# Contraction Theory of Output Regulation

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# Outline

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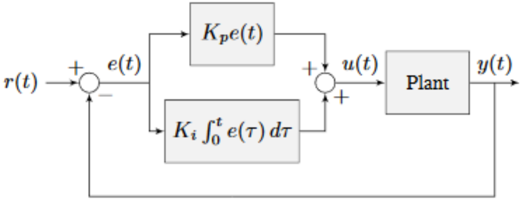
# Introduction to Integral Action

- Goal of feedback: achieve a prescribed goal in presence of model uncertainties
- For set-point tracking: P, PI, PID
- Example of standard PI controller:




- Taught in any basic course of control
- Industrial applications

# Some milestones

1868: James Clerk Maxwell: "On Governors"
Scottish physicist and mathematician

1911: Elmer Ambrose Sperry: PID controller using a marine gyro compass for automatic steering of ships.
American inventor and entrepreneur

1922: Minorsky: "Directional stability of automatically steered bodies"
American mathematician
*"the second class of controllers (= PI) has the remarkable result that such a (constant) disturbance has no influence upon the device"*

1931: Foxboro, USA (Schneider Electric Company since 2014): pneumatic differential PI controller named Stabilog Model 10

1942: Ziegler-Nichols method for optimal tuning of a PID controller

# Output Regulation Theory

Integral action as a special case of robust output regulation (servomechanism problem)

$$\begin{aligned} \dot{w} &= Sw \\ \dot{x} &= Ax + Bu + Pw \\ e &= Cx + Qw \end{aligned}$$

$w$ : perturbations and references

Regulation objective: $\lim_{t\to\infty} e(t) = 0$

1976: Francis, Wohnam and Davison: the internal model principle

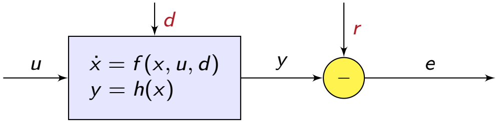
*"the output regulation property is insensitive to plant parameter variations only if the controller utilizes feedback of the regulated variable, and incorporates in the feedback path a suitably reduplicated model of the dynamic structure of the exogenous signals which the regulator is required to process"*

$\Longrightarrow$ Necessity and Sufficiency of an integral action for set-point tracking and constant perturbation rejection

- The adjective "robust" has a different meaning w.r.t. to robust control theory

From 80's: Development of nonlinear output regulation theory

# How the Integral Action works




- $x$: state
- $u$: control
- $e$: regulated output
- $(r, d)$: constant references and perturbations

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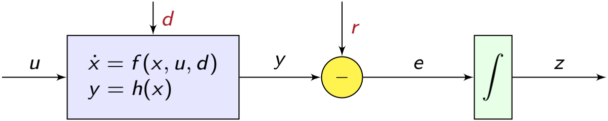


1. Add the integral action on the regulated output $e$
2. Design a feedback so that the for a given pair $(r, d)$, the closed-loop system admits an asymptotically stable equilibrium $(x^\circ, z^\circ)$
3. On the equilibrium $(x^\circ, z^\circ)$, we have

$$0 = \dot{z} = e \qquad \Longleftrightarrow h(x^\circ) = r$$

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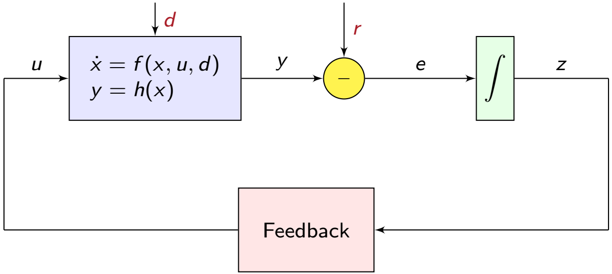


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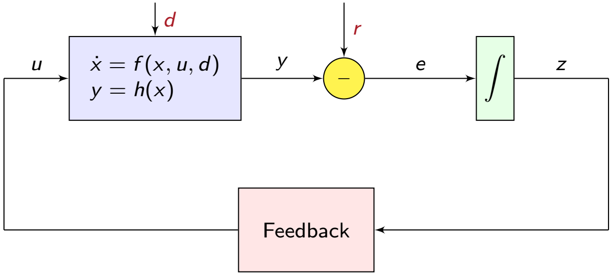


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# Main difficulties

- Design of a stabilizing feedback for the extended system $(x, z)$ to guarantee the existence of an equilibrium
  - domain of attraction (DoA)?
  - uniformity with respect to $(d, r)$?

