CONVERGENCE ANALYSIS OF THE SIGN ALGORITHM FOR ADAPTIVE FILTERING

by

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Abstract

We consider the convergence analysis of the sign algorithm for adaptive filtering when the input processes are uncorrelated and Gaussian. Asymptotic time-averaged convergence results for the mean deviation error, mean-square deviation error, and for the signal estimation error are established. These results are shown to hold for arbitrary step size $\mu > 0$.

Key Words: Adaptive filtering sign algorithm, convergence analysis.

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I. INTRODUCTION

Adaptive linear estimation methods based on the principle of steepest descent and its variations have been applied to a wide range of problems such as filtering, noise canceling, line enhancement, antenna processing, and interference suppression. The most widely used algorithm is the LMS algorithm whose convergence properties have been extensively studied. In this paper we are concerned with the convergence analysis of the sign algorithm [1], [3], [6],[8], [9]. The updated equation for the vector $\mathbf{h}(j)$ of the estimated filter's coefficients at iteration j is given by

$$\mathbf{h}(j+1) = \mathbf{h}(j) + \mu\,\mathbf{x}(j)\,\mathrm{sgn}\,[e(j)], \quad j = 1, 2, \cdots \tag{1.1}$$

where $\mathbf{x}(j)$ is the data at iteration j, $e(j)$ is the error in estimating the desired signal $d(j)$ using the data vector $\mathbf{x}(j)$,

$$e(j) = d(j) - \mathbf{h}^T(j)\,\mathbf{x}(j)\,, \tag{1.2}$$

and μ is the adaptation size. Here $\mathbf{h}(j)$ and $\mathbf{x}(j)$ are column vectors with dimension N; $\mathbf{h}^T(j)$ is the transpose of $\mathbf{h}(j)$. The initial estimate $\mathbf{h}(1)$ is assumed to be nonrandom. It is assumed that the processes $\{d(j)\}_{j=1}^{\infty}$ and $\{\mathbf{x}(j)\}_{j=1}^{\infty}$ are jointly stationary with finite second moments. Let $R = E[\mathbf{x}(j)\,\mathbf{x}^T(j)]$ and $\mathbf{b} = E[\mathbf{x}(j)\,d(j)]$. Throughout it is assumed that R is positive definite with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$. The optimal linear Wiener-Hopf filter $\mathbf{h}_{opt}$ is the solution of the linear equation $R\,\mathbf{h}_{opt} = \mathbf{b}$ with corresponding minimum mean-square error $\varepsilon_{\min}^2$.

Let

$$\mathbf{v}(j) = \mathbf{h}(j) - \mathbf{h}_{opt} \tag{1.3}$$

be the deviation error in estimating the Wiener-Hopf coefficients and let $K(j)$ be its covariance matrix

$$K(j) = E\,[\mathbf{v}(j)\,\mathbf{v}^T(j)]. \tag{1.4}$$

If $\|\mathbf{v}(j)\|$ denotes the Euclidean norm of the vector $\mathbf{v}(j)$, then the mean square deviation error for the filter's coefficients (MSD) is given by

$$E[\,\|\mathbf{v}(j)\|^2] = \mathrm{tr}\,[K(j)]\,. \tag{1.5}$$

The standard measures of performance of the adaptive algorithm are

$$\limsup_{j\to\infty}\; E[\|\mathbf{v}(j)\|^2]\,,$$

for the MSD error, and

$$\limsup_{j\to\infty} E[e^2(j)]\,,$$

for the signal estimation error. In view of the known results for the LMS algorithm (see, for example [4], [5], [7]), the usual goal is to show that for the sign algorithm we have

$$\limsup_{j\to\infty}\; E[\|\mathbf{v}(j)\|^2] \le \mu\, C_1 \tag{1.6}$$

and

$$E[e^2(j)] = \varepsilon^2_{\min} + \varepsilon^2(j) \tag{1.7a}$$

with

$$\limsup_{j\to\infty} \varepsilon^2(j) \le \mu\, C_2 \tag{1.7b}$$

where C_1 and C_2 are positive constants.

Gersho [6] provided a rigorous analysis of the sign algorithm and showed that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} E[\,|e(j)|\,] \le \tilde{\varepsilon}_{\min} + \frac{1}{2}\mu\; \mathrm{tr}\,[R] \tag{1.8}$$

for <u>any</u> step size $\mu > 0$, where $\tilde{\varepsilon}_{\min}$ is the least mean-absolute error minimizing $E\,|\,d(j) - \mathbf{h}^T\,\mathbf{x}(j)\,|$. In Mathews and Cho [8] it is further assumed that $\{\mathbf{x}(j), d(j)\}_{j=1}^{\infty}$ are jointly Gaussian i.i.d. random variables with zero means. Results of the form (1.7) (with equality in (1.7b) and limsup replaced by limit) are obtained in [8] under the following crucial assumption. Let $z_{j-1} = \{\mathbf{x}(i), d(i)\}_{i=1}^{j-1}$ be the past of the input processes. Define the conditional expectation

$$\sigma^2_{e\,|\,z}(j) \triangleq E[e^2(j)\;|\;z_{j-1}] \tag{1.9}$$

and

$$\sigma_e^2(j) \triangleq E[e^2(j)] = E[\sigma_{e|z}^2(j)]\,. \tag{1.10}$$

The analysis in [8] is based on the assumption that for small μ, the random variable $\sigma_{e|z}^2(j)$ is in fact a constant, i.e.,

