CONFIGURATION CONTROLLABILITY OF MECHANICAL SYSTEMS ON LIE GROUPS

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ABSTRACT. The recent introduction of a new notion of controllability and a corresponding computable test has provided a deeper insight into mechanical control systems. This paper focuses on systems whose Lagrangian is the kinetic energy and whose configuration manifold is a Lie group. The theory developed by the Lewis and Murray, coupled with the notion of invariant affine connection, leads naturally to a simple algebraic test. Relevant examples illustrate the theory.

1. INTRODUCTION

Thanks to their importance in applications, mechanical control systems have recently received a great deal of attention in the control literature. While they belong to the class of nonlinear system, a rich additional structure can be exploited in the analysis of control theoretical questions.

This paper constitutes an application of the controllability theory for mechanical systems recently proposed by Lewis and Murray [LM95] and [Lew95]. We consider the class of simple mechanical systems, that is systems whose Lagrangian is the sum of potential and kinetic energy. It is then of interest to introduce controllability notions that only involve configuration variables: for example we want to characterize systems that can be steered between two equilibrium points. Indeed, for these weaker notions of controllability, computable tests have been successfully introduced and applied to instructive examples, see [LM95]. The main computational tool is the *symmetric product*, a new object defined on the configuration manifold.

Within this framework, we here study the case in which the configuration space is a Lie group. In particular, we introduce the notion of an invariant

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affine connection to gain insight into these new controllability tests and in particular into the symmetric product. As our main result, we characterize the notion of equilibrium controllability in terms of an *algebraic* test that involves metric and Lie group structure. This result is obtained thanks to the original controllability definitions, which are in this way shown to be well-suited for the class of mechanical systems. To better illustrate the theory, we then apply this test to two notable examples, the rigid body with external torques and the forced planar rigid body and relate our results to the standard ones described for example in [NvdS90].

The literature on the subject is quite vast. The interest in control systems on Lie groups dates back to the work of Brockett [Bro72]. Later, Crouch [Cro84] works on the rigid body with internal and external torques and obtains conditions for accessibility. In the work of Bonnard [Bon84], the compactness of $SO(3)$ is exploited in proving controllability results (as opposed to accessibility ones). A standard treatment is contained in [NvdS90, page 88].

The paper is organized as follows. In Section 2 we introduce the notion of an affine connection and an invariant affine connection on a Lie group. In Section 3 we review the definition of simple mechanical control systems. In Section 4 we describe the controllability definitions and tests introduced in [LM95] and then we specialize the results to systems defined on Lie groups obtaining an algebraic test. Finally, Section 5 contains two relevant examples: the forced planar rigid body and the rigid body with external torques. In Section 6, future avenues of research are outlined.

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2. Invariant affine connections on Lie groups

We refer to [AM87, Section 2.7] for an introduction to Riemannian geometry and to [Hel78, Section II.3] and [Arn89, Appendix B] for the notion of left-invariant connections on a Lie group.

Let M be a Riemannian manifold, denote with $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ its metric tensor and with the symbols $^\flat : TM \to T^*M$ and $^\sharp : T^*M \to TM$ the musical isomorphisms. An *affine connection* on M is a map that assigns to each pair of smooth vector fields X, Y a smooth vector field $\nabla_X Y$ such that

i) $\nabla_{fX} Y = f \nabla_X Y$ and

ii) $\nabla_X fY = f\nabla_X Y + \mathcal{L}_X f\, Y$ for all $f \in C^\infty(M)$.

Given any three vector fields X, Y, Z on M, we say that the affine connection ∇ on M is *torsion-free* if $[X, Y] = \nabla_X Y - \nabla_Y X$, and is *compatible* with the metric $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ if $\mathcal{L}_X \langle\!\langle Y, Z \rangle\!\rangle = \langle\!\langle \nabla_X Y, Z \rangle\!\rangle + \langle\!\langle Y, \nabla_X Z \rangle\!\rangle$.

