CONTROL ON THE SPHERE AND REDUCED ATTITUDE STABILIZATION

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Abstract. This paper focuses on a new geometric approach to (fully actuated) control systems on the sphere. Our control laws exploit the basic and intuitive notions of geodesic direction and of distance between points, and generalize the classical proportional plus derivative feedback (PD) without the need of arbitrary local coordinate charts. The stability analysis relies on an appropriate Lyapunov function, where the notion of distance and its properties are exploited. This methodology then applies to spin-axis stabilization of a spacecraft actuated by only two control torques: discarding the rotation about the unactuated axis, a reduced system is considered, whose state is in fact defined on the sphere. For this reduced stabilization problem our approach allows us not only to deal optimally with the inevitable singularity, but also to achieve simplicity, versatility and (coordinate independent) adaptive capabilities.

Key Words. spin-axis stabilization, attitude control, nonlinear control systems, adaptive control.

1. INTRODUCTION

The purpose of this paper is twofold. On one hand we design control laws for fully actuated systems defined on the sphere $\mathbb{S}^2$. On the other, we apply these laws to the model of a spacecraft actuated by only two control torques and we give complete solution to a reduced attitude stabilization problem, i.e. we stabilize the spacecraft attitude up to a rotation about the unactuated axis. This problem is of practical importance, since it models for example the failure of an actuator, and is a classic, very instructive issue in nonlinear control theory. Indeed new applications to visual tracking problems (Swain and Stricker, 1991) seem to offer new examples of systems on spheres.

Within the vast literature on attitude control, Crouch (1984) shows positive controllability results for the case of three independent control torques and various smooth, stabilizing, control laws have been proposed (Wen and Kreutz-Delgado, 1991). The case of only two independent controls is more difficult. Indeed, Byrnes and Isidori (1991) show the non-stabilizability of the system: no smooth feedback control law can locally, asymptotically stabilize the full state of a spacecraft with only two actuators. Both non-continuous (Krishnan *et al.*, 1994) and smooth time-varying control laws (Walsh *et al.*, 1994; Morin *et al.*, 1994) can overcome this limitation by using ideas from the theory of nonholonomic stabilization, but are limited to the case of gas jet actuators.

Following Byrnes and Isidori (1991) and Tsiotras and Longuski (1994), we employ here a reduced approach: by discarding the rotation about the unactuated axis we come down to stabilizing a two dimensional system. The new reduced system is fully controllable (actuated), in that at each position variable corresponds an independent control, and its state is naturally defined on the sphere $\mathbb{S}^2$. Therefore our attention turns to the study of control laws on this manifold. Note that Brockett (1973) introduces a quite complete theory of control systems defined on spheres, in that he discusses controllability, observability and optimal control issues. Here instead, we concern ourselves with the explicit search for control laws. Since the manifold $\mathbb{S}^2$ is compact, has no boundary and its Euler characteristic is two, no smooth control law with only one stable equilibrium point exists; therefore we must be satisfied with control laws defined (and stabilizing) on a open dense subset of $\mathbb{S}^2$. Following Koditschek (1989) we call such feedback law almost–global.

The main contribution of this paper is a novel, general approach to fully actuated control systems defined on the sphere $\mathbb{S}^2$. The novelty is based on exploiting the metric properties of the Riemannian manifold $\mathbb{S}^2$. For a first order model, our control law exerts an action which has intensity proportional to the distance between the state of the system and the goal and is directed along the geodesic direction connecting these two points. An appropriate Lyapunov function based on the Riemannian notion of distance allows us to prove exponential stability. We then extend this control law to second order models through a standard procedure in the robotics literature: we couple the proportional action with a derivative term, i.e. with a term proportional to the "velocity". Again, the exponential stability of this proportional plus derivative (PD) control is based on the metric properties of $\mathbb{S}^2$. Our new approach shows two main advantages. First of all our control laws are coordinate invariant (no arbitrary choice of local chart is necessary) so that they allow us to solve the trajectory tracking problem in a global way. Second of all, our ideas can be generalized in a straightforward manner so that "geodesic" control laws can be designed for more general Riemannian manifolds (see Bullo and Murray (1994)).

Regarding the control of the underactuated spacecraft, we cast the problem into this well–suited framework and we give a complete solution to the reduced stabilization problem. Many differences exist with respect to the approach described in Tsiotras and Longuski (1994). First of all we respect and exploit the (geo)metric properties of the sphere instead of relying on a choice of local coordinates. As a result, our almost-global control laws confine the inevitable singularity as far away as possible from the equilibrium (i.e. at the antipodal point) and indeed sufficient conditions on the initial state of the system are provided in order to confine the closed-loop trajectories away from the singularity (instead of simply assuming this as an hypothesis). Additionally the control action remains always bounded even for big errors. From a practical viewpoint, our feedback laws allow positive definite matrix gains rather than simple positive constants, and the final expression of the control is somewhat simpler than the one given by Tsiotras and Longuski, where unusual cross terms (position-velocity) are present.

