Performance Analysis of Adaptive Filters Using the Sign Algorithm

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Abstract — **Convergence analysis of the sign algorithm for adaptive filtering with Gaussian uncorrelated input processes is presented. Asymptotic time-averaged results for the mean-square deviation error and for the signal estimation error are established. The results hold for arbitrary step size $\mu > 0$.**

I. INTRODUCTION

Adaptive linear estimation methods based on the principle of steepest descent and its variations have been applied to a wide range of problems such as filtering, noise canceling, line enhancement, antenna processing, and interference suppression. In this paper we are concerned with the convergence analysis of the sign algorithm [1] [3]. The updated equation for the vector $\mathbf{h}(j)$ of the estimated filter's coefficients at iteration j is given by

$$\mathbf{h}(j+1) = \mathbf{h}(j) + \mu\, \mathbf{x}(j) \operatorname{sgn}[e(j)], \quad j = 1,2,\cdots \tag{1}$$

where $\mathbf{x}(j)$ is the data at iteration j, $e(j)$ is the error in estimating the desired signal $d(j)$ using the data vector $\mathbf{x}(j)$,

$$e(j) = d(j) - \mathbf{h}^T(j)\mathbf{x}(j), \tag{2}$$

and μ is the adaptation size. Here $\mathbf{h}(j)$ and $\mathbf{x}(j)$ are column vectors with dimension N; $\mathbf{h}^T(j)$ is the transpose of $\mathbf{h}(j)$. The initial estimate $\mathbf{h}(1)$ is assumed to be nonrandom. The processes $\{d(j)\}_{j=1}^{\infty}$ and $\{\mathbf{x}(j)\}_{j=1}^{\infty}$ are jointly stationary with finite second moments. Let $R = E[\mathbf{x}(j)\mathbf{x}^T(j)]$ assumed to be positive definite with eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N$. Let $\mathbf{h}_{opt}$ be the optimal linear Wiener-Hopf filter with corresponding minimum mean-square error $\varepsilon^2_{\min}$. Let

$$\mathbf{v}(j) = \mathbf{h}(j) - \mathbf{h}_{opt} \tag{3}$$

be the deviation error in estimating the Wiener-Hopf coefficients. Then the mean-square deviation error for the filter's coefficients (MSD) is given by $E[\|\mathbf{v}(j)\|^2]$; $E[e^2(j)]$ is the mean-square error for the signal estimation. We have

$$E[e^2(j)] = \varepsilon^2_{\min} + \varepsilon^2(j)$$

where $\varepsilon^2(j)$ is the signal estimation "excess mean-square error". Gersho [1] provided a rigorous analysis of the sign algorithm and showed that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} E[\,|e(j)|\,] \le \tilde{\varepsilon}_{\min} + \frac{1}{2}\mu \operatorname{tr}[R] \tag{4}$$

for <u>any</u> step size $\mu > 0$, where $\tilde{\varepsilon}_{\min}$ is the least mean-absolute error minimizing $E\,|d(j) - \mathbf{h}^T\mathbf{x}(j)|$. In Mathews and Cho [3] it is further assumed that $\{\mathbf{x}(j), d(j)\}_{j=1}^{\infty}$ are jointly Gaussian i.i.d. random variables with zero means. They provided a heuristic argument, based on an unverifiable approximation, and obtained convergence results for small step size μ. The purpose of this paper is to establish the asymptotic time-averaged convergence for the mean-square deviation error $E[\|\mathbf{v}(j)\|^2]$ and for the signal estimation error $E[e^2(j)]$. These results hold for <u>any</u> step size $\mu > 0$.

II. RESULTS

We make the assumption that $\{\mathbf{x}(j), d(j)\}_{j=1}^{\infty}$ are jointly Gaussian i.i.d. random variables with zero means. The proof of the following results can be found in [2].

<u>Theorem 1.</u> For any initial weight vector $\mathbf{h}(1)$ and for <u>any positive step size μ</u> we have

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} E[\|\mathbf{v}(j)\|^2] \le C_1\mu + C_2\mu^2 \tag{5a}$$

where the constants C_1 and C_2 are given by

$$C_1 = \frac{9}{2}\sqrt{\frac{\pi}{2}}\,\frac{\operatorname{tr}[R] + 2\lambda_N}{\lambda_1}\,\varepsilon_{\min} \tag{5b}$$

$$C_2 = \frac{27\pi}{8}\,\frac{\lambda_N}{\lambda_1^2}\,[\operatorname{tr}[R] + 2\lambda_N]^2 + \frac{m_4}{\operatorname{tr}[R] + 2\lambda_N}\,. \tag{5c}$$

with $m_4 = E[\|\mathbf{x}(j)\|^4]$.

<u>Theorem 2.</u> For any initial weight vector $\mathbf{h}(1)$ and for <u>any positive step size μ</u> we have

$$\frac{1}{n}\sum_{j=1}^{n} E[e^2(j)] = \varepsilon^2_{\min} + \frac{1}{n}\sum_{j=1}^{n}\varepsilon^2(j)$$

with

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n}\varepsilon^2(j) \le C_3\mu + C_4\mu^2 \tag{6}$$

where $C_3 = \lambda_N C_1$ and $C_4 = \lambda_N C_2$.

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