There exists a unique torsion-free affine connection ∇ on M compatible with the metric. We call this ∇ the Riemannian (or Levi-Civita) connection. □

We now specialize these results to Lie groups. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. Given $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathfrak{g}}$ an an inner product on $\mathfrak{g}$, we obtain a metric structure on TG by left-translation. Such a Riemannian metric is by construction *left-invariant*, as it is preserved by all left translations L_g.

Definition 1. An affine connection ∇ is said to be *left-invariant* if

$$(\nabla_X Y)_g = T_e L_g (\nabla_X Y)_e$$

for each pair of left-invariant vector fields X, Y on G.

Proposition 1. *The Riemannian connection of a left-invariant metric is also left-invariant. We denote by $\overline{\nabla} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ its restriction to the identity and we call it the* ***reduced connection****. For all $\xi, \eta \in \mathfrak{g}$, we have*

$$\overline{\nabla}_\xi \eta = \frac{1}{2}[\xi, \eta]_{\mathfrak{g}} - \frac{1}{2}\left(\operatorname{ad}^*_\xi \eta^\flat + \operatorname{ad}^*_\eta \xi^\flat \right)^\sharp, \tag{2.1}$$

where $[\cdot, \cdot]_{\mathfrak{g}}$ is the Lie bracket on $\mathfrak{g}$, $\operatorname{ad}_\xi \eta = [\xi, \eta]_{\mathfrak{g}}$ and ad^*_ξ is the dual operator of ad_ξ on $\mathfrak{g}^*$.

Remark 1 (Symmetric product)**.** On the Lie algebra $\mathfrak{g}$, note the decomposition of the covariant derivative into skew-symmetric and symmetric terms

$$2\overline{\nabla}_\xi \eta = [\xi, \eta]_{\mathfrak{g}} + \langle \xi : \eta \rangle_{\mathfrak{g}},$$

where we call

$$\langle \xi : \eta \rangle_{\mathfrak{g}} \triangleq \overline{\nabla}_\xi \eta + \overline{\nabla}_\eta \xi \; = \; -\left(\operatorname{ad}^*_\xi \eta^\flat + \operatorname{ad}^*_\eta \xi^\flat \right)^\sharp$$

the *symmetric product* of ξ and η. We shall see later the meaning of this definition in a control theoretic setting.

3. Mechanical systems: Euler-Lagrange and Euler-Poincaré equations

Using the notion of affine connections introduced in the previous section, we review mechanical control systems and write the Euler-Lagrange equations in an invariant form, see [AM87]. In the Lie group case, the notion of invariant connections may be used to rederive the Euler-Poincaré equations.

A *simple mechanical control system* is defined by a Riemannian metric $\langle\langle\cdot,\cdot\rangle\rangle$ on a configuration manifold Q (defining the kinetic energy), a function V on Q (defining the potential energy), and m one-forms, $F^1,\dots,F^m$, on Q (defining the inputs).

Let us denote with $q(t)\in Q$ the configuration of the system and with $\dot q(t)\in T_qQ$ its velocity. Using the formalism introduced in the previous section, the Euler-Lagrange equations for a simple mechanical control system can be written as

$$\nabla_{\dot q(t)}\dot q(t) = \boldsymbol{d}V^\sharp(q(t)) + u^a(t)Y_a(q(t)), \tag{3.1}$$

where ∇ is the Riemannian connection associated with $\langle\langle\cdot,\cdot\rangle\rangle$ and $Y_a=(F^a)^\sharp$ are the input vector fields. □

A *simple mechanical control system on a Lie group* is defined by a left-invariant Riemannian metric $\langle\langle\cdot,\cdot\rangle\rangle$ on the configuration group G (defining the kinetic energy), and m left-invariant one-forms, $F^1,\dots,F^m$, on G (defining the inputs).

Note that no non-trivial potential energy can be defined in a left-invariant fashion. Since the forms F^a are left-invariant, they are determined by their values at the identity $f^a=(F^a)_e\in\mathfrak{g}^*$. In particular we call $y_a=(f^a)^\sharp\in\mathfrak{g}$ the *input vectors.*

Proposition 2 (Euler-Poincaré equations)**.** *Consider a simple mechanical control system on a Lie group. For a curve $g(t)$ in G, define a curve ξ in $\mathfrak{g}$ by $t\mapsto\xi(t)=T_{g(t)}L_{g(t)^{-1}}(\dot g(t))$. Then the following are equivalent:*

i) $g(t)$ satisfies the Euler-Lagrange equations (3.1) *on G;*

*ii) the **Euler-Poincaré equations** hold:*

$$\dot\xi = (\mathrm{ad}^*_\xi\,\xi^\flat)^\sharp + u^a y_a. \tag{3.2}$$

Note that if $g\in G$ is the system configuration and $\dot g\in T_gG$ the velocity, $\xi=g^{-1}\dot g\in\mathfrak{g}$ is the velocity in "body-frame". We omit the proof for brevity and refer to [SW86, Section 27, "Variations on a theme by Euler"].

4. Configuration controllability of Mechanical Systems: in general and in the Lie group case

We start by reviewing the results obtained in [LM95] and we restrict ourselves to simple mechanical systems with no potential energy. Let the manifold Q be the configuration space, $q_0 \in Q$ and U be a neighborhood of q_0. Denote with $0_{q_0} \in T_{q_0}Q$ the zero tangent vector at q_0. We define

$$\begin{aligned}\mathcal{R}_Q^U(q_0, T) = \{q \in Q \mid \ & \text{there exists a solution } (c, u) \text{ of } (3.1) \\ & \text{such that } c'(0) = 0_{q_0},\ c(t) \in U \text{ for } t \in [0, T], \text{ and } c'(T) \in T_qQ\}\end{aligned}$$

and denote

$$\mathcal{R}_Q^U(q_0, \leq T) = \bigcup_{0 \leq t \leq T} \mathcal{R}_Q^U(q_0, t).$$

Notice that in the definition of $\mathcal{R}_Q^U(q_0, \leq T)$ we restrict our interest to initial conditions in the zero section of TQ, that is the set of zero tangent vectors. We now introduce our notions of controllability.

Definition 2. We shall say that (3.1) is *locally configuration accessible* at $q_0 \in Q$ if there exists $T > 0$ such that $\mathcal{R}_Q^U(q_0, \leq t)$ contains a non-empty open subset of Q for all neighborhoods U of q_0 and all $0 < t \leq T$. If this holds for any $q_0 \in Q$ then the system is called *locally configuration accessible.*

We shall say that (3.1) is *small-time locally configuration controllable* (STLCC) at q_0 if it is locally configuration accessible at q_0 and if there exists $T > 0$ such that q_0 is in the interior of $\mathcal{R}_Q^U(q_0, \leq t)$ for every neighborhood U of q_0 and $0 < t \leq T$. If this holds for any $q_0 \in Q$ then the system is called *small-time locally configuration controllable.*

We shall say that (3.1) is *equilibrium controllable* if, for q_1, q_2 equilibrium points of L, there exists a solution (c, u) of (3.1) where $c : [0, T] \to Q$ is such that $c(0) = q_1$, $c(T) = q_2$ and both $c'(0)$ and $c'(T)$ are zero.

Note that the definitions only refer to configuration variables, since we are interested in zero velocity states. □

We now need to recall the definition of the symmetric product. If X and Y are vector fields on Q, we define

$$\langle X : Y \rangle = \nabla_X Y + \nabla_Y X$$

to be the *symmetric product* of X and Y. If $\mathcal{V}$ is a family of vector fields on Q, we shall denote by $\overline{\mathrm{Lie}}(\mathcal{V})$ the *involutive closure* of $\mathcal{V}$, i.e. the set of vector fields on Q defined by taking iterated Lie brackets of vector fields in $\mathcal{V}$. In like fashion we define $\overline{\mathrm{Sym}}(\mathcal{V})$ to be the collection of vector fields obtained by taking iterated symmetric products of vector fields from $\mathcal{V}$ and we call this collection the *symmetric closure* of $\mathcal{V}$. In the following, let $\mathcal{Y} = \{Y_1, \ldots, Y_m\}$, where the Y_i are the input vector fields of the simple mechanical control system in equation (3.1). We say that a symmetric product from $\overline{\mathrm{Sym}}(\mathcal{Y})$ is *bad* if it contains an even number of each of the vector fields in $\mathcal{Y}$. A symmetric product which is not bad is called *good.* [1] The main result in [LM95] can be stated as follows:

Theorem 1 (Lewis-Murray). *The system* (3.1) *is*

i) locally configuration accessible at $q \in Q$ if $\mathrm{rank}(\overline{\mathrm{Lie}}(\overline{\mathrm{Sym}}(\mathcal{Y}))(q)) = \dim(Q)$,

ii) STLCC at $q \in Q$ if it is locally configuration accessible at q and if every bad symmetric product can be written as a linear combination of good symmetric products of lower order at q, and

iii) equilibrium controllable if it is STLCC at each $q \in Q$.

Note that these controllability tests are expressed on the configuration manifold Q and not on the full phase space TQ. This reflect the definitions, which also were expressed only in terms of configuration variables. □

We now focus our attention on mechanical control systems on Lie groups defined by the forced Euler-Poincaré equations (3.2). Recall (from Remark 1) the definition of the symmetric product $\langle \cdot : \cdot \rangle_{\mathfrak{g}} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ as

$$\langle \xi : \eta \rangle_{\mathfrak{g}} = -\left(\mathrm{ad}^*_{\xi} \eta^{\flat} + \mathrm{ad}^*_{\eta} \xi^{\flat} \right)^{\sharp}.$$

In the following, all the vector fields involved in the computations are left-invariant. Therefore, symmetric and involutive closures can be computed using linear algebra on $\mathfrak{g}$. Given a family of Lie algebra elements $\mathcal{V}$, we denote by $\overline{\mathrm{Lie}}_{\mathfrak{g}}(\mathcal{V})$ and by $\overline{\mathrm{Sym}}_{\mathfrak{g}}(\mathcal{V})$ the involutive and symmetric closure of $\mathcal{V}$ in $\mathfrak{g}$. Let $\mathcal{Y} = \{y_1, \dots, y_m\}$ be the set of the input vectors in the Lie algebra $\mathfrak{g}$. We can now state a stronger version of the previous theorem:

Proposition 3. *The system* (3.2) *is*

i) locally configuration accessible if $\mathrm{rank}(\overline{\mathrm{Lie}}_{\mathfrak{g}}(\overline{\mathrm{Sym}}_{\mathfrak{g}}(\mathcal{Y}))) = \dim(G)$ *and*

ii) equilibrium controllable if it is locally configuration accessible and if every bad symmetric product can be written as a linear combination of good symmetric products of lower order.

Remark 2 (Invariance implies algebraic computation scheme). The controllability properties stated in the theorem are independent of the base point $g \in G$, as the invariance of the original system suggests. As a consequence, the conditions for configuration controllability of the original nonlinear system are now expressed in a purely algebraic way (no differentiation is required).

[1] Note that to make this definitions more precise, we need the notion of symmetric free algebras, as described in [LM95].

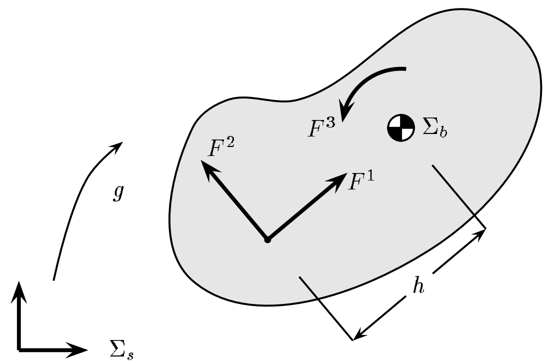

FIGURE 1. Forced planar rigid body: the configuration variable $g \in SE(2)$ determines the position of the body frame Σ_b with respect to the spatial frame Σ_s. Notice the positions of application of the various forces.

5. Examples

We illustrate the results through two examples: the forced planar rigid body and the rigid body with external torques.

We start by writing matrix expressions for the Euler-Poincaré equations and the symmetric product. Let $\{e_1, \dots, e_n\}$ be a basis for $\mathfrak{g}$ and let the metric tensor have matrix representation $\mathbb{I}$ (moment of inertia). Recall the definition of the input vectors $y_a = (f^a)^\sharp = \mathbb{I}^{-1} f^a$, where f^a and y_a are now understood to be the components of these quantities with respect to the basis. Then we have

$$\mathbb{I}\,\dot{\xi} = \mathrm{ad}_\xi^T (\mathbb{I}\,\xi) + u_a f^a,$$

(where we redefine $u_a = u^a$) and

$$\langle x : y \rangle_{\mathfrak{g}} = -\mathbb{I}^{-1}\big(\,\mathrm{ad}_x^T\,\mathbb{I} y + \mathrm{ad}_y^T\,\mathbb{I} x\big). \tag{5.1}$$

Example 1 (Forced planar rigid body)**.** The mechanical system is depicted in Figure 1. Let $g \in SE(2)$ be the configuration of the system. Let (x, y, θ) be the standard coordinate chart for $SE(2)$, let

$$\begin{aligned}
\widetilde{e}_1 &= \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \\
\widetilde{e}_2 &= \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y} \\
\widetilde{e}_3 &= \frac{\partial}{\partial \theta}
\end{aligned} \tag{5.2}$$

be a left-invariant basis for $T_g SE(2)$ and let $\{\widetilde{e}^1, \widetilde{e}^2, \widetilde{e}^3\}$ be its dual basis. Denote with $\langle \cdot, \cdot \rangle$ the pairing of vector fields and one-forms on $SE(2)$ and its value at the identity $\mathfrak{se}(2)$. Then we write $\langle \widetilde{e}^i, \widetilde{e}_j \rangle = \langle e^i, e_j \rangle = \delta^i_j$, where $\{e_i\}$ and $\{e^i\}$ are the corresponding bases of $\mathfrak{se}(2)$ and $\mathfrak{se}(2)^*$. Consider the left-invariant metric tensor

$$\begin{aligned}\mathbb{I} &= m dx \otimes dx + m dy \otimes dy + J d\theta \otimes d\theta \\ &= m(\widetilde{e}^1 \otimes \widetilde{e}^1 + \widetilde{e}^2 \otimes \widetilde{e}^2) + J\widetilde{e}^3 \otimes \widetilde{e}^3.\end{aligned}$$

The control inputs consist of forces applied at a distance $h \neq 0$ from the center of mass and a torque about the center of mass. See Figure 1. We write the control one-forms as

$$F^1 = \widetilde{e}^1, \qquad F^2 = \widetilde{e}^2 - h\widetilde{e}^3, \qquad F^3 = \widetilde{e}^3$$

from which we compute the input vectors as

$$y_1 = \frac{1}{m} e_1, \qquad y_2 = \frac{1}{m} e_2 - \frac{h}{J} e_3, \qquad y_3 = \frac{1}{J} e_3.$$

We can now compute the meaningful symmetric products and Lie brackets; see [LM95]. With the aid of Mathematica, we have

$$\begin{aligned}
&\langle y_1 : y_2\rangle_{\mathfrak{se}(2)} = -(h/Jm)e_2 \quad (1{:}2) && \langle y_1 : y_1\rangle_{\mathfrak{se}(2)} = 0 \quad (1{:}1)\\
&\langle y_1 : y_3\rangle_{\mathfrak{se}(2)} = (1/Jm)e_2 \quad (1{:}3) && \langle y_2 : y_2\rangle_{\mathfrak{se}(2)} = (2h/Jm)e_1 \quad (2{:}2)\\
&\langle y_2 : y_3\rangle_{\mathfrak{se}(2)} = -(1/Jm)e_1 \quad (2{:}3) && \langle y_3 : y_3\rangle_{\mathfrak{se}(2)} = 0. \quad (3{:}3)
\end{aligned}$$

The controllability results for the forced planar rigid body are summarized in the following table; "no" means failure of the rank test.