- Persistence of an equilibrium in presence of model uncertainties $\Delta_f, \Delta_h$

# Outline

# Revising the linear approach

Consider the linear system

$$\begin{cases} \dot{x} = Ax + Bu + d & x \in \mathbb{R}^n, \; u \in \mathbb{R}^m \\ e = Cx - r & e \in \mathbb{R}^p \end{cases}$$

## Theorem

*Suppose that*

- $(A, B)$ *is stabilizable;*
- *non-resonance condition* $\operatorname{rank} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = n + p$ *holds*[1].

*Then, there exists* $K, L$ *such that the controller*

$$\begin{cases} \dot{z} = e \\ u = Kx + Lz \end{cases}$$

*solves the robust output regulation problem, i.e.* $\lim_{t \to \infty} e(t) = 0$ *for all initial conditions* $(x_0, z_0) \in \mathbb{R}^n \times \mathbb{R}^p$*, all constant references and disturbances* $(r, d)$ *and for small model perturbations* $\Delta_A, \Delta_B, \Delta_C$.

[1] This conditions is equivalent to ask that the transfer function between $u$ and $e$ has no zeros at the origin.

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# Sketch of the proof (1)

- $(A, B)$ stabilizable and the non-resonance condition implies the extended system

$$\left( \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, \begin{pmatrix} B \\ 0 \end{pmatrix} \right)$$

is stabilizable, i.e., there exists $K, L$ so that $F := \begin{pmatrix} A + BK & BL \\ C & 0 \end{pmatrix}$ is Hurwitz.

- For any $(d, r)$, the system

$$\begin{cases} \dot{x} = (A + BK)x + BLz + d \\ \dot{z} = Cx - r \end{cases}$$

admits an equilibrium $(x^\circ, z^\circ)$ given by

$$0 = \begin{pmatrix} A + BK & BL \\ C & 0 \end{pmatrix} \begin{pmatrix} x^\circ \\ z^\circ \end{pmatrix} + \begin{pmatrix} d \\ -r \end{pmatrix} \implies \begin{pmatrix} x^\circ \\ z^\circ \end{pmatrix} = F^{-1} \begin{pmatrix} d \\ -r \end{pmatrix}$$

- This equilibrium $(x^\circ, z^\circ)$ is GAS due to the stability of $F$. This can be shown in error coordinates $(x - x^\circ, z - z^\circ)$.

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- Note that if

$$\begin{pmatrix} A+BK & BL \\ C & 0 \end{pmatrix}$$

is Hurwitz, for small perturbations $\Delta_A, \Delta_B, \Delta_C$, stability of

$$\begin{pmatrix} (A+\Delta_A)+(B+\Delta_B)K & (B+\Delta_B)L \\ (C+\Delta_C) & 0 \end{pmatrix}$$

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## Takeaway message from linear world

**Linear lesson:**

- The design of the feedback is independent of $(d, r)$
- Stability guarantees the existence of a unique equilibrium for any $(d, r)$ and in presence of model uncertainties

**Question:**

- Can we extend such a paradigm to the nonlinear ODEs?

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# Outline

# A Stabilization of cascade systems

- Consider a system of the form

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ \dot{z} = h(x) \end{cases}$$

- **Problem:** design a feedback $u = \alpha(x, z)$ guaranteeing origin GAS (and LES)

- **A possible solution:** forwarding design

  see, e.g., Praly, Mazenc, A. Astolfi, Ortega, Teel, Sepulchre, Kokotovic, Kristic, ..

- **Remark:** a necessary condition for the existence of $\alpha$ is the existence of a $\alpha_0$:

$$\dot{x} = f(x) + g(x)\alpha_0(x) \qquad \text{is GAS}$$

  In the following we suppose this step has already been done:

$$\dot{x} = f(x) \qquad \text{is GAS and LES}$$

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# Forwarding design: main ideas

- Consider a system of the form

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ \dot{z} = h(x) \end{cases}$$

- Let $x$ be the fast stable dynamics and $z$ the slow one

- If the origin of $\dot{x} = f(x)$ is GAS and LES, we can define an invariant-manifold for $z = M(x)$ satisfying

$$L_f M(x) := \frac{\partial M}{\partial x}(x) f(x) = h(x)$$

- Consider the change of coordinates $\zeta := z - M(x)$ giving-

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ \dot{\zeta} = -L_g M(x) u \end{cases}$$

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# Forwarding design: main ideas (2)

- We have

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- With the Lypaunov function $W(x,\zeta) = V(x) + \frac{1}{2}\zeta^2$ we obtain

$$\dot{W} \leq L_f V(x) + (L_g V(x) - \zeta L_g M(x))\, u$$

$\Longrightarrow$ Select $u = -(L_g V(x) - \zeta L_g M(x))$ to obtain a negative derivative

- We obtain

$$\dot{W} \leq L_f V(x) - u^2$$

- If $L_g M(0) \neq 0$ we can conclude stability of the origin $(x,\zeta) = 0$.

