2024 CDC Workshop on "Contraction Theory for Systems, Control, Optimization, and Learning"

Full-Day Workshop, in conjunction with the 2024 Conference on Decision and Control in Milano, Italy

Organizer: Francesco Bullo, UC Santa Barbara

Schedule. Sunday, December 15, 2024, 8:30–17:45

There will be 13 presentations, each lasting a total of 25 minutes. This time includes 20 minutes for the talk, 5 minutes for questions, transition and informal conversations. There will also be a Rapid Presentations session.

08:30–08:40: Introduction by Francesco Bullo (10 minutes)

Morning Session: 4 talks

- 08:40–09:05: *A Quarter Century of Contraction Analysis*, **Jean-Jacques Slotine**, MIT, USA
- 09:05–09:30: *Contraction Theory of Output Regulation* **Daniele Astolfi**, Université de Lyon, France
- 09:30–09:55: *Contractivity of Interconnected Continuous and Discrete-time Systems*, **Emiliano Dall'Anese**, Boston University, USA
- 09:55–10:20: *Computation of Contraction Metrics with Meshfree Collocation*, **Peter Giesl**, University of Sussex, UK

10:20–11:00: Coffee Break (40 minutes), CDC coffee will be served between 10:00 and 11:00

Midday Session: 3 talks

- 11:00–11:25: *On 2-Contraction and Non-Oscillatory Systems: Some Theory and Applications*, **David Angeli**, Imperial College, UK
- 11:25–11:50: *Towards Contracting Biologically Plausible Neural Networks* **Giovanni Russo**, Universita di Salerno, Italy
- 11:50–12:15: *Youla-Kucera Parametrization in the Contraction Framework*, **Yu Kawano**, Hiroshima University, Japan

12:15–13:45: Group Photo and Lunch Break (1 hour 30 minutes)

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- 13:45-14:10: *A Robust Learning Framework built on Contraction and Monotonicity*, **Ian Manchester**, University of Sydney, Australia
- 14:10–14:35: *Compound Matrices and Dynamical Systems* (PDF), **Michael Margaliot**, Tel Aviv University, Israel
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- 16:05–16:30: *Contractions for Interaction Networks*, **Eduardo Sontag**, NorthEastern University, USA
- 16:30–16:55: *Time-Varying Convex Optimization: A Contraction and Equilibrium Tracking Approach*, **Francesco Bullo**, UC Santa Barbara, USA
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  - **Li Qiu**, Presidential Chair Professor, School of Science and Engineering, The Chinese University of Hong Kong, Shenzhen, China, *Five Minutes on Phase Theory*
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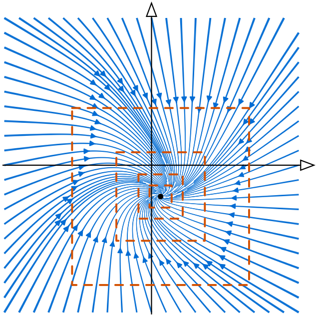
  - **Alexander (Sasha) Davydov**, URL, Mechanical Engineering, University of California, Santa Barbara, *Learning Globally Contracting Dynamics from Demonstrations* (PDF)

contractivity = robust, modular, computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces

highly-ordered transient and asymptotic behavior, no anonymous constants/functions:



1. sharp analysis for numerous example systems
2. unique globally exponential stable equilibrium & two natural Lyapunov functions
3. robustness properties
   - bounded input, bounded output (iss)
   - finite input-state gain
   - robustness margin wrt unmodeled dynamics
4. periodic input, periodic output
5. modularity and interconnection properties
6. accurate numerical integration and equilibrium point computation

search for contraction properties
design engineering systems to be contracting
verify correct/safe behavior via known Lipschitz constants

Example contracting systems

1. **gradient descent flows** under strong convexity assumptions
   (proximal, primal-dual, distributed, Hamiltonian, saddle, pseudo, best response, etc)
2. **neural network dynamics** under assumptions on synaptic matrix
   (recurrent, implicit, reservoir computing, etc)
3. **Lur'e-type systems** under assumptions on nonlinearity and LMI conditions
   (Lipschitz, incrementally passive, monotone, conic, etc)
4. **interconnected systems** under contractivity and small-gain assumptions
   (Hurwitz Metzler matrices, network small-gain theorem, etc)

5. data-driven learned models (imitation learning)
6. feedback linearizable systems with stabilizing controllers
7. incremental ISS systems
8. Giesl Converse Theorem: nonlinear systems with a locally exponentially stable equilibrium are contracting with respect to appropriate Riemannian metric

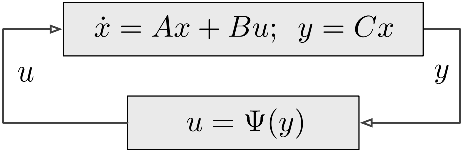
Example #1: Parametric convex optimization and contracting dynamics

Many convex optimization problems can be solved with contracting dynamics

$$\dot{x} = \mathsf{F}(x, \theta)$$

	Convex Optimization	Contracting Dynamics
Unconstrained	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$	$\dot{x} = -\nabla_x f(x, \theta)$
Constrained	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ s.t. $\quad x \in \mathcal{X}(\theta)$	$\dot{x} = -x + \mathrm{Proj}_{\mathcal{X}(\theta)}(x - \gamma \nabla_x f(x, \theta))$
Composite	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta) + g(x, \theta)$	$\dot{x} = -x + \mathrm{prox}_{\gamma g_\theta}(x - \gamma \nabla_x f(x, \theta))$
Equality	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ s.t. $\quad Ax = b(\theta)$	$\dot{x} = -\nabla_x f(x, \theta) - A^\top \lambda,$ $\dot{\lambda} = Ax - b(\theta)$
Inequality	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ s.t. $\quad Ax \leq b(\theta)$	$\dot{x} = -\nabla f(x, \theta) - A^\top \nabla M_{\gamma, b(\theta)}(Ax + \gamma\lambda),$ $\dot{\lambda} = \gamma(-\lambda + \nabla M_{\gamma, b(\theta)}(Ax + \gamma\lambda))$

Example #2: Systems in Lur'e form



For $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ and $C \in \mathbb{R}^{n\times m}$, **nonlinear system in Lur'e form**

$$\dot{x} = Ax + B\Psi(Cx) \quad =: \mathsf{F}_{\text{Lur'e}}(x)$$

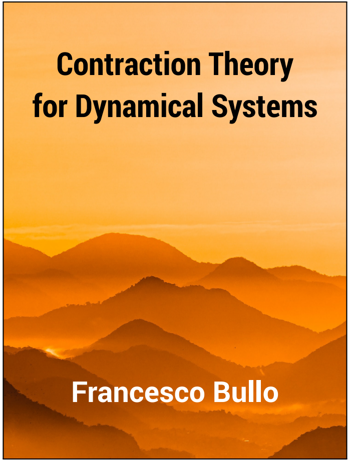
where $\Psi : \mathbb{R}^m \to \mathbb{R}^m$ is described by an **incremental multiplier matrix** M, that is,

$$\begin{bmatrix} y_1 - y_2 \\ \Psi(y_1) - \Psi(y_2) \end{bmatrix}^\top M \begin{bmatrix} y_1 - y_2 \\ \Psi(y_1) - \Psi(y_2) \end{bmatrix} \geq 0 \qquad \text{for all } y_1, y_2 \in \mathbb{R}^m$$

For $P = P^\top \succ 0$, following statements are equivalent:

1. $\mathsf{F}_{\text{Lur'e}}$ infinitesimally contracting wrt $\|\cdot\|_{2,P}$ with rate $\eta > 0$ for each Ψ described by M,
2. $\exists \lambda \geq 0$ s.t. $\begin{bmatrix} PA + A^\top P + 2\eta P & PB \\ B^\top P & \mathbb{0}_{m\times m} \end{bmatrix} + \lambda \begin{bmatrix} C^\top & \mathbb{0}_{n\times m} \\ \mathbb{0}_{m\times m} & I_m \end{bmatrix} M \begin{bmatrix} C & \mathbb{0}_{m\times m} \\ \mathbb{0}_{m\times n} & I_m \end{bmatrix} \preceq 0$

Ongoing education and research on contraction theory



"Continuous improvement is better than delayed perfection"
Mark Twain

- Textbook: Contraction Theory for Dynamical Systems, Francesco Bullo, rev 1.2, Aug 2024. (Book and slides freely available) **https://fbullo.github.io/ctds**
- Tutorial slides: **https://fbullo.github.io/ctds**
- Youtube lectures: "Minicourse on Contraction Theory" **https://youtu.be/FQV5PrRHks8** 6 lectures, total 12h

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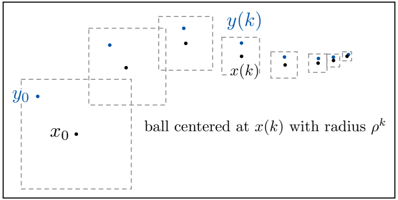
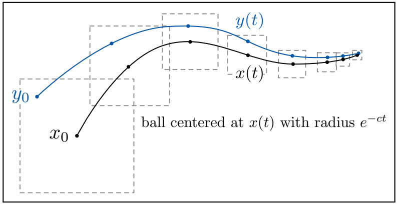
Contracting Dynamics for Optimization

Francesco Bullo

Center for Control,
Dynamical Systems & Computation
University of California at Santa Barbara
https://fbullo.github.io/ctds

2024 IEEE CDC Contraction Theory Workshop, 2024/12/15. This version: 2024/12/15





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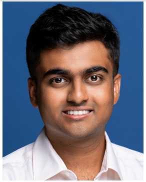

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Michael Dorothy @ARO W911NF-24-1-0228

Outline

Continuous-time dynamics and one-sided Lipschitz constants

$$\dot{x} = \mathsf{F}(x) \qquad \text{on } \mathbb{R}^n \text{ with norm } \|\cdot\| \text{ and induced log norm } \mu(\cdot)$$

One-sided Lipschitz constant (≈ maximum expansion rate)

$$\begin{aligned} \mathrm{osLip}(\mathsf{F}) &= \inf\{b \in \mathbb{R} \text{ such that } [\![\mathsf{F}(x) - \mathsf{F}(y), x - y]\!] \leq b\|x - y\|^2 \quad \text{for all } x, y\} \\ &= \sup_x \mu(D\mathsf{F}(x)) \qquad \text{(when F differentiable)} \end{aligned}$$

For **scalar map** f, $\mathrm{osLip}(f) = \sup_x f'(x)$

For **affine map** $\mathsf{F}_A(x) = Ax + a$

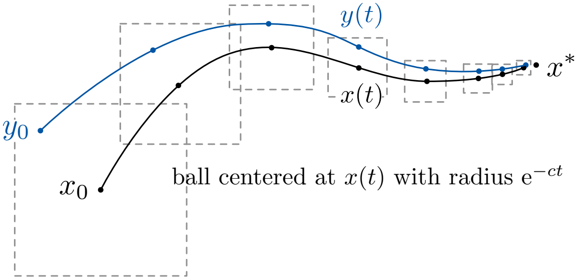
$$\mathrm{osLip}_{2,P^{1/2}}(\mathsf{F}_A) = \mu_{2,P^{1/2}}(A) \leq \ell \qquad \iff \qquad A^\top P + PA \preceq 2\ell P$$

$$\mathrm{osLip}_\infty(\mathsf{F}_A) = \mu_\infty(A) \leq \ell \qquad \iff \qquad a_{ii} + \sum_{j \neq i} |a_{ij}| \leq \ell$$

Banach contraction theorem for continuous-time dynamics:

If $-c := \mathrm{osLip}(\mathsf{F}) < 0$, then

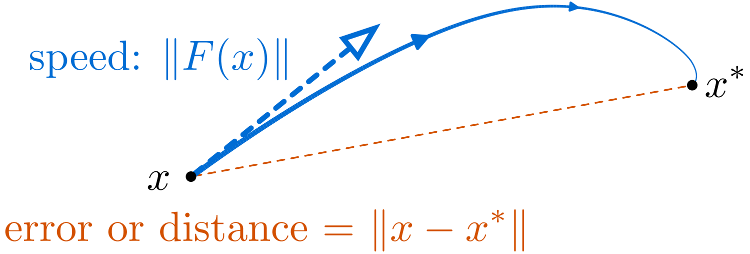
1. F is **infinitesimally contracting** = distance between trajectories decreases exp fast (e^{-ct})
2. F has a unique, glob exp stable equilibrium x^*



Outline

Speed and error

osLip(F) $= -c < 0$ and x^* is equilibrium



The two canonical Lyapunov functions and their relationship

Given $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$ and a norm $\|\cdot\|$ (with induced norm $\|\cdot\|$ and log norm $\mu(\cdot)$)

$$\dot{x}(t) = \mathsf{F}(x(t))$$

If $\operatorname{osLip}(\mathsf{F}) = -c < 0$ and $\mathsf{F}(x^*) = \mathbb{0}_n$, then

1. two global Lyapunov functions:

$$x \mapsto \|x - x^*\| \qquad \textbf{(error)}$$

$$x \mapsto \|\mathsf{F}(x)\| \qquad \textbf{(speed)}$$

2. for each $x(0) = x_0$ and $t \in \mathbb{R}_{\geq 0}$,

$$\|x(t) - x^*\| \;\leq\; \mathrm{e}^{-ct}\|x_0 - x^*\| \qquad \textbf{(error)}$$

$$\|\mathsf{F}(x(t))\| \;\leq\; \mathrm{e}^{-ct}\|\mathsf{F}(x_0)\| \qquad \textbf{(speed)}$$