Available control inputs	Local configuration accessibility	Equilibrium controllability
y_1, y_2	yes (1:2)	yes
y_1, y_3	yes (1:3)	yes
y_2, y_3	yes (2:3)	yes
y_1	no (1:1)	no (of course)
y_2	yes (2:2)	? (sufficient test fails)
y_3	no (3:3)	no (of course)

□

Example 2 (Rigid body with external torques)**.** Let $g \in SO(3)$ be the configuration of the system (that is the attitude of the rigid body). Let $\{e_1, e_2, e_3\}$ be the canonical basis of $\mathfrak{so}(3)$ and $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3\}$ be the corresponding left-invariant basis for $T_g SO(3)$. Consider the metric tensor

$$\mathbb{I} = \sum_{i=1}^{3} J_i \widetilde{e}^i \otimes \widetilde{e}^i.$$

The control inputs consist of external torques about the center of mass. We write the control one-forms as $F^i = \widetilde{e}^i$, [2] and we compute the input vectors as $y_i = e_i / J_i$.

We can now check for local configuration accessibility and equilibrium controllability: note that our results show a close similarity to the standard treatment in [NvdS90, Example 3.23 page 88], where controllability for the full system is considered.

Two actuators: Assuming the two actuators are aligned along the first two principal axes,

$$\langle y_1 : y_2 \rangle_{\mathfrak{so}(3)} = \frac{J_2 - J_1}{J_1 J_2 J_3} e_3,$$

and local configuration accessibility follows for $J_1 \neq J_2$, that is for an *asymmetric rigid body*. Also, since $\langle y_i : y_i \rangle_{\mathfrak{so}(3)} = 0$ for $i = 1, 2$, all "bad" symmetric products vanish and the system is equilibrium controllable.

For the *symmetric rigid body*, that is when $J_1 = J_2$, the system is locally configuration controllable since

$$[y_1, y_2]_{\mathfrak{so}(3)} = \frac{1}{J_1 J_2} e_3.$$

Hence equilibrium controllability is achieved through the involutive closure. Indeed it is clear that, due to the additional symmetry of the inertia tensor, there is a conserved quantity: $\omega_3 = 0$.

One actuator: Assuming that the available actuator is aligned with any of the principal axes, we have

$$\langle y_1 : y_1 \rangle_{\mathfrak{so}(3)} = 0 \quad \text{and} \quad [y_1, y_1]_{\mathfrak{so}(3)} = 0,$$

so that the system is not locally configuration accessible.

Assume now that y is a generic input vector, not aligned along any of the principal axes. We write it as $y = \alpha^i e_i = \begin{bmatrix} \alpha^1 & \alpha^2 & \alpha^3 \end{bmatrix}^T$ and compute

$$\langle y : y \rangle_{\mathfrak{so}(3)} = \alpha^1 \alpha^2 \alpha^3 \begin{bmatrix} (J_3 - J_2)/(J_1 \alpha^1) \\ (J_1 - J_3)/(J_2 \alpha^2) \\ (J_2 - J_1)/(J_3 \alpha^3) \end{bmatrix}.$$

Local configuration accessibility is achieved as long as $\langle y : y \rangle_{\mathfrak{so}(3)}$ is not parallel to y. The sufficient conditions for STLCC fails.

□

[2]Hence the inputs are exerted along the principal inertia axes. We do this for the sake of simplicity.

6. Conclusion

In this paper controllability problems for mechanical systems on Lie groups have been analyzed using invariant affine connections. For these systems, the general theory developed in [LM95] leads to a simple algebraic test and to a detailed treatment of instructive examples, like the rigid body with external torques and the forced planar rigid body. Future directions of research will focus on the following themes:

1. Theoretical controllability issues on Lie groups deserve some dedicated attention. In particular we plan to relate our results to the ones contained in [Bon84] and [Bai81].
2. Systems with symmetry breaking forces also constitute an important extension. In particular, the forced planar rigid body with gravity is a simple example with interesting features. Also, we would like to capture examples like the rigid body with internal rotors, which belong to the more general class of mechanical systems with symmetry.
3. The constructive controllability problem is to design a methodology for the stabilization of (equilibrium controllable) systems with time-varying inputs, see for example [Leo95].

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