The paper is organized as follows: Section 2 deals with some basic Riemannian notions and with the design of PD control laws on the sphere. In Section 3 reduced attitude stabilization is formulated as a control problem on the sphere. In Section 4 we apply our design techniques to kinematic and dynamic models of a spacecraft. Finally we report some simulations in Section 5 and Section 6 contains a brief discussion. Due to space limitations, the stability proofs of all theorems are contained in Bullo *et al.* (1995).

2. GEODESIC CONTROL OF FIRST AND SECOND ORDER SYSTEMS ON THE SPHERE

Our goal is to design optimal, in the sense of geodesic, control laws for fully actuated control system of first and second order, whose state lies on the manifold $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$. We start by describing the geometric properties of the sphere and by applying some basic results of Riemannian geometry (Boothby, 1975). In the following, denote standard inner and outer product on $\mathbb{R}^3$ with $\langle\cdot,\cdot\rangle$ and $[\cdot,\cdot]$. Let p be a generic point on the sphere and $-p$ its opposite. For all points $p \in \mathbb{S}^2 \subset \mathbb{R}^3$, every tangent vector $X_p \in T_p\mathbb{S}^2$ can be uniquely represented as a vector $X_p \in \mathbb{R}^3$ such that $X_p \perp p$ (using the standard inner product on $\mathbb{R}^3$) and more generally $T_p\mathbb{S}^2 = \operatorname{span}\{p\}^\perp$. The canonical inner product on $\mathbb{R}^3$ induces a Riemannian structure on $\mathbb{S}^2$ (i.e. an inner product on $T_p\mathbb{S}^2$) in the natural way:

$$\langle X_p, Y_p\rangle_{T_p\mathbb{S}^2} \triangleq \langle X_p, Y_p\rangle \quad \forall X_p, Y_p \in T_p\mathbb{S}^2 \subset \mathbb{R}^3.$$

The geodesics of this natural metric are great circles and the *distance* between two generic points $p, q \in \mathbb{S}^2$ is the angle between the two directions:

$$d(p,q) \equiv \arccos(\langle p,q\rangle_{\mathbb{R}^3}), \tag{1}$$

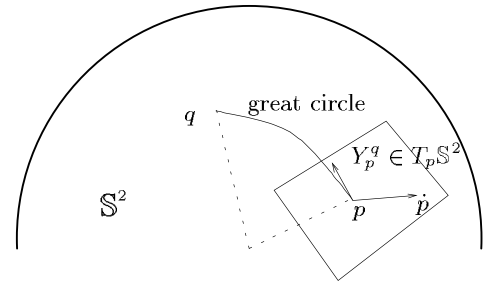

Fig. 1. The sphere and its tangent plane.

with arccos taking values in $[0,\pi]$. Additionally, provided p and q are neither equal nor opposite, there is a uniquely defined unit vector $Y_p^q \in T_p\mathbb{S}^2$ called the *geodesic versor* that gives the geodesic direction in p towards q:

$$Y_p^q \triangleq [\operatorname{vers}([p,q]), p] = \operatorname{vers}(q_\perp),$$

where $q_\perp$ is the component of q orthogonal to p and the operator versor simply means: $\operatorname{vers}(x) = x/\|x\|$. The notion of distance and of geodesic versor are related by a useful extension of Gauss's Lemma:

Lemma 1 (Derivative of distance function)
Consider a trajectory $p = p(t) \in \mathbb{S}^2$, such that $p(t)$ never passes through the fixed points q or $-q$. Then

$$\frac{d}{dt} d(p(t), q) = -\langle \dot p, Y_p^q\rangle_{T_p\mathbb{S}^2}.$$

The proof follows by differentiation of equation (1).

The terms of Lemma 1 can be intuitively interpreted as follows: the distance between state of the system p and fixed goal q evolves in time only depending on the component of $\dot p$ along the geodesic direction Y_p^q. In particular for a trajectory $p(t)$ such that $d(p,q)$ is constant, Lemma 1 reduces to Gauss's Lemma and asserts the orthogonality between $\dot p(t)$ and the geodesic versor Y_p^q. Since we are now able to compute its time derivative, the Riemannian notion of distance appears suitable for stability analysis; in the following let $B_p = \{X_p^1, X_p^2\}$ be a smooth orthonormal basis of $T_p\mathbb{S}^2$. The regulation problem for a fully controllable, first order system defined on the sphere reads:

Problem 2.1 (Regulation of first order systems)
Given the system

$$\dot p = X_p^1 v_1 + X_p^2 v_2 \quad \in T_p\mathbb{S}^2, \tag{2}$$

find a control $v = v(p)$ such as to steer asymptotically the state $p \in \mathbb{S}^2$ to the fixed goal q.

Note that B_p cannot be smoothly defined on all $\mathbb{S}^2$. A canonical choice in the regulation problem would be

$$B_p = \left\{Y_p^q, \operatorname{vers}([p,q])\right\},$$

where the base B_p is defined for all p neither equal nor opposite to q. More generally a natural choice B_p

might be suggested by the particular control problem in question; this is the case in Section 4.

To solve this first problem, we generalize the classical proportional control to the manifold $\mathbb{S}^2$ as follows: the control action has intensity proportional to the distance between state and goal and is applied along the geodesic direction (connecting state and goal) skewed by a positive definite gain K_p. Let $\lambda_{\min}(K_p)$ ($\lambda_{\max}(K_p)$) be the minimum (maximum) eigenvalue of the positive definite matrix K_p.

Theorem 2.1 (Regulation of first order systems) Consider the system in equation (2). Then the control law

$$v = d(p,q)\, K_p \begin{bmatrix} \langle Y_p^q, X_p^1 \rangle \\ \langle Y_p^q, X_p^2 \rangle \end{bmatrix}, \tag{3}$$

exponentially stabilizes the state p at q from any initial condition $p(0) \neq -q$ and with time-constant at least $1/\,\lambda_{\min}(K_p)$.

The stability analysis is based on the Lyapunov function $W := \frac{1}{2} d(p,q)^2$; for details see Bullo *et al.* (1995). Note that for $p = q$ the geodesic versor Y_p^p is not defined. This is no problem, since the control law v can be easily prolonged continuously and defined equal to zero at q: $v(q) = 0$. Hence the control law in equation (3) is smooth on the whole $\mathbb{S}^2 \setminus \{-q\}$ and has a single exponentially stable equilibrium point q. As already explained, this is the most we can achieve.

We now consider the more general problem of controlling a system via accelerations (or forces) instead of velocities. The regulation problem for a fully controllable, second order system defined on the sphere reads:

Problem 2.2 (Regulation of second order systems) Given the system

$$\left\{ \begin{array}{rcl} \dot p &=& X_p^1 v_1 + X_p^2 v_2 \qquad X_p^1, X_p^2 \in T_p\mathbb{S}^2 \\ \dot v &=& u, \end{array} \right. \tag{4}$$

find a control $u = u(p, \dot p)$ such as to steer asymptotically the state $p \in \mathbb{S}^2$ to the fixed goal q.

As typically done in the robotics literature, we now combine proportional and derivative (PD) action. The closed-loop system will behave as a nonlinear spring with a velocity damper and correspondingly the Lyapunov function will be the sum of pseudo-kinetic and pseudo-potential energy terms. With respect to a standard PD controller in local coordinates, the novelty here consists in the form of the proportional action (we have a geodesic spring) and of the corresponding pseudo-potential energy term (Riemannian distance squared).

Theorem 2.2 (Regulation of second order systems) Consider the system in equation (4). Given the positive definite matrix gains K_p and K_d, the control law

$$u = d(p,q)\, K_p \begin{bmatrix} \langle Y_p^q, X_p^1 \rangle \\ \langle Y_p^q, X_p^2 \rangle \end{bmatrix} - K_d v,$$

exponentially stabilizes the equilibrium point q from any initial condition $p(0) \neq -q$ and for all K_p and $\dot p(0)$ such that

$$\lambda_{\min}(K_p) > \frac{\| \dot p(0) \|^2}{\pi^2 - d(p(0), q)^2}. \tag{5}$$

The stability analysis relies on the Lyapunov function $W := \frac{1}{2} d(p,q)^2 + \frac{1}{2} \|v\|^2_{K_p^{-1}}$. Condition (5) confines the closed-loop trajectories away from singularity.

As last result of the section, we state the trajectory tracking version of our PD control law. Let $q = q(t) \in \mathbb{S}^2$ be the desired goal and $\dot q, \ddot q$ its velocity and acceleration[1] belonging to $T_q\mathbb{S}^2$. We assume $\|\dot q\|$ to be bounded. Define the rotation matrix $\mathcal{R}$ such that $\mathcal{R}q = p$ and $\mathcal{R}[p,q] = [p,q]$. Additionally let us define the scalar quantity ξ as

$$\xi = \langle \tan[\tfrac{1}{2} d(p,q)] Y_p^q - p \;\;,\;\; [\dot p, q] + [p, \dot q] \rangle.$$

Theorem 2.3 (Tracking of second order systems) Consider the system in equation (4). Given the positive k_p and the positive definite K_d, the control law

$$u = \mathcal{R}\ddot q + k_p d(p,q)\, Y_p^q - K_d(\dot p - \mathcal{R}\dot q) + \xi[\dot p, p],$$

exponentially stabilizes $\{d(p(t), q(t))\ ,\ \dot p - \mathcal{R}\dot q\}$ to zero from any initial condition $p(0) \neq -q(0)$ and for all $k_p, \dot p(0), \dot q(0)$ such that

$$k_p > \frac{\|\dot p(0) - \mathcal{R}(0)^T \dot q(0)\|^2}{\pi^2 - d(p(0), q(0))^2}.$$

Note the strong similarity with the $\mathbb{R}^n$ case; only in the parameter ξ the curvature of the manifold $\mathbb{S}^2$ comes into play. For details see Bullo *et al.* (1995).

3. SPACECRAFT MODELS WITH TWO CONTROL TORQUES: PROJECTION ONTO THE SPHERE

In this section we review kinematic and dynamic models of a spacecraft actuated by two momentum wheels. We employ the following standard assumptions: the control torques are applied along the principal axes of the spacecraft and the body frame is along these principal axes, so that $J = \mathrm{diag}(J_1, J_2, J_3)$. Let $R \in SO(3)$ be the rotation matrix, state of the full system. It holds

$$\dot R = R(\omega \times), \tag{6}$$

where ω is the angular velocity expressed in the body frame and the operator $\times$ is defined such that $(\omega\times)v = \omega \times x$ for all $x \in \mathbb{R}^3$. Following Marsden (1992), we neglect the dynamics of the actuators and we start

[1] Note the slight abuse of notation: by $\ddot q$ we here mean the time derivative of the components of $\dot q$ expressed with respect to the basis B_q. More formally, adopting the notation in Boothby (1975), we have $\ddot q = \frac{D}{dt}\dot q$.

with a kinematic analysis. It holds

$$J\omega = R^T m_0 + e_1 v_1 + e_2 v_2, \tag{7}$$

where m_0 is the total constant angular momentum (vector), $e_1 = [1,0,0]^T$, $e_2 = [0,1,0]^T$ and the v_i are the velocities of the wheels (scaled by the moment of inertia of the wheels about their own rotation axes). Combining (6) and (7) we have the kinematic model

$$\dot{R} = R\left[J^{-1}(R^T m_0 + e_1 v_1 + e_2 v_2)\right]\times\,. \tag{8}$$

A dynamic analysis (Crouch, 1984) leads instead to the standard second order model

$$\left\{\begin{array}{rcl} \dot{R} & = & R(\omega\times), \\ J\dot{\omega} & = & [R^T m_0, \omega] + e_1\tau_1 + e_2\tau_2, \end{array}\right. \tag{9}$$

where the τ_i, $i = 1,2$ are the torques applied to the wheels (scaled by the momentum of inertia of the wheels about their own rotation axes). Note that this model also applies to the case of gas jet actuators by replacing the internal drift $[R^T m_0, \omega]$ with the term $[J\omega, \omega]$ (Euler equations).

The reduced control problem for the spacecraft models in equations (8) and (9) consists in the design of a feedback control law that stabilizes the state $R \in SO(3)$ up to a rotation about the unactuated principal axis $e_1 \times e_2 = [0,0,1]^T =: e_0$. To simplify the formulation of the problem, define the projection maps $\pi_i : SO(3) \to \mathbb{S}^2$ as $\pi_i(R) := Re_i$ (this is the same projection operator introduced in Walsh and Sastry (1995)). Stabilizing R up to a rotation about e_0 is equivalent to stabilizing the direction of the axis Re_0 and discarding the residual drift about this direction. Thus we can restate our control problem in terms of the point $\pi_0 \in \mathbb{S}^2$.

Problem 3.1 (Reduced Attitude Stabilization)
Given the models in equations (8) and (9), find a feedback control law such as to steer asymptotically the reduced state $\pi_0 \in \mathbb{S}^2$ to a fixed point $q \in \mathbb{S}^2$.

We now derive the reduced dynamic system corresponding to the state π_0. Projecting equation (6):

$$\begin{array}{rcl} \dot{\pi}_0 & = & \dot{R}e_0 = R([\omega, e_0]) = -\pi_2\omega_1 + \pi_1\omega_2 \\ & = & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}, \end{array}$$

with respect to the orthonormal basis $\{\pi_1, \pi_2\}$. By discarding the variable ω_3 we write our *kinematic model* as

$$\dot{\pi}_0 = \begin{bmatrix} 0 & J_2^{-1} \\ -J_1^{-1} & 0 \end{bmatrix}\left\{\begin{bmatrix} \langle m_0, \pi_1\rangle \\ \langle m_0, \pi_2\rangle \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right\}, \tag{10}$$

and our *dynamic model* as

$$\left\{\begin{array}{rcl} \dot{\pi}_0 & = & \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ \begin{bmatrix} J_1\dot{\omega}_1 \\ J_2\dot{\omega}_2 \end{bmatrix} & = & \begin{bmatrix} \langle m_0, [g\omega, \pi_1]\rangle \\ \langle m_0, [g\omega, \pi_2]\rangle \end{bmatrix} + \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}. \end{array}\right. \tag{11}$$

Eventually, note that the same reduction procedure applies to the gas jet actuators case. The dynamic equation in system (11) changes to

$$\begin{bmatrix} J_1\dot{\omega}_1 \\ J_2\dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} (J_2 - J_3)\omega_2\omega_3 \\ (J_3 - J_1)\omega_1\omega_3 \end{bmatrix} + \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}.$$

4. EXPLICIT FORM OF CONTROL LAWS FOR THE SPACECRAFT MODELS

Since the spacecraft models introduced in Section 3 are fully controllable, we can apply the stability results obtained in Section 2.

Subsection 4.1 deals briefly with the kinematic model. Assuming perfect knowledge of the plant's parameters, the first proposed control law relies on a feedforward term which exactly compensates for the dynamics of the system. The main drawback of this cancellation strategy is that various external disturbances may actually affect the rate of change of the plant's parameter m_0. These external disturbances include gravity gradients, solar radiation pressure, and Earth's magnetic field (see, for example, Slafer and Seidenstucker (1991)). Therefore, since the dependence of the internal dynamics is linear on m_0, we propose an indirect adaptive control scheme; for details on the standard procedure see Sastry and Bodson (1989).

Subsection 4.2 deals in full detail with the dynamic model. We give a complete solution to the reduced (spin-axis) stabilization problem through three different strategies: model independent control law (PD without feedforward term), model dependent control law (PD plus exact feedforward cancellation) and indirect adaptive control law (PD plus feedforward and adaptation law). The set of stability results that our laws achieve is very similar to what usually obtained in the robotics literature (Wen and Kreutz-Delgado, 1991) for passive mechanical systems: Lyapunov stability for the model independent law, exponential convergence in case of exact feedforward cancellation and asymptotic stability for the indirect adaptive control scheme. A complete discussion on the proposed control laws is included.

4.1. *PROPORTIONAL CONTROL LAWS FOR THE KINEMATIC MODEL*

Theorem 4.1 (Regulation of kinematic model)
Consider the kinematic model in equation (10) and let $q \in \mathbb{S}^2$ be the desired goal. Given the positive definite gain K_p, the control law

$$v = d(\pi_0, q)\begin{bmatrix} 0 & -J_1 \\ J_2 & 0 \end{bmatrix} K_p \begin{bmatrix} \langle Y^q_{\pi_0}, \pi_1\rangle \\ \langle Y^q_{\pi_0}, \pi_2\rangle \end{bmatrix} - \begin{bmatrix} \langle m_0, \pi_1\rangle \\ \langle m_0, \pi_2\rangle \end{bmatrix},$$

exponentially stabilizes the state $\pi_0(t)$ at the goal q from any initial condition $\pi_0(0) \neq -q$ and with time-constant at least $1/\lambda_{\min}(K_p)$.

In case only an estimate of the angular momentum m_0 is available, a classical indirect adaptive control scheme can be designed:

Theorem 4.2 (Adaptive regulation of kinematic model) Consider system in equation (10) and let $\widehat{m}_0$ be the current estimate of the unknown parameter m_0. Let the control gain K_p and the adaptation gain Γ be symmetric, positive definite matrices. Then the control law (based on the certainty equivalence principle):

$$v = d(\pi_0, q) \begin{bmatrix} 0 & -J_1 \\ J_2 & 0 \end{bmatrix} K_p \begin{bmatrix} \langle Y^q_{\pi_0}, \pi_1 \rangle \\ \langle Y^q_{\pi_0}, \pi_2 \rangle \end{bmatrix} - \begin{bmatrix} \langle \pi_1, \widehat{m}_0 \rangle \\ \langle \pi_2, \widehat{m}_0 \rangle \end{bmatrix},$$

and the update law:

$$\frac{d}{dt} \widehat{m}_0 = -d(\pi_0, q) \, \Gamma g J^{-1} R^T \text{ vers}[\pi_0, q],$$

locally, asymptotically stabilize the state $\pi_0(t)$ at the goal $q \in \mathbb{S}^2$ and make $m_0 - \widehat{m}_0$ go to a constant.

4.2. *PROPORTIONAL PLUS DERIVATIVE CONTROL LAWS FOR THE DYNAMIC MODEL*

All the stability results in this section rely on a skewed mechanical metric on $\mathbb{S}^2$ to design the Lyapunov function. Again, see Bullo *et al.* (1995) for more details. The simplest control law we propose is a PD controller without feedforward cancellation; since no knowledge of the model is required we call this control model independent.