3. if additionally $\operatorname{Lip}(\mathsf{F}) = \ell$,

$$c\|x - x^*\| \;\leq\; \|\mathsf{F}(x)\| \;\leq\; \ell\|x - x^*\|$$

A third Lyapunov function for symmetric Jacobians

If additionally $D\mathsf{F}(x) = D\mathsf{F}(x)^\top$ for all x, then

$$f(x) = -\int_0^1 x^\top \mathsf{F}(tx) dt$$

is a global Lyapunov function, c-strongly convex, and $\mathsf{F} = -\nabla f$

E.g.: for $\mathsf{F}_Q(x) = -Qx + q$ with $Q = Q^\top \succ 0$ and $q \in \mathbb{R}^n$,

$$x \mapsto \|x - Q^{-1}q\|_2^2, \qquad x \mapsto \|Qx - q\|_2^2, \qquad x \mapsto \frac{1}{2}x^\top Qx - q^\top x$$

More generally, $\mathsf{Cost} : \mathbb{R}^n \to \mathbb{R}$ such that

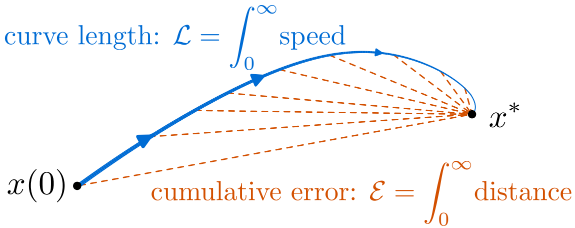
$$x^* = \operatorname*{argmin}_x \mathsf{Cost}(x)$$

$$\ell_{\mathsf{Cost}} = \mathsf{Lip}(\mathsf{Cost})$$

Outline

Cumulative error and curve length

osLip(F) $= -c < 0$ and x^* is equilibrium



$$\text{Length}(x_{[0,\infty)}) = \int_0^\infty \|\mathsf{F}(x(t))\| dt \leq \frac{1}{c}\|\mathsf{F}(x_0)\| \qquad \textbf{(curve length)}$$

$$\text{CErr}(x_{[0,\infty)}) = \int_0^\infty \|x(t) - x^*\| dt \leq \frac{1}{c}\|x_0 - x^*\| \qquad \textbf{(cumulative error)}$$

$$\text{CCost}(x_{[0,\infty)}) = \int_0^\infty \text{Cost}(x(t)) - \text{Cost}(x^*) dt \leq \frac{\ell_{\text{Cost}}}{c}\|x_0 - x^*\| \qquad \text{(cumulative cost)}$$

Outline

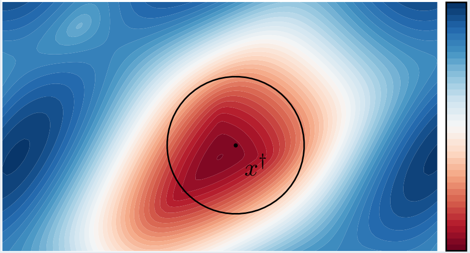
§1. Time-invariant contracting dynamics

- The continuous-time Banach contraction theorem
- Canonical Lyapunov functions
- Cumulative error and curve length
- Local contractivity and the small-residual theorem

§2. Time-varying contracting dynamics

- Incremental input-state-stability
- Equilibrium tracking and tube invariance
- Exact equilibrium tracking with feedforward control
- Dynamic regret
- Gradient dynamics and online feedback optimization

$$\dot{x}(t) = \mathsf{F}\big(x(t)\big)$$

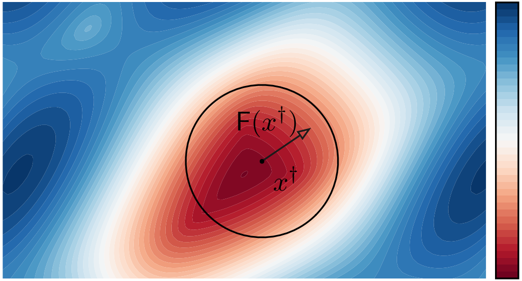


Example contour plot of $x \mapsto \mu(D\mathsf{F}(x))$
Red values are points x where $\mu(D\mathsf{F}(x)) < 0$
Blue values are points where $\mu(D\mathsf{F}(x)) > 0$

contracting set $S :=$ red region
closed ball $\overline{B}_r(x^\dagger) = \{x \text{ such that } \|x - x^\dagger\| \leq r\}$

Lemma: if contracting region S is invariant and convex,
then restrict F to S and usual contractivity properties apply

1. invariance of contracting set S?
2. convexity of contracting set S?



The small-residual theorem

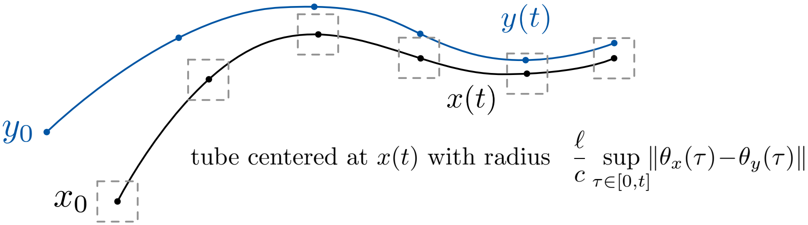
For $\dot{x} = F(x)$ infinitesimally contracting with rate $c > 0$ in region S

$$\overline{B}_r(x^\dagger) \subset S \quad \text{and} \quad \|F(x^\dagger)\| \leq cr \quad \Longrightarrow \quad \overline{B}_r(x^\dagger) \text{ is invariant}$$

Intuition: if $\|F(x^\dagger)\|/c \leq r$,
then solution from $x^\dagger$ remains inside $\overline{B}_r(x^\dagger)$ and converges to equilibrium point

Outline

Time-varying contracting dynamics



$$\dot{x}(t) = \mathsf{F}(x(t), \theta_x(t))$$
$$\dot{y}(t) = \mathsf{F}(y(t), \theta_y(t))$$

If $\|\theta_x(t) - \theta_y(t)\| \leq \delta$ for all t,

then $y(t)$ approaches or remains inside the tube with center $x(t)$ and radius $\dfrac{\ell\delta}{c}$

Incremental input-state-stability

For parameter-dependent vector field $\mathsf{F} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ and differentiable $\theta : \mathbb{R}_{\geq 0} \to \Theta \subset \mathbb{R}^d$

$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t))$$

Assume there exist norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\Theta}$ s.t.

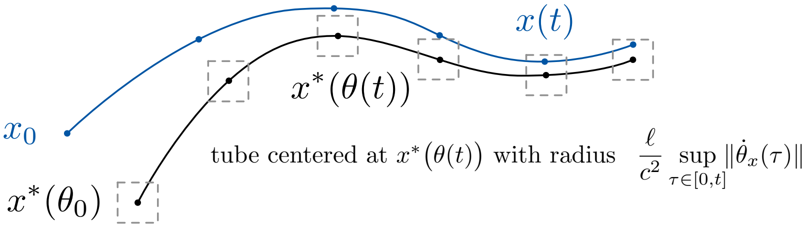
- **contractivity wrt x:** $\mathrm{osLip}_x(\mathsf{F}) \leq -c < 0,$ uniformly in θ
- **Lipschitz wrt θ:** $\mathrm{Lip}_\theta(\mathsf{F}) \leq \ell,$ uniformly in x

Theorem: Incremental ISS. Any two soltns: $x(t)$ with input θ_x and $y(t)$ with input θ_y

$$\|x(t) - y(t)\| \;\leq\; \mathrm{e}^{-ct}\|x_0 - y_0\| \;+\; \frac{\ell}{c} \sup_{\tau \in [0,t]} \|\theta_x(\tau) - \theta_y(\tau)\| \qquad \text{(error)}$$

Outline

Equilibrium tracking and tube invariance



$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t))$$
$$x^*(\theta(t)) = \text{equilibrium trajectory}$$

If $\|\dot{\theta}(t)\| \leq \delta$ for all t,

then $x(t)$ approaches or remains inside the tube with center $x^\star(\theta(t))$ and radius $\frac{\ell\delta}{c^2}$

Equilibrium tracking

For parameter-dependent vector field $\mathsf{F} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ and differentiable $\theta : \mathbb{R}_{\geq 0} \to \Theta \subset \mathbb{R}^d$

$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t))$$

Assume there exist norms $\|\cdot\|$ and $\|\cdot\|$ s.t.

- **contractivity wrt x**: $\mathrm{osLip}_x(\mathsf{F}) \leq -c < 0,$ uniformly in θ
- **Lipschitz wrt θ**: $\mathrm{Lip}_\theta(\mathsf{F}) \leq \ell,$ uniformly in x

Theorem: Equilibrium tracking for contracting dynamics.