$$\sigma_{e|z}^2(j) = \sigma_e^2(j) \quad \text{almost surely.} \tag{1.11}$$

The purpose of this paper is twofold: First we establish in Section II exact recursive equation for the covariance matrix $K(j)$ of (1.4) and its trace $E[\|\mathbf{v}(j)\|^2]$ (under the Gaussian i.i.d assumption but <u>without</u> using the unverifiable assumption (1.11)) and discuss its implications regarding the convergence of the sign algorithm. The primary contribution of the paper is given is Section III where asymptotic time-averaged convergence is established for the mean deviation error $E[\|\mathbf{v}(j)\|]$ (Theorem 3.1), mean-square deviation error $E[\|\mathbf{v}(j)\|^2]$ (Theorem 3.2), and for the signal estimation error $E[e^2(j)]$ (Theorem 3.3). These results are shown to hold for <u>all</u> step sizes $\mu > 0$ in contrast to the behavior (1.6)-(1.7) of the LMS algorithm and the analysis of the sign algorithm in [8] where μ is assumed small.

Throughout this paper it is assumed that the $\{\mathbf{x}(j), d(j)\}_{j=1}^{\infty}$ are jointly Gaussian i.i.d. random variables with zero means, positive definite covariance matrix $R = E[\mathbf{x}(j)\,\mathbf{x}^T(j)]$ with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$.

II. EXACT RECURSION FOR THE COVARIANCE MATRIX OF THE DEVIATION ERROR

Multiplying $\mathbf{v}(j+1)$ by its transpose, using (1.1), and taking expectation we have

$$K(j+1) = K(j) + \mu^2 R + \mu\, E[\mathbf{x}(j)\,\mathbf{v}^T(j)\,\mathrm{sgn}[e(j)]] + \mu\, E[\mathbf{v}(j)\,\mathbf{x}^T(j)\,\mathrm{sgn}[e(j)]]. \tag{2.1}$$

Let

$$J = E[\mathbf{x}(j)\,\mathbf{v}^T(j)\,\mathrm{sgn}[e(j)]].$$

Conditioning on z_{j-1} and noting that $\mathbf{h}(j)$ is a measurable function of z_{j-1}, we have

$$J = E\left\{ E\left[\mathbf{x}(j)\,\mathrm{sgn}[e(j)] \mid z_{j-1} \right] \mathbf{v}^T(j) \right\}. \tag{2.2}$$

Now given z_{j-1}, the random variables $\mathbf{x}(j)$ and $e(j)$ are conditionally jointly Gaussian with zero means. Hence (cross covariance property),

$$E[\mathbf{x}(j)\,\mathrm{sgn}[e(j)] \mid z_{j-1}] = E[\mathbf{x}(j)\,e(j) \mid z_{j-1}]\frac{E[e(j)\,\mathrm{sgn}[e(j)] \mid z_{j-1}]}{E[e^2(j) \mid z_{j-1}]}$$

$$= E[\mathbf{x}(j)\,e(j) \mid z_{j-1}]\frac{E[\,|e(j)| \mid z_{j-1}]}{E[e^2(j) \mid z_{j-1}]}. \tag{2.3}$$

We evaluate the terms on the right side of (2.3). Using (1.2) we have

$$E[\mathbf{x}(j)\,e(j) \mid z_{j-1}] = E[\mathbf{x}(j)\,d(j) \mid z_{j-1}] - E[\mathbf{x}(j)\,\mathbf{x}^T(j)\,\mathbf{h}(j) \mid z_{j-1}]$$

and since $\{\mathbf{x}(j), d(j)\}$ is independent of z_{j-1} and $\mathbf{h}(j)$ is a measurable function of z_{j-1}, we obtain

$$E[\mathbf{x}(j)\,e(j) \mid z_{j-1}] = E[\mathbf{x}(j)\,d(j)] - E[\mathbf{x}(j)\,\mathbf{x}^T(j)]\,\mathbf{h}(j)$$

$$= \mathbf{b} - R\,\mathbf{h}(j) = \mathbf{b} - R\,\mathbf{h}_{opt} - R\,[\mathbf{h}(j) - \mathbf{h}_{opt}]$$

$$= -R\,\mathbf{v}(j). \tag{2.4}$$

Next, we write

$$e(j) = [d(j) - \mathbf{x}^T(j)\,\mathbf{h}_{opt}] - \mathbf{x}^T(j)\,\mathbf{v}(j).$$

$$\equiv e_{\min} - \mathbf{x}^T(j)\,\mathbf{v}(j)\,. \tag{2.5}$$

The first term $e_{\min}$ on the right side of (2.5) is orthogonal to $\mathbf{x}(j)$ so that, given z_{j-1}, it is orthogonal to $\mathbf{x}^T(j)\,\mathbf{v}(j)$. Thus

$$\sigma^2_{e\,|z}(j) = E[e^2(j)\mid z_{j-1}] = E[(d(j) - \mathbf{x}^T(j)\,h_{opt})^2 \mid z_{j-1}] + E[\mathbf{v}^T(j)\,\mathbf{x}(j)\,\mathbf{x}^T(j)\,\mathbf{v}(j)\mid z_{j-1}]$$

$$= \varepsilon^2_{\min} + \mathbf{v}^T(j)\,R\,\mathbf{v}(j)\,. \tag{2.6}$$

Finally, since $e(j)$ is conditionally Gaussian, given z_{j-1}, we have

$$E[\,|e(j)|\mid z_{j-1}] = (2/\pi)^{1/2}\{E[e^2(j)\mid z_{j-1}]\}^{1/2}\,. \tag{2.7}$$

It then follows from (2.3)-(2.7) that

$$E[\mathbf{x}(j)\,\mathrm{sgn}[e(j)]\mid z_{j-1}] = -(2/\pi)^{1/2}\,\frac{R\,\mathbf{v}(j)}{[\varepsilon^2_{\min} + \mathbf{v}^T(j)\,R\,\mathbf{v}(j)]^{1/2}} \tag{2.8}$$

and by (2.2)

$$E[\mathbf{x}(j)\mathbf{v}^T(j)\mathrm{sgn}[e(j)]] = -(2/\pi)^{1/2}\,E\left[\frac{R\,\mathbf{v}(j)\,\mathbf{v}^T(j)}{[\varepsilon^2_{\min} + \mathbf{v}^T(j)\,R\,\mathbf{v}(j)]^{1/2}}\right].$$

Thus, by (2.1) we have the exact expression

$$K(j+1) = K(j) + \mu^2 R - (2/\pi)^{1/2}\,\mu\,E\left[\frac{R\,\mathbf{v}(j)\,\mathbf{v}^T(j) + \mathbf{v}(j)\,\mathbf{v}^T(j)\,R}{[\varepsilon^2_{\min} + \mathbf{v}^T(j)\,R\,\mathbf{v}(j)]^{1/2}}\right]. \tag{2.9}$$

From (2.9) we immediately obtain the recursive equation for the MSD error: By (1.5) and $\mathrm{tr}\,[\mathbf{v}(j)\,\mathbf{v}^T(j)\,R] = \mathbf{v}^T(j)\,R\,\mathbf{v}(j)$ we have

$$E[\,\|\mathbf{v}(j+1)\|^2] = E[\,\|\mathbf{v}(j)\|^2] + \mu^2\mathrm{tr}\,[R] - 2(2/\pi)^{1/2}\,\mu\,E\left[\frac{\mathbf{v}^T(j)\,R\,\mathbf{v}(j)}{[\varepsilon^2_{\min} + v^T(j)\,R\,\mathbf{v}(j)]^{1/2}}\right]. \tag{2.10}$$

Note that the expressions (2.9) and (2.10) are exact and hold for any step size $\mu > 0$. If we could establish an asymptotic result of the form (1.6) for the MSD error by using (2.10), then the corresponding result for the signal estimation error (1.7) would be obtained easily:

$$E[e^2(j)] = \varepsilon_{\min}^2 + \varepsilon^2(j) \tag{2.11}$$

and by (2.6)

$$\varepsilon^2(j) = E[\mathbf{v}^T(j) R \,\mathbf{v}(j)] \le \lambda_N E[\,\|\mathbf{v}(j)\|^2] \tag{2.12}$$

so that

$$\limsup_{j \to \infty} \varepsilon^2(j) \le \lambda_N \limsup_{j \to \infty} E[\,\|\mathbf{v}(j)\|^2] \le \lambda_N C_1 \mu \equiv C_2 \mu \,.$$

In order to obtain the asymptotic result for the MSD error, we need to evaluate the expectation of the ratio in the third term on the right side of (2.10). This is not feasible since the distribution of $\mathbf{v}(j)$ is unknown. It would suffice of course to obtain a lower bound on the third term on the right side of (2.10). Consider the function

$$g(x) = \frac{x}{(a^2 + x)^{1/2}} \,, \quad x \ge 0 \,.$$

It is seen that

$$g''(x) = -\frac{a^2 + x/4}{(a^2 + x)^{5/2}} < 0, \quad \text{for all } \; x \ge 0 \,.$$

It follows that $g(x)$ is concave on $[0, \infty)$. Hence by Jensen's inequality [2], $E[g(Y)] \le g(E[Y])$, for any positive random variable Y for which $E[Y] < \infty$, with equality if and only if $Y = E[Y]$ almost surely. By (2.10) and Jensen's inequality we then have

$$E[\,\|\mathbf{v}(j+1)\|^2] > E[\,\|\mathbf{v}(j)\|^2] + \mu^2 \mathrm{tr}[R] - 2(2/\pi)^{1/2} \mu \, \frac{E[\mathbf{v}^T(j) R \,\mathbf{v}(j)]}{\{\varepsilon_{\min}^2 + E[\mathbf{v}^T(j) R \,\mathbf{v}(j)]\}^{1/2}} \tag{2.13}$$

for <u>any</u> $\mu > 0$ and thus no convergence result can be obtained from (2.13) by a limiting argument as $j \to \infty$. The simplified analysis in [8] corresponds to replacing the strict inequality sign in (2.13) by an equality sign. This can be justified analytically if and only if the random variable $\mathbf{v}^T(j) R \,\mathbf{v}(j)$ is actually a constant, i.e.,

$$\mathbf{v}^T(j) R \,\mathbf{v}(j) = E[\mathbf{v}^T(j) R \,\mathbf{v}(j)] \quad \text{almost surely.} \tag{2.14}$$

However, it is seen that

$$0 < \lambda_1 \|\mathbf{v}(j)\|^2 \leq \mathbf{v}^T(j)\, R\, \mathbf{v}(j) \leq \lambda_N \|\mathbf{v}(j)\|^2$$

and since the input processes are Gaussian, then for any $\mu > 0$ the positive random variable $\mathbf{v}^T(j)\, R\, \mathbf{v}(j)$ takes values in $(0, \infty)$ and cannot be equal to its expected value almost surely. The intuitive idea that the mean of the random variable $\mathbf{v}^T(j)\, R\, \mathbf{v}(j)$ is small when μ is "small" does not imply that (2.14) holds; the random variable can still take arbitrarily large values (possibly with small probabilities).

The above discussion implies that the rigorous establishment of convergence results having the classical form (1.6) for the sign algorithm remains an open question even for i.i.d. Gaussian data. However, in the next section we adopt Gersho’s notion of convergence as a long term time average and, using the recursive equations of Section II, we establish convergence results which are valid for <u>any</u> $\mu > 0$.

III. CONVERGENCE ANALYSIS

We first define the function $\beta_1(j)$ as follows:

$$\beta_1(j) \triangleq E\left[\frac{\mathbf{v}^T(j)\, R\, \mathbf{v}(j)}{[\varepsilon_{\min}^2 + \mathbf{v}^T(j)\, R\, \mathbf{v}(j)]^{1/2}} \right]. \tag{3.1}$$

Note that $\beta_1(j)$ approximates $E[\mathbf{v}^T(j)\, R\, \mathbf{v}(j)]^{1/2}$ when $\varepsilon_{\min}^2$ is negligible. We have

Lemma 3.1. For any initial weight vector $\mathbf{h}(1)$ and for any positive step size μ we have

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \beta_1(j) \le \frac{1}{2(2/\pi)^{1/2}}\, \mu \operatorname{tr}[R].$$

The theorem has the flavor of Gersho's result (1.8).

Proof. Writing (2.10) as

$$E[\,\|\mathbf{v}(j+1)\|^2] = E[\,\|\mathbf{v}(j)\|^2] + \mu^2 \operatorname{tr}[R] - 2(2/\pi)^{1/2}\mu\, \beta_1(j)$$

and iterating backward $n-1$ steps we obtain

$$E[\,\|\mathbf{v}(n+1)\|^2] = \|\mathbf{v}(1)\|^2 + n\,\mu^2 \operatorname{tr}[R] - 2(2/\pi)^{1/2}\mu \sum_{j=1}^{n} \beta_1(j)$$

where we used the fact that $\mathbf{h}(1)$ is nonrandom. Hence

$$\frac{1}{n}\sum_{j=1}^{n}\beta_1(j) = \frac{1}{2(2/\pi)^{1/2}} \left\{ \frac{\|\mathbf{v}(1)\|^2}{n\,\mu} + \mu \operatorname{tr}[R] - \frac{1}{n\,\mu} E[\,\|\mathbf{v}(n+1)\|^2] \right\}$$

$$\le \frac{1}{2(2/\pi)^{1/2}} \left\{ \frac{\|\mathbf{v}(1)\|^2}{n\,\mu} + \mu \operatorname{tr}[R] \right\} \tag{3.2}$$

for every $n \ge 1$ and $\mu > 0$. The result of the theorem now follows from (3.2). □

We next establish the following result for the convergence of the mean deviation error $E[\,\|\mathbf{v}(j)\|\,]$.

Theorem 3.1. For any initial weight vector $\mathbf{h}(1)$ and for any positive step size μ we have

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} E[\,\|\mathbf{v}(j)\|\,] \le B_1 \sqrt{\mu} + B_2\, \mu \tag{3.3}$$

where the constants B_1 and B_2 are given by

$$B_1 = \left[\frac{\varepsilon_{\min}^2 \, \mathrm{tr}\,[R]}{2\lambda_1 \sqrt{2/\pi}} \right]^{1/2} , \quad B_2 = \frac{\mathrm{tr}\,[R]}{2(\sqrt{2\lambda_1/\pi}\,)}.$$

Remark 3.1. Note that the right side of (3.3) is proportional to $\sqrt{\mu}$ — this is not unexpected since Theorem 1 gives the long term behavior of $E[\,\|\mathbf{v}(j)\|\,]$ rather than that of $E[\,\|\mathbf{v}(j)\|^2]$.

Proof. Let $Y = \mathbf{v}(j)\, R\, \mathbf{v}(j)$ and $a^2 = \varepsilon_{\min}^2$. Write

$$E[\sqrt{Y}\,] = E\left[\frac{\sqrt{Y}}{(a^2 + Y)^{1/4}} \, (a^2 + Y)^{1/4} \right].$$

By the Cauchy-Schwarz inequality we then have

$$E[\sqrt{Y}\,] \le \left\{ E\left[\frac{Y}{\sqrt{a^2 + Y}} \right] E[\sqrt{a^2 + Y}\,] \right\}^{1/2} = \beta_1^{1/2}(j) \, \{E[\sqrt{a^2 + Y}\,]\}^{1/2} .$$

Using $\sqrt{a^2 + Y} \le a + \sqrt{Y}$, we obtain

$$E[\sqrt{Y}\,] \le \beta_1^{1/2}(j) \, (a + E[\sqrt{Y}\,])^{1/2} . \tag{3.4}$$

With $u = E[\sqrt{Y}\,]$ we have the quadratic equation $u^2 - \beta_1 u - \beta_1 a \le 0$. The roots of the equation are $\beta_1/2 \pm \sqrt{(\beta_1/2)^2 + \beta_1 a}$. Hence we have

$$0 \le E[\sqrt{Y}\,] \le (\beta_1/2) + \sqrt{(\beta_1/2)^2 + \beta_1 a} \; . \tag{3.5}$$

Now $\sqrt{Y} \ge \sqrt{\lambda_1}\, \|\mathbf{v}(j)\|$ so that $E[\sqrt{Y}\,] \ge \sqrt{\lambda_1}\, E[\,\|\mathbf{v}(j)\|\,]$. Hence by (3.5)

$$E[\,\|\mathbf{v}(j)\|\,] \le \frac{1}{\sqrt{\lambda_1}} \left\{ \frac{1}{2} \beta_1(j) + \sqrt{(\beta_1(j)/2)^2 + \beta_1(j)\, a} \right\}. \tag{3.6}$$

Next consider the time-averaged error

$$\frac{1}{n} \sum_{j=1}^{n} E[\,\|\mathbf{v}(j)\|\,] \le \frac{1}{\sqrt{\lambda_1}} \left\{ \frac{1}{2n} \sum_{j=1}^{n} \beta_1(j) + \frac{1}{n} \sum_{j=1}^{n} \sqrt{(\beta_1(j)/2)^2 + \beta_1(j)\, a} \right\}. \tag{3.7}$$

We upper bound the second term on the right side of (3.7) as follows: Using the inequality

$\sqrt{x^2+y} \le x + \sqrt{y}$ and the Cauchy-Schwarz inequality

$$\frac{1}{n}\sum_{j=1}^{n}\sqrt{\beta_1(j)} \le \left\{\frac{1}{n}\sum_{j=1}^{n}\beta_1(j)\right\}^{1/2},$$

we obtain

$$\frac{1}{n}\sum_{j=1}^{n}\sqrt{(\beta_1(j)/2)^2+\beta_1(j)\,a} \;\le\; \frac{1}{2n}\sum_{j=1}^{n}\beta_1(j)+\sqrt{a}\left(\frac{1}{n}\sum_{j=1}^{n}\beta_1(j)\right)^{1/2}. \tag{3.8}$$

It follows by (3.7) and (3.8) that

$$\frac{1}{n}\sum_{j=1}^{n}E[\,\|\mathbf{v}(j)\|\,] \le \frac{1}{\sqrt{\lambda_1}}\left\{\frac{1}{n}\sum_{j=1}^{n}\beta_1(j)+\sqrt{a}\left(\frac{1}{n}\sum_{j=1}^{n}\beta_1(j)\right)^{1/2}\right\}. \tag{3.9}$$

The result now follows by substituting the upper bound (3.2) on the right side of (3.9) and taking limits. □

We now consider the asymptotic behavior of the time averaged mean-square deviation error of the filter's coefficients. We need to utilize recursive equations for the fourth order moment $E[\|\mathbf{v}(j)\|^4]$. Define

$$m_4 \triangleq E[\|\mathbf{x}(j)\|^4] \tag{3.10}$$

and

$$\beta_2(j) \triangleq E\left[\frac{\|\mathbf{v}(j)\|^2\,\mathbf{v}^T(j)\,R\,\mathbf{v}(j)}{[\varepsilon_{\min}^2+\mathbf{v}^T(j)\,R\,\mathbf{v}(j)]^{1/2}}\right]. \tag{3.11}$$

Lemma 3.2. For any initial weight vector $\mathbf{h}(1)$ and for any positive step size μ we have

$$\frac{1}{n}\sum_{j=1}^{n}\beta_2(j) \le a_1 + a_2\,\frac{1}{n}\sum_{j=1}^{n}E[\|\mathbf{v}(j)\|^2]$$

where a_1 and a_2 (which depend on μ) are given by

$$a_1 = \frac{1}{4}\sqrt{\pi/2}\left\{\frac{\|\mathbf{v}(1)^4}{\mu\,n}+m_4\mu^3\right\},\quad a_2 = \frac{1}{2}\sqrt{\pi/2}\,(\mathrm{tr}[R]+2\lambda_N)\,\mu\,. \tag{3.12}$$

Proof. Using (1.1) we have for the second moment

$$\|\mathbf{v}(j+1)\|^2 = \|\mathbf{v}(j)\|^2 + \mu^2\|\mathbf{x}(j)\|^2 + 2\mu\, \mathbf{v}^T(j)\,\mathbf{x}(j)\,\mathrm{sgn}[e(j)]\,,$$

and squaring both side, taking expectations, and noting the independence of $\mathbf{x}(j)$ and $\mathbf{v}(j)$ we obtain

$$E[\|\mathbf{v}(j+1)\|^4] = E[\|\mathbf{v}(j)\|^4] + \mu^4 E[\|\mathbf{x}(j)\|^4] + 4\mu^2 E[(\mathbf{v}^T(j)\,\mathbf{x}(j))^2] + 2\mu^2 E[\|\mathbf{x}(j)\|^2]\,E[\mathbf{v}(j)\|^2]$$

$$+ 4\mu\, E\left\{\|\mathbf{v}(j)\|^2\, \mathbf{v}^T(j)\,\mathbf{x}(j)\,\mathrm{sign}[e(j)]\right\} + 4\mu^3 E\left\{\|\mathbf{x}(j)\|^2 \mathbf{v}^T(j)\,\mathbf{x}(j)\,\mathrm{sign}[e(j)]\right\}$$

$$\equiv I_1 + I_2 + I_3 + I_4 + I_5 + I_6\,. \tag{3.13}$$

By the independence of $\mathbf{x}(j)$ and $\mathbf{v}(j)$,

$$I_3 = 4\mu^2 E[(\mathbf{v}^T(j)\,\mathbf{x}(j))^2] = 4\mu^2 E[\mathbf{v}^T(j)\,\mathbf{x}(j)\,\mathbf{x}^T(j)\,\mathbf{v}(j)] = 4\mu^2 E[\mathbf{v}^T(j)\,R\,\mathbf{v}(j)] \le 4\mu^2 \lambda_N E[\|\mathbf{v}(j)\|^2]\,. \tag{3.14}$$

By conditioning on z_{j-1}, we have for the term I_5 in (3.13)

$$I_5 = 4\mu\, E\left[\|\mathbf{v}(j)\|^2\, \mathbf{v}^T(j) E[\mathbf{x}(j)\,\mathrm{sign}[e(j)] \mid z_{j-1}]\right]$$

and using (2.8) for the inner conditional expectation we obtain

$$I_5 = -4\mu\sqrt{2/\pi}\, E\left[\frac{\|\mathbf{v}(j)\|^2\, \mathbf{v}^T(j)\,R\,\mathbf{v}(j)}{(\varepsilon_{\min}^2 + \mathbf{v}^T(j)\,R\,\mathbf{v}(j))^{1/2}}\right] = -4\sqrt{2/\pi}\,\mu\,\beta_2(j)\,. \tag{3.15}$$

We finally show that the term I_6 in (3.13) is nonpositive: Note that $e(j) = e_{\min} - \mathbf{x}^T(j)\mathbf{v}(j)$ and let A be the event

$$A = \{\, |e_{\min}| > |\mathbf{x}^T(j)\mathbf{v}(j)|\,\}$$

and A^c be its complement. Note that

$$|e_{\min}(j)| > |\mathbf{x}^T(j)\mathbf{v}(j)| \Rightarrow \mathrm{sign}[e_{\min} - \mathbf{x}^T(j)\mathbf{v}(j)] = \mathrm{sign}[e_{\min}(j)]$$

$$|e_{\min}(j)| < |\mathbf{x}^T(j)\mathbf{v}(j)| \Rightarrow \mathrm{sign}[e_{\min} - \mathbf{x}^T(j)\mathbf{v}(j)] = \mathrm{sign}[-\mathbf{x}^T(j)\mathbf{v}(j)]\,.$$

Hence

$$I_6 = E\left\{ \|\mathbf{x}(j)\|^2 \mathbf{x}^T(j)\,\mathbf{v}(j)\,\mathrm{sign}[e_{\min}]\,1_A \right\} - E\left\{ \|\mathbf{x}(j)\|^2 \;|\mathbf{x}^T(j)\mathbf{v}(j)|\; 1_{A^c} \right\} = I_6{}' + I_6{}''. \tag{3.16}$$

Note that $e_{\min}(j)$ is independent of $\{z_{j-1}, \mathbf{x}(j)\}$, while $\{\mathbf{v}(j), \mathbf{x}(j)\}$ are measurable functions of it. Thus,

$$I_6{}' = E\left\{ \|\mathbf{x}(j)\|^2 \mathbf{x}^T(j)\mathbf{v}(j)\, E\left[\mathrm{sign}[e_{\min}] 1_A \mid z_{j-1}, \mathbf{x}(j) \right] \right\}.$$

Since $e_{\min}$ is (conditionally) zero mean Gaussian random variable, the inner conditional expectation is equal to zero. Thus, $I_6{}' = 0$. Hence

$$I_6 = -E\left\{ \|\mathbf{x}(j)\|^2 \,|\mathbf{x}^T(j)\mathbf{v}(j)|\; 1_{A^c} \right\} \le 0. \tag{3.17}$$

It then follows from (3.13)-(3.17) that

$$E[\|\mathbf{v}(j+1)\|^4] \le E[\|\mathbf{v}(j)\|^4] + m_4\mu^4 + [2\mathrm{tr}[R] + 4\lambda_N]\mu^2 E[\|\mathbf{v}(j)\|^2] - 4\mu\sqrt{2/\pi}\,\beta_2(j)$$

and iterating backward $n-1$ steps we obtain

$$4\mu\sqrt{2/\pi}\sum_{j=1}^{n}\beta_2(j) \le \|\mathbf{v}(1)\|^4 + m_4\mu^4 n + [2\mathrm{tr}[R] + 4\lambda_N]\mu^2 \sum_{j=1}^{n} E[\|\mathbf{v}(j)\|^2],$$

and the result follows. □

We now establish the asymptotic time-average convergence of the mean-square deviation error $E[\|\mathbf{v}(j)\|^2]$.

<u>Theorem 3.2.</u> For any initial weight vector $\mathbf{h}(1)$ and for <u>any positive step size μ</u> we have

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} E[\,\|\mathbf{v}(j)|^2] \le C_1\mu + C_2\mu^2 \tag{3.18a}$$

where the constants C_1 and C_2 are given by

$$C_1 = \frac{9}{2}\sqrt{\frac{\pi}{2}}\,\frac{\mathrm{tr}[R] + 2\lambda_N}{\lambda_1}\,\varepsilon_{\min} \tag{3.18b}$$

$$C_2 = \frac{27\pi}{8}\,\frac{\lambda_N}{\lambda_1^2}\,[\mathrm{tr}[R] + 2\lambda_N]^2 + \frac{m_4}{\mathrm{tr}[R] + 2\lambda_N}. \tag{3.18c}$$

Remark 3.2. The convergence results of Theorems 3.1 and 3.2 have the strength that they hold for any value of $\mu > 0$ (in contrast to the behavior (1.6)-(1.7) of the LMS algorithm and the heuristic result in [8] for the sign algorithm where μ must be "small").

Proof. We divide the proof in two steps. In the first step we obtain an upper bound on the mean-square deviation error $E[\|\mathbf{v}(j)\|^2]$ in terms of fractional powers of $\beta_2(j)$ defined in (3.11). In the second step, we combine this result with Lemma 3.2 to establish the theorem.

Step 1. Let Y be a real-valued random variable with finite fourth order moment and let a be a positive constant. Define

$$\gamma(a) \triangleq E\left[\frac{Y^4}{\sqrt{a^2+Y^2}}\right]. \tag{3.18}$$

We prove that

$$E[Y^2] \le \gamma^{2/3}(a) + \sqrt{a}\,\gamma^{1/2}(a). \tag{3.19}$$

We proceed as follows. Write

$$E[Y^2] = E\left[\frac{Y^2}{(a^2+Y^2)^{1/4}}(a^2+Y^2)^{1/4}\right] \le \left\{E\left[\frac{Y^4}{\sqrt{a^2+Y^2}}\right]\right\}^{1/2}\left\{E[\sqrt{a^2+Y^2}]\right\}^{1/2}$$

by the Cauchy-Schwarz inequality. Now $E[\sqrt{a^2+Y^2}] \le E[a + |Y|] \le a + \{E[Y^2]\}^{1/2}$. Thus

$$(E[Y^2])^2 \le \gamma(a)(a + \{E[Y^2]\}^{1/2}).$$

We wish to solve for $E[Y^2]$ in terms of $\gamma(a)$. With $u = (E[Y^2])^{1/2}$, the above equation is of the form

$$g(u) \triangleq u^4 - \gamma u - a\gamma \le 0.$$

It is seen that $g(0) < 0$, $g(u)$ is convex since $g''(u) \ge 0$, and $g(u) \to \infty$ as $u \to \infty$. It follows that there is a unique value $\alpha_0 > 0$ such that $g(u) \le 0$ for $0 \le u \le \alpha_0$. Hence

$$E[Y^2] \le \alpha_0^2. \tag{3.20}$$

We cannot obtain α_0 analytically but we can find an upper bound for it: it suffices to find $\alpha_1 > 0$ such that

$g(\alpha_1) > 0$. Choose

$$\alpha_1 = \sqrt{\gamma^{2/3} + (a\gamma)^{1/2}}\,. \tag{3.21}$$

Then

$$g(\alpha_1) = \alpha_1(\alpha_1^3 - \gamma) - a\gamma = \alpha_1[\alpha_1\gamma^{2/3} + \alpha_1(a\gamma)^{1/2} - \gamma] - a\gamma \geq \alpha_1^2(a\gamma)^{1/2} - a\gamma$$

since $\alpha_1 \geq \gamma^{1/3}$. Hence,

$$g(\alpha_1) \geq [\gamma^{2/3} + (a\gamma)^{1/2}](a\gamma)^{1/2} - a\gamma = a^{1/2}\gamma^{7/6} > 0\,.$$

Thus

$$E[Y^2] \leq \alpha_0^2 \leq \alpha_1^2 \tag{3.22}$$

and (3.19) follows by (3.21)-(3.22). Now by (3.11) we have

$$\beta_2(j) \geq E\left[\frac{\lambda_1 \|\mathbf{v}(j)\|^4}{\sqrt{\varepsilon_{\min}^2 + \lambda_N \|\mathbf{v}(j)\|^2}}\right] = \frac{\lambda_1}{\sqrt{\lambda_N}} E\left[\frac{\|\mathbf{v}(j)\|^4}{\sqrt{\varepsilon_{\min}^2/\lambda_N + \|\mathbf{v}(j)\|^2}}\right].$$

Hence with $Y \triangleq \|\mathbf{v}(j)\|$ and $a^2 \triangleq \varepsilon_{\min}^2/\lambda_N$ we have

$$\gamma(a) \leq \frac{\sqrt{\lambda_N}}{\lambda_1}\beta_2(j)\,. \tag{3.23}$$

Combining (3.19) and (3.23) we obtain

$$E[\|\mathbf{v}(j)\|^2] \leq \left\{\sqrt{\lambda_N}\,\beta_2(j)\right\}^{2/3} + \left\{\frac{\varepsilon_{\min}}{\lambda_1}\beta_2(j)\right\}^{1/2}. \tag{3.24}$$

We have thus obtained an upper bound on $E[\|\mathbf{v}(j)\|^2]$ in terms of fractional powers of $\beta_2(j)$.

<u>Step 2.</u> Summing up (3.24) over $j = 1, \cdots, n$ and dividing by n, we have

$$\frac{1}{n}\sum_{j=1}^{n} E[\|\mathbf{v}(j)\|^2 \leq c_1\left[\frac{1}{n}\sum_{j=1}^{n}\beta_2^{2/3}(j)\right] + c_2\left[\frac{1}{n}\sum_{j=1}^{n}\beta_2^{1/2}(j)\right]$$

and applying Holder's inequality to the first sum on the right side and the Cauchy-Schwarz inequality to the second sum, we obtain

$$\frac{1}{n}\sum_{j=1}^{n} E[\|\mathbf{v}(j)\|^2 \le c_1 \left[\frac{1}{n}\sum_{j=1}^{n}\beta_2(j)\right]^{2/3} + c_2 \left[\frac{1}{n}\sum_{j=1}^{n}\beta_2(j)\right]^{1/2}, \tag{3.25a}$$

where the constants c_1 and c_2 are given by

$$c_1 = (\sqrt{\lambda_N}/\lambda_1)^{2/3}, \quad c_2 = (\varepsilon_{\min}/\lambda_1)^{1/2.} \tag{3.25b}$$

Note that (3.25a) holds for any positive value of the step size μ and for any $n \ge 1$. Define

$$S \triangleq \frac{1}{n}\sum_{j=1}^{n} E[\|\mathbf{v}(j)\|^2], \quad U \triangleq \frac{1}{n}\sum_{j=1}^{n}\beta_2(j)$$