Theorem 4.3 (Model independent regulation) Consider the dynamic model in equation (11). Given the positive k_p and the positive definite K_d, the control law

$$\tau = k_p d(\pi_0, q) \begin{bmatrix} -\langle Y^q_{\pi_0}, \pi_2 \rangle \\ \langle Y^q_{\pi_0}, \pi_1 \rangle \end{bmatrix} - K_d \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix},$$

renders the equilibrium point $(q, [0\ 0]^T)$ Lyapunov stable from any initial condition $\pi_0(0) \neq -q$ and for all k_p such that

$$k_p > \frac{\langle \omega, J\omega \rangle}{\pi^2 - d(\pi_0(0), q)^2}, \tag{12}$$

Additionally the distance $d(\pi_0(t), q)$ converges to a constant $\bar{d} \leq \|m_0\|^2/(2J_3 k_p)$.

The second proposed control law assumes exact cancellation between internal dynamics and feedforward control so as to satisfy the hypothesis of Theorem 2.2.

Theorem 4.4 (Model dependent regulation) Consider the system in equation (11). Given the positive definite gains K_p and K_d, the control law

$$\begin{aligned} \tau \;\; = \;\; & d(\pi_0, q)\, K_p \begin{bmatrix} -\langle Y^q_{\pi_0}, \pi_2 \rangle \\ \langle Y^q_{\pi_0}, \pi_1 \rangle \end{bmatrix} - K_d \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ & - \begin{bmatrix} \langle m_0, [g\omega, \pi_1] \rangle \\ \langle m_0, [g\omega, \pi_2] \rangle \end{bmatrix}, \end{aligned}$$

exponentially stabilizes the state $\pi_0(t)$ at the goal q from any initial condition $\pi_0(0) \neq -q$ and for all K_p and ω such that

$$\lambda_{\min}(K_p) > \frac{J_1 \omega_1^2 + J_2 \omega_2^2}{\pi^2 - d(\pi_0(0), q)^2}. \tag{13}$$

As already explained, exact cancellation of internal drift is not a robust procedure, in that the total angular momentum m_0 might either be unknown or change slowly in time. Therefore, as for the kinematic model, we design an indirect adaptive control scheme:

Theorem 4.5 (Indirect Adaptive Regulation) Consider the system in equation (11). Let $\widehat{m}_0$ be the current estimate of m_0, let the control gain K_p, K_d and the adaptation gain Γ be symmetric, positive definite matrices. Then the control law

$$\begin{aligned} \tau \;\; = \;\; & d(\pi_0, q)\, K_p \begin{bmatrix} -\langle Y^q_{\pi_0}, \pi_2 \rangle \\ \langle Y^q_{\pi_0}, \pi_1 \rangle \end{bmatrix} - K_d \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ & - \begin{bmatrix} \langle \widehat{m}_0, [g\omega, \pi_1] \rangle \\ \langle \widehat{m}_0, [g\omega, \pi_2] \rangle \end{bmatrix}, \end{aligned}$$

and the update law

$$\begin{aligned} \frac{d}{dt} \widehat{m}_0 \;\; = \;\; & \Gamma g(\omega \times) \begin{bmatrix} K_p^{-1} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ 0 \end{bmatrix} \\ & - \epsilon\, d(\pi_0, q)\, g(\omega \times) J^{-1} \text{ vers}[e_0, R^T q], \end{aligned}$$

locally, asymptotically stabilize the equilibrium point $q \in \mathbb{S}^2$ for sufficiently small ϵ.

Remark 4.1 (Trade-off between the proposed laws) Each of the three proposed strategies has its own strengths and weaknesses. From an applicative viewpoint, the choice of control law can be taken on the basis of meaningful parameters, such as controller complexity and stability properties versus computational load, or required a priori knowledge of the plant's parameter and of the external disturbances. For a complete discussion on this issue we refer for example to Wen and Kreutz-Delgado (1991), where a full set of model independent, model dependent and indirect adaptive control laws is also proposed (but for the attitude stabilization problem).

Remark 4.2 (A family of simplified control laws) The proportional action, as stated in Section 2, is:

$$d(p, q)\, Y^q_p \equiv \frac{\theta}{\sin \theta} [[p, q], p],$$

where $\theta = d(p, q)$. This expression immediately suggests a simplification; assuming $\sin(\theta) \approx \theta$, the model dependent control law looks like

$$\tau = -\text{drift} + K_p \begin{bmatrix} -\langle q, \pi_2 \rangle \\ \langle q, \pi_1 \rangle \end{bmatrix} - K_d \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}. \tag{14}$$

The stability properties of this sin-like control law can be verified through a Lyapunov function based on a potential-like energy term of the form: $(1 - \cos(d(\pi_0, q)))$; indeed local exponential stability can be easily proved. This simplified control law is smooth on all $\mathbb{S}^2$ and has an instable equilibrium point at $-q$;

additionally it has the drawback of exerting a decreasing control for an increasing distance of state and goal when the state is distant more than $\pi/2$.

Remark 4.3 (Gas jet actuators) So far we have dealt with momentum wheels actuators, but the proposed control laws also apply to the case of gas jets. The model independent control law remains unchanged:

$$\tau = k_p d(\pi_0, q) \begin{bmatrix} -\langle Y^q_{\pi_0}, \pi_2 \rangle \\ \langle Y^q_{\pi_0}, \pi_1 \rangle \end{bmatrix} - K_d \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix},$$

and Lyapunov stability can be proved easily through the same proof of Theorem 4.3. In the model dependent control law, we simply compensate for the different drift:

$$\begin{aligned} \tau &= d(\pi_0, q)\, K_p \begin{bmatrix} -\langle Y^q_{\pi_0}, \pi_2 \rangle \\ \langle Y^q_{\pi_0}, \pi_1 \rangle \end{bmatrix} - K_d \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ &\quad - \begin{bmatrix} a_1 \omega_2 \omega_3 \\ a_2 \omega_1 \omega_3 \end{bmatrix}, \end{aligned} \tag{15}$$

where $a_1 = J_2 - J_3$, $a_2 = J_3 - J_1$. As before, condition (13) ensures the smoothness of the control law.

Note that control law in equation (15) relies on exact knowledge of the inertia matrix and is not robust with respect to retrieval or deployment of unknown payloads. Hence we design a locally, asymptotically adaptive control scheme based on a certainty equivalence control

$$\begin{aligned} \tau &= d(\pi_0, q)\, K_p \begin{bmatrix} -\langle Y^q_{\pi_0}, \pi_2 \rangle \\ \langle Y^q_{\pi_0}, \pi_1 \rangle \end{bmatrix} - K_d \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \\ &\quad - \begin{bmatrix} \widehat{a}_1 \omega_2 \omega_3 \\ \widehat{a}_2 \omega_1 \omega_3 \end{bmatrix}, \end{aligned}$$

coupled with the update law

$$\frac{d}{dt} \begin{bmatrix} \widehat{a}_1 \\ \widehat{a}_2 \end{bmatrix} = \gamma \begin{bmatrix} \omega_1 \omega_2 \omega_3 \\ \omega_1 \omega_2 \omega_3 \end{bmatrix}, \quad \forall \gamma > 0.$$

5. SIMULATIONS

To verify our theoretical predictions, we run simulations for the dynamic model of a spacecraft with momentum wheels. We simulated the full system defined on $SO(3) \times \mathbb{R}^3$ as described in equation (9). As is well-known, the attitude matrix R is not a suitable parametrization of $SO(3)$ for computational goals; we therefore relied on unit quaternions, enforcing the magnitude constraint through a projection procedure. We implemented the model independent control law and the adaptive scheme as in Theorems 4.3 and 4.5. An explicit expression for the adaptive control law is

$$\begin{aligned} \tau &= \begin{bmatrix} \langle [\omega, R^T \widehat{m}_0], e_1 \rangle \\ \langle [\omega, R^T \widehat{m}_0], e_2 \rangle \end{bmatrix} + \frac{\arcsin \|z\|}{\|z\|} K_p z \\ &\quad - K_d \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} \quad \text{with} \quad z \triangleq \begin{bmatrix} -\langle R^T q, e_2 \rangle \\ \langle R^T q, e_1 \rangle \end{bmatrix}, \end{aligned}$$

where we used arcsin rather than arccos to avoid numerical singularities. The update law for $\widehat{m}_0$ is as in Theorem 4.5. We assumed the moments of inertia to be $(1.0, 0.63, 0.87)$, the constant angular momentum $[1, 1, 1]$; initial conditions were $R(0) = I_3$ and $\omega(0) = J^{-1} R^T m_0$ and the final desired orientation was $q = [1, 0, 0]^T$. The parameters of the various controller were: $k_p = 5$, $\gamma = 5$, $\epsilon = 1$ and

$$K_p = \begin{bmatrix} 3.0 & .5 \\ .5 & 1.5 \end{bmatrix}, \qquad K_d = \begin{bmatrix} 3.0 & .3 \\ .3 & 3.0 \end{bmatrix}.$$

Thus, given that $\pi_0(0) = e_0$ and $q = e_1$, we have $d(\pi_0(0), q) = \pi/2$. The initial kinetic energy of the spacecraft is $K(0) = \langle \omega(0), J\omega(0) \rangle = 2.5$ so that $5 = k_p \geq \frac{\langle \omega(0), J\omega(0) \rangle}{\pi^2 - d(\pi_0(0), q)} \approx 1.0$ is a sufficient condition for the smoothness of the feedback (see Theorem 4.5).