The equilibrium map $x^\star(\cdot)$ is Lipschitz with constant $\dfrac{\ell}{c}$ and

$$\|x(t) - x^\star(\theta(t))\| \;\leq\; \mathrm{e}^{-ct}\big\|x_0 - x^\star(\theta_0)\big\| \;+\; \frac{\ell}{c^2}\sup_{\tau>0}\|\dot{\theta}(\tau)\| \qquad \textbf{(error)}$$

$$\|\mathsf{F}\big(x(t), \theta(t)\big)\| \;\leq\; \mathrm{e}^{-ct}\big\|\mathsf{F}\big(x_0, \theta_0\big)\big\| \;+\; \frac{\ell}{c}\sup_{\tau>0}\|\dot{\theta}(\tau)\| \qquad \textbf{(speed)}$$

Outline

Exact equilibrium tracking with feedforward control

For parameter-dependent vector field $\mathsf{F} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ and differentiable $\theta : \mathbb{R}_{\geq 0} \to \Theta \subset \mathbb{R}^d$

$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t))$$

- **contractivity wrt x**: $\mathrm{osLip}_x(\mathsf{F}) \leq -c < 0$, uniformly in θ
- Lipschitz wrt θ: $\mathrm{Lip}_\theta(\mathsf{F}) \leq \ell$, uniformly in x

If additionally F is differentiable in both arguments, then **inverse function theorem**

$$D_\theta x^\star(\theta) = -\big(D_x\mathsf{F}(x^\star(\theta), \theta)\big)^{-1} D_\theta \mathsf{F}(x^\star(\theta), \theta).$$

Exact equilibrium tracking with feedforward control

Time-varying contracting dynamics with feedforward prediction

$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t)) - \big(D_x\mathsf{F}(x(t), \theta(t))\big)^{-1} D_\theta\mathsf{F}(x(t), \theta(t))\, \dot{\theta}(t)$$

Asymptotically exact equilibrium tracking

$$\big\|\mathsf{F}\big(x(t), \theta(t)\big)\big\| \;\leq\; \mathrm{e}^{-ct}\|\mathsf{F}\big(x_0, \theta_0\big)\| \qquad \textbf{(speed)}$$

$$\|x(t) - x^\star(\theta(t))\| \;\leq\; \frac{1}{c}\mathrm{e}^{-ct}\|\mathsf{F}(x_0, \theta_0)\| \;\overset{\ell_x = \mathsf{Lip}_x(\mathsf{F})}{\leq}\; \frac{\ell_x}{c}\mathrm{e}^{-ct}\|x_0 - x^\star(\theta_0)\| \qquad \textbf{(error)}$$

E.g., if $F = -\nabla_x f$, then $\dot{x} = -\nabla_x f(x, \theta) + \big(\mathrm{Hess}\, f(x, \theta)\big)^{-1} D_\theta \nabla_x f(x, \theta)\, \dot{\theta}$

Exact equilibrium tracking with feedforward control – discrete time

Conjecture: no exact tracking is possible in discrete time

A. Simonetto and E. Dall'Anese. Prediction-correction algorithms for time-varying constrained optimization. *IEEE Transactions on Signal Processing*, 65(20):5481–5494, 2017.

Outline

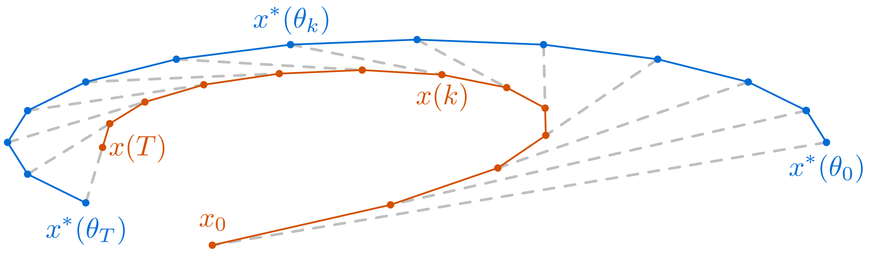
§1. Time-invariant contracting dynamics

- The continuous-time Banach contraction theorem
- Canonical Lyapunov functions
- Cumulative error and curve length
- Local contractivity and the small-residual theorem

§2. Time-varying contracting dynamics

- Incremental input-state-stability
- Equilibrium tracking and tube invariance
- Exact equilibrium tracking with feedforward control
- Dynamic regret
- Gradient dynamics and online feedback optimization

Dynamic regret



$$\text{CErr}\big(x_{[0,T]}, \theta_{[0,T]}\big) \;=\; \int_0^T \big\|x(t) - x^*\big(\theta(t)\big)\big\| dt \qquad \textbf{(cumulative error)}$$

$$\text{Regret}\big(x_{[0,T]}, \theta_{[0,T]}\big) \;=\; \int_0^T \text{Cost}\big(x(t), \theta(t)\big) - \text{Cost}\big(x^*(\theta(t)), \theta(t)\big) dt \qquad \textbf{(dynamic regret)}$$

Note: $\text{Length}\big(x^*_{[0,T]}\big) = \int_0^T \|\dot{x}^*(\theta(t))\| dt \overset{\text{Lip}(x^*) = \ell/c}{\leq} \frac{\ell}{c} \int_0^T \|\dot{\theta}(t)\| dt = \frac{\ell}{c} \text{Length}\big(\theta_{[0,T]}\big)$

Dynamic regret

For parameter-dependent vector field $\mathsf{F} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ and differentiable $\theta : \mathbb{R}_{\geq 0} \to \Theta \subset \mathbb{R}^d$

$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t))$$

Assume there exist norms $\|\cdot\|_{\mathcal{X}}$ and $\|\cdot\|_{\Theta}$ s.t.

- **contractivity wrt x**: $\mathrm{osLip}_x(\mathsf{F}) \leq -c < 0$, uniformly in θ
- **Lipschitz wrt θ**: $\mathrm{Lip}_\theta(\mathsf{F}) \leq \ell$, uniformly in x

Cumulative tracking error

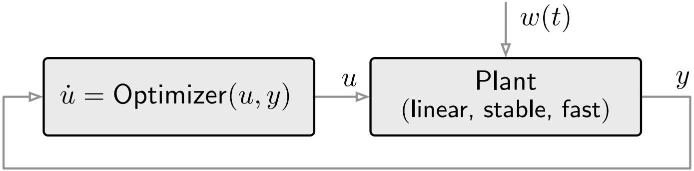
$$\mathsf{CErr}\big(x_0, \theta_{[0,T]}\big) \;\leq\; \frac{1}{c}\big\|x_0 - x^*(\theta_0)\big\| + \frac{\ell}{c^2}\mathsf{Length}\big(\theta_{[0,T]}\big)$$

Dynamic regret (for a $\mathsf{Cost}(x,\theta)$ that is ℓ_{Cost}-Lipschitz in x)

$$\mathsf{Regret}\big(x_0, \theta_{[0,T]}\big) \;\leq\; \ell_{\mathsf{Cost}}\mathsf{CErr}\big(x_0, \theta_{[0,T]}\big) \;=\; \mathcal{O}\big(1 + \mathsf{Length}\big(\theta_{[0,T]}\big)\big)$$

Outline

Solving optimization problems via dynamical systems



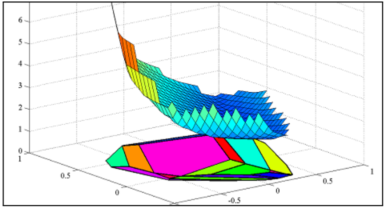
- studies in linear and nonlinear programming (Arrow, Hurwicz, and Uzawa 1958)
- neural networks (Hopfield and Tank 1985) and analog circuits (Kennedy and Chua 1988)
- optimization on manifolds (Brockett 1991)
- ...
- online and dynamic feedback optimization (Dall'Anese, Dörfler, Simonetto, ...)

A. Davydov, V. Centorrino, A. Gokhale, G. Russo, and F. Bullo. Time-varying convex optimization: A contraction and equilibrium tracking approach. *IEEE Transactions on Automatic Control*, June 2023. doi. Conditionally accepted

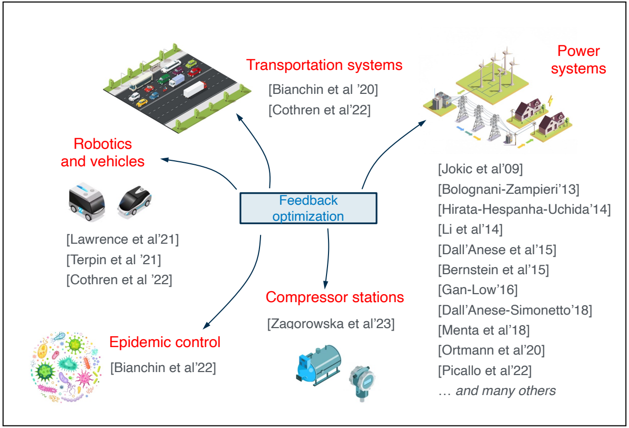
L. Cothren, F. Bullo, and E. Dall'Anese. Online feedback optimization and singular perturbation via contraction theory. *SIAM Journal on Control and Optimization*, Aug. 2024. doi. Submitted

Motivation: Optimization-based control

1. parametric optimization
2. **online feedback optimization**
3. model predictive control
4. control barrier functions
5. ...



parametric QP. YALMIP + Multi-Parametric Toolbox



Online feedback optimization. Courtesy of Emiliano Dall'Anese.

Parametric and time-varying convex optimization

$$\min \mathcal{E}(x) \quad \Longleftrightarrow \quad \dot{x} = \mathsf{F}(x) \qquad \rightsquigarrow \quad x^*$$

Parametric and time-varying convex optimization

1. **parametric contracting dynamics for parametric convex optimization**

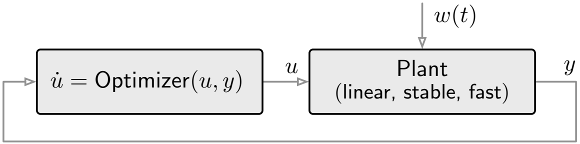
$$\min \mathcal{E}(x,\theta) \quad \Longleftrightarrow \quad \dot{x} = \mathsf{F}(x,\theta) \qquad \rightsquigarrow \quad x^*(\theta)$$

2. **contracting dynamics for time-varying strongly-convex optimization**

$$\min \mathcal{E}\big(x,\theta(t)\big) \quad \Longleftrightarrow \quad \dot{x} = \mathsf{F}\big(x,\theta(t)\big) \qquad \rightsquigarrow \quad x^*\big(\theta(t)\big)$$

A. Davydov, V. Centorrino, A. Gokhale, G. Russo, and F. Bullo. Time-varying convex optimization: A contraction and equilibrium tracking approach. *IEEE Transactions on Automatic Control*, June 2023. doi. Conditionally accepted

Application: Online feedback optimization



$$\begin{cases} \min & \text{cost}_1(u) + \text{cost}_2(y) \\ \text{subj. to} & y = \text{Plant}\big(u, w(t)\big) \end{cases} \implies \begin{cases} \dot{u} = \text{Optimizer}(u, y) \\ y = \text{Plant}\big(u, w(t)\big) \end{cases}$$

Gradient controller

Online feedback optimization

$$u^*\big(w(t)\big) \;:=\; \underset{u}{\operatorname{argmin}} \;\; \phi(u) + \psi(y(t)) \qquad \text{(c-strongly convex ϕ, convex ψ)}$$
$$\text{subj to} \;\; y(t) = Y_u u + Y_w w(t)$$

gradient controller

$$\dot{u} \;=\; \mathsf{F}_{\mathsf{GradCtrl}}(u,w) := -\nabla_u\big(\phi(u) + \psi(y(t))\big) \;=\; -\nabla\phi(u) - Y_u^\top \nabla\psi(Y_u u + Y_w w)$$

Contractivity of the gradient controller $\Longrightarrow$ eq. tracking + regret estimates

1. $u(t)$ approaches or remains inside

   the tube with center $u^*\big(w(t)\big)$ and radius $\dfrac{\ell_w}{c^2} \sup_{\tau\le t} \|\dot{w}(\tau)\|$

2. Regret $\le \dfrac{\ell_x}{c}\big\|u_0 - u^*\big(w_0\big)\big\| + \dfrac{\ell_x \ell_w}{c^2}\mathsf{Length}\big(\theta_{[0,T]}\big)$

Selected references from my group

Contracting neural networks:

- A. Davydov, A. V. Proskurnikov, and F. Bullo. Non-Euclidean contractivity of recurrent neural networks. In *American Control Conference*, pages 1527–1534, Atlanta, USA, May 2022.
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Optimization:

- A. Davydov, S. Jafarpour, A. V. Proskurnikov, and F. Bullo. Non-Euclidean monotone operator theory and applications. *Journal of Machine Learning Research*, 25(307):1–33, 2024. . URL **http://jmlr.org/papers/v25/23-0805.html**
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Conclusion

1. canonical properties of contracting dynamics for optimization
    - error and speed Lyapunov functions (and the gradient case)
    - curve length and cumulative error
    - incremental ISS
    - equilibrium tracking and feedforward control
    - dynamic regret
2. gradient controller
3. local invariance

search for	contraction properties
design	engineering systems to be contracting
verify	correct/safe behavior via known Lipschitz constants