Then Lemma 3.2 takes the form

$$U \le a_1 + a_2 S \tag{3.26a}$$

By Step 1, eq. (3.25a) takes the form

$$S \le c_1 U^{2/3} + c_2 U^{1/2}, \tag{3.26b}$$

where the constants $\{a_i\}$ and $\{c_i\}$ are given in (3.12) and (3.25b), respectively. It is easily seen that for eqs. (3.26) to be satisfied, (S, U) lie in a bounded region. The largest values ($S_{\max}$, $U_{\max}$) are obtained as the solution of (3.26) written with equality signs. Hence $S_{\max}$ is the solution of

$$S_{\max} = c_1 (a_1 + a_2 S_{\max})^{2/3} + c_2 (a_1 + a_2 S_{\max})^{1/2} \tag{3.27}$$

or, equivalently, $U_{\max}$ is the solution of

$$U_{\max} = c_1 a_2 U_{\max}^{2/3} + c_2 a_2 U_{\max}^{1/2} + a_1 .$$

Consider the equation

$$\phi(u) \triangleq u - a_1 - c_1 a_2 u^{2/3} - c_2 a_2 u^{1/2}, \quad u \ge 0,$$

and note that $\phi(0) < 0$, $\phi(u)$ is convex since $\phi''(u) \ge 0$, and $\phi(u) \to \infty$ as $u \to \infty$. Hence, $U_{\max}$ is the unique solution of $\phi(u) = 0$. While we are unable to find $U_{\max}$ analytically, we can easily obtain an upper bound for it: it suffices to find u_0 such that $\phi(u_0) > 0$. write

$$\phi(u) = \frac{u^{2/3}}{3}(u^{1/3} - 3c_1 a_2) + \frac{u^{1/2}}{3}(u^{1/2} - 3c_2 a_2) + \frac{1}{3}(u - 3a_1).$$

Hence

$$u_0 = \max\{(3c_1 a_2)^3, (3c_2 a_2)^2, 3a_1\} \le 27(c_1 a_2)^3 + 9(c_2 a_2)^2 + 3a_1$$

Thus

$$U_{\max} \le u_0 \le 27(c_1 a_2)^3 + 9(c_2 a_2)^2 + 3a_1 \tag{3.28}$$

Now by (3.26a) and (3.27)

$$S_{\max} = \frac{U_{\max} - a_1}{a_2} \le 27c_1^3 a_2^2 + 9c_2^2 a_2 + 2a_1/a_2 .$$

Substituting the values of $\{a_i\}$ and $\{c_i\}$ from (3.12) and (3.25b) we have

$$\frac{1}{n}\sum_{j=1}^{n} E[\|\mathbf{v}(j)\|^2] \le \frac{27\pi}{8}\frac{\lambda_N}{\lambda_1^2}(\mathrm{tr}[R] + 2\lambda_N)^2\mu^2 + \frac{9}{2}\sqrt{\pi/2}\,\frac{\varepsilon_{\min}}{\lambda_1}(\mathrm{tr}[R] + 2\lambda_N)\mu + \left\{\frac{\|\mathbf{v}(1)\|^4}{n\mu} + m_4\mu^3\right\}/\{(\mathrm{tr}[R] + 2\lambda_N)\mu\} .$$

Taking limsup as $n \to \infty$ we obtain

$$\limsup_{n\to\infty} \sum_{j=1}^{n} E[\|\mathbf{v}(j)\|^2] \le C_1\mu + C_2\mu^2$$

where the constants C_1 and C_2 are given by (3.18b) and (3.18c), respectively. □

Remark 3.3. Note that the constants C_1 and C_2 are not tight since we upper bounded $U_{\max}$ by u_0 and furthermore, we replaced in (3.28) the maximum of three terms by their sum. If one is able to solve (3.27) analytically, tighter values for C_1 and C_2 will result.

We finally establish asymptotic time-averaged result for the signal estimation error $E[e^2(j)]$.

Theorem 3.3. For any initial weight vector $\mathbf{h}(1)$ and for any positive step size μ we have

$$\frac{1}{n}\sum_{j=1}^{n} E[e^2(j)] = \varepsilon_{\min}^2 + \frac{1}{n}\sum_{j=1}^{n}\varepsilon^2(j)$$

with

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}\varepsilon^2(j) \le C_3\mu + C_4\mu^2 \qquad (3.29)$$

where $C_3 = \lambda_N C_1$ and $C_4 = \lambda_N C_2$.

Remark 3.4. Note that the asymptotic time-averaged excess mean-square signal estimation error is bounded as in (3.29). This is of similar character to Gersho's result (1.8) for the asymptotic time average excess mean absolute value of the signal estimation error. Also note that Theorem 3.3 holds for any positive value of the step size μ.

Proof. By (3.12) we have

$$\frac{1}{n}\sum_{j=1}^{n}\varepsilon^2(j) \le \lambda_N \left[\frac{1}{n}\sum_{j=1}^{n} E[\|\mathbf{v}(j)\|^2]\right]$$

and the result follows by Theorem 3.2. □

IV. CONCLUDING REMARKS

We considered in this paper the convergence analysis of the sign algorithm for Gaussian i.i.d. data. We established asymptotic time-averaged convergence results for the mean deviation error $E[\|\mathbf{v}(j)\|]$ (Theorem 3.1), mean-square deviation error $E[\|\mathbf{v}(j)\|^2]$ (Theorem 3.2), and for the signal estimation error $E[e^2(j)]$ (Theorem 3.3). The results were shown to hold for all step sizes $\mu > 0$ in contrast to the behavior (1.6)-(1.7) of the LMS algorithm and the approximate analysis of the sign algorithm in [8] where μ is assumed small. The theorems were established by utilizing the exact recursions for the covariance matrix of the deviation error developed in Section II of the paper. It was pointed out that these recursive equations do not normally lead to convergence results of the form (1.6) for the mean-square deviation error.

It would be of interest to generalize the results of this paper to dependent Gaussian data and we are currently examining this issue.

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