The numerical simulations are reported in Figure 2 (model independent control), and Figure 3 (indirect adaptive control). In both cases we include: $\pi_0 \in \mathbb{S}^2 \subset \mathbb{R}^3$ expressed in inertial coordinates (unitless), the distance $d(\pi_0, q)$ and the angular velocity ω. For the second simulation (adaptive control scheme) we also show the estimate of the angular momentum $\widehat{m}_0$. Regarding the model independent control law: As in the theoretical analysis (Theorem 4.3), the distance between π_0 and q goes to a constant (second picture), which satisfies bound (12). Indeed the first two components of the angular velocity go to zero, while the third one becomes a constant (third picture). Regarding the adaptive control law: As in the theoretical analysis (Theorem 4.5), the state of the spacecraft converges (at least) asymptotically to the desired equilibrium configuration (see second picture for $d(\pi_0, q)$ and third picture for ω_1 and ω_2), while the estimation error goes to a steady state generically different from zero (see fourth picture).

6. CONCLUSIONS

In this paper we have dealt with fully actuated control systems defined on the sphere $\mathbb{S}^2$ and in such setting we have proposed a novel approach to regulation and trajectory tracking problems. Our results have then applied to a reduced attitude stabilization problem (spin-axis stabilization). We have designed a comprehensive set of control laws, which differ in stability properties, computational complexity and required model knowledge. By and large, our (differential) geometric approach has lead to a family of simple, versatile and robust control laws.

The work proposed here can be seen as a development of previous investigations on the correct Lyapunov function's design (Koditschek, 1989). We rely on the Riemannian notion of distance to achieve a simple and successful solution to global problems such as trajectory tracking. The simplicity and efficacy of this approach can then apply to more general Riemannian manifolds (e.g. see Bullo *et al.* (1995) for the $\mathbb{S}^n$ case). Indeed, control systems defined on Lie groups belong to this class and provide a very instructive example. Here the topological properties of the group, such as the compactness, influence its metric structure and only in certain cases our approach applies straightforwardly; for an introduction see Bullo and Murray (1994).

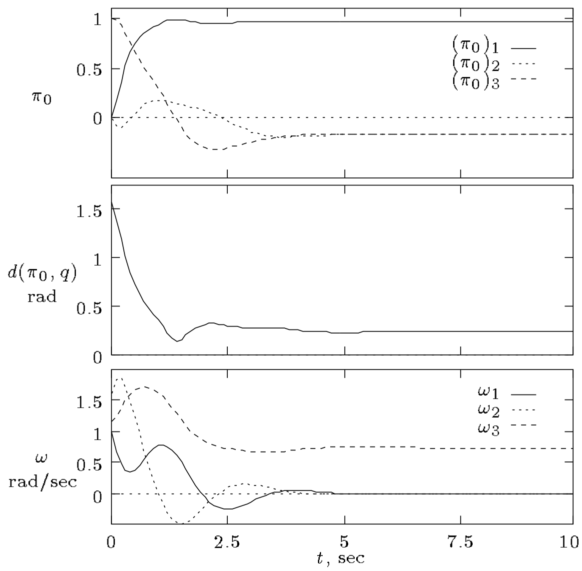

Fig. 2. Simulation for the model independent control law (as in Theorem 4.3).

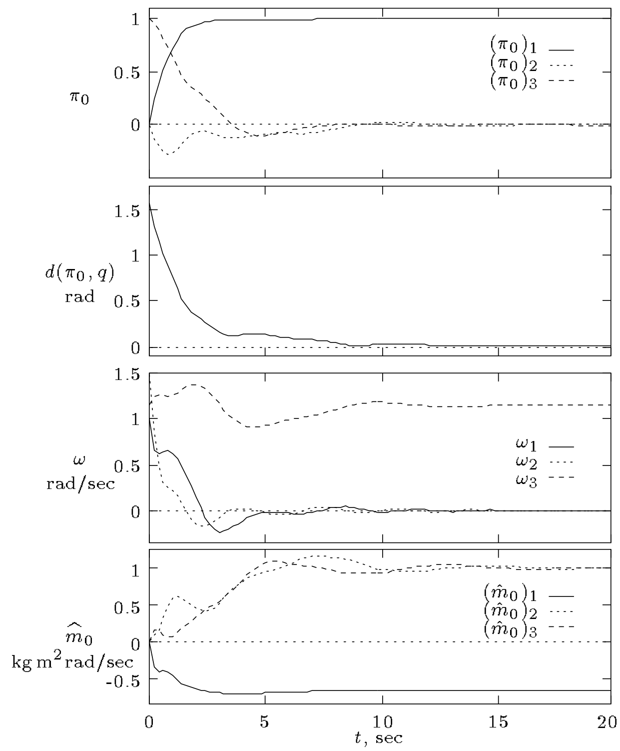

Fig. 3. Simulation for indirect adaptive control scheme (as in Theorem 4.5).

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