumptions (A2), (A4), (A9) and (A7).

▼

[Final sum] As last step, we now upper bound the time derivative of the Lyapunov function $W = W_{\text{total}} + \epsilon W_{\text{cross}}$. We have

$$\frac{d}{dt}W \;\le\; -\begin{bmatrix}\sqrt{\varphi}\\ \|\dot e\|\end{bmatrix}^t \mathcal{Q} \begin{bmatrix}\sqrt{\varphi}\\ \|\dot e\|\end{bmatrix},$$

where the matrix $\mathcal{Q}$ is positive definite for small enough ϵ, since

$$\begin{aligned}\mathcal{Q}_{1,1} &= \epsilon(\mathcal{Q}_{\text{cross}})_{1,1}\\ \mathcal{Q}_{1,2} \;=\; \mathcal{Q}_{2,1} &= \epsilon(\mathcal{Q}_{\text{cross}})_{1,2}\\ \mathcal{Q}_{2,2} &= \inf_{q\in Q}\|K_d\| + \epsilon(\mathcal{Q}_{\text{cross}})_{2,2},\end{aligned}$$

and $\mathcal{Q}_{2,2}$ is bounded away from zero thanks to (A4). Hence, there exist a $\lambda > 0$ such that $\dot W < -\lambda W$. Therefore both φ and $\|e\|$ converge to zero exponentially. ■

E-mail address: bullo@indra.caltech.edu

CALTECH, MAIL CODE 104-44, PASADENA, CA 91125