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# Forwarding design: main result

## Lemma (Existence of the Invariant Manifold)

*Suppose*

- *The origin of* $\dot{x} = f(x)$ *is GAS and LES,* $f(0) = 0$, $h(0) = 0$
- *The non-resonance condition* $CA^{-1}B \neq 0$ *holds, with*

$$A := \frac{\partial f}{\partial x}(0), \quad B := g(0), \quad C := \frac{\partial h}{\partial x}(0).$$

*Then, there exists a* $C^2$ *function* $M$ *satisfying* $M(0) = 0$, $\frac{\partial M}{\partial x}(0) = CA^{-1}$ *and* [2]

$$L_f M(x) = h(x), \qquad L_g M(0) \neq 0.$$

## Theorem (Forwarding Stabilization)

*There exists a feedback* $u = \alpha(x, z)$ *such that the origin of*

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[2] $L_g M(0)$ is the DC-gain at the origin of $\dot{x} = f(x) + g(x)u$, $y = h(x)$.

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# Forwarding and perturbations

Consider now the perturbed system:

$$\dot{x} = f(x, d) + g(x, d)\alpha(x, z)$$
$$\dot{z} = h(x, r)$$

with $\alpha$ designed with the forwarding as shown before.

- When $(d, r) = 0$ then the origin is GAS and LES
- Can we conclude the existence of an equilibrium for $(d, r) \neq 0$ ?
- In general, only for small values, i.e. $|(d, r)| \leq \varepsilon$

  Total stability: equilibria are preserved under small perturbations

  [Astolfi Praly, TAC 2017]

- To achieve global results, we need stronger properties:

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# Towards the existence of a "global" controller

- Consider again the closed-loop system and use a compact notation

$$\left\{ \begin{array}{l} \dot{x} = f(x, \alpha(x,z), d) \\ \dot{z} = h(x) - r \end{array} \right. \qquad \Longrightarrow \qquad \dot{\xi} = F(\xi, (d, r))$$

- Global regulation can be obtained if the vector field $F$ admits a fixed point for any $(d, r)$

- **Main Idea:** use Banach fixed point theorem..

$\Longrightarrow$ we need $F$ to be a (uniform) contraction

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# Some highlights on Contraction Theory

## Contraction with Riemannian metric

The vector field $F : \mathbb{R}^n \to \mathbb{R}^n$ is contracting if (and only if, for $F$ $C^2$ globally Lipschitz) there exists a (Riemannian) metric $P : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ taking SPD values and $\bar{p}, \underline{p}, q > 0$ such that[3]

$$\underline{p}I \leq P(\xi) \leq \bar{p}I, \quad L_F P(\xi) := \dot{P}(\xi) + P(\xi)\frac{\partial F}{\partial \xi}(\xi) + \frac{\partial F}{\partial \xi}(\xi)^\top P(\xi) \preceq -qI, \quad \forall \xi \in \mathbb{R}^n.$$

**Remark:** generalization of Lyapunov Matrix Inequality $PA + A^\top P \preceq -Q$

## Contraction and Incremental Stability

Suppose $F$ is a contraction. Then $\exists k, \lambda > 0$ such that solutions to system $\dot{\xi} = F(\xi)$ satisfy

$$|\phi(\xi_a, t) - \phi(\xi_b, t)| \leq k\,|\xi_a - \xi_b| \exp(-\lambda t) \qquad \forall \xi_a, \xi_b \in \mathbb{R}^n.$$

In other words, the system is Incrementally Globally Exponentially Stable ($\delta$GES).

[3] The notation $\dot{P}$ defines a matrix with its $ij$-th elements defined as $\dot{P}_{ij} = \frac{\partial P_{ij}}{\partial \xi} F(\xi)$

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# Contraction and equilibria

## Incremental Stability and Equilibria

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**Sketch of the proof:**

- Select $\tau$ such that $k \exp(-\lambda\tau) < 1$.
- The application $\xi \mapsto \phi(\xi, \tau)$ defines a contraction:

$$|\phi(\xi_a, \tau) - \phi(\xi_b, \tau)| < |\xi_a - \xi_b| \qquad \forall\, \xi_a, \xi_b \in \mathbb{R}^n.$$

- Banach fixed point theorem gives existence and uniqueness of $\xi^\circ$.

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## Incremental Stability and Equilibria

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In order to apply previous theorem we need the system

$$\left\{ \begin{array}{l} \dot{x} = f(x, \alpha(x, z), d) \\ \dot{z} = h(x) - r \end{array} \right. \qquad \Longleftrightarrow \qquad \dot{\xi} = F(\xi, (d, r))$$

to be $\delta$GES uniformly in $(d, r)$.

$\Longrightarrow$ we need a uniform contraction

$$\dot{P}(\xi, (d, r)) + P(\xi)\frac{\partial F}{\partial \xi}(\xi, (d, r)) + \frac{\partial F}{\partial \xi}(\xi, (d, r))^\top P(\xi) \leq -qI \qquad \forall\ \xi, (d, r)$$

# Killing vector and uniform contraction

## Killing vector

Given a 2-tensor $P : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, a $C^1$ function $G : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a Killing vector for $P$ if $L_G P(\xi) = 0$ for all $\xi \in \mathbb{R}^n$.

- Consider a system

$$\dot{\xi} = F(\xi) + G(\xi) w$$

- Suppose there exist a $P$ such that that
  - $F$ is contractive w.r.t $P$: $L_F P(\xi) \preceq -qI$
  - $G$ is a Killing vector for $P$: $L_G P(\xi) = 0$

- Then,

$$L_F P(\xi) + L_G P(\xi)\, w \preceq -qI \qquad \forall\, \xi, w$$

$\Longrightarrow$ the system defines a uniform contraction $\Longrightarrow$ is $\delta$GES uniformly $\forall w$!

**Problem:** design of the feedback $\alpha(x, z)$ satisfying previous conditions

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# Sufficient conditions for contractive forwarding

Consider the system

$$\begin{cases} \dot{x} = f(x) + g(x)(u + d) \\ \dot{z} = h(x) - r \end{cases}$$

## Theorem (Incremental Uniform Global Forwarding Stabilization)

Suppose that

- $f$ is contraction[4] for $P$ and $g$ is a Killing for $P$
- there exists a function $M : \mathbb{R}^n \to \mathbb{R}$ and a constant $\gamma > 0$ satisfying

$$L_f M(x) = h(x), \qquad L_g M(x) = \gamma$$

Then, for any $k > 0$ the control law

$$u = k \left[ z - M(x) \right]$$

makes the closed-loop system uniformly contractive and $\lim_{t\to\infty} h(x) = r$ for any initial condition $(x_0, z_0) \in \mathbb{R}^n \times \mathbb{R}$ and any $(d, r)$.

[Giaccagli, Astolfi Andrieu, Marconi, TAC 2022]

[4] This can be also obtained after a preliminary state-feedback with Control Contraction Metrics: $L_f P(x) + P(x)g(x)g(x)^\top P(x) \preceq -qI$

# Some Remarks

- The first conditions corresponds to the stabilizability of $(A, B)$ in the proposed contractive framework
- The second condition correspond to a global uniform non-resonance condition, i.e., a controllability (contractive) condition for the extended system $(x, z)$
- Design based on the construction of a contraction metric for the closed-loop dynamics

✓ The control law depends on the solution of a PDE

$$L_f M(x) = h(x)$$

but there exist alternative designs to rely only on an approximation of $M$

✗ Conditions are restrictive due to the nature of the problem we aim at solving:
$\Longrightarrow$ The result is global in the initial conditions and in $(d, r)$

✗ We considered only disturbances $d$ satisfying a matching-condition

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# Outline

# Conclusions

**Takeaway messages:**

- We proposed sufficient conditions for the design of a global integral action based on forwarding and contraction analysis
- Integral action has many applications: PI-control, tracking, and also optimization

$$\begin{matrix} \min f(x) \\ Ax = b \end{matrix} \quad \Longrightarrow \quad \begin{aligned} \dot{x} &= \nabla f(x) - A^\top \lambda \\ \dot{\lambda} &= Ax - b \end{aligned}$$

16:30–16:55: Time-Varying Convex Optimization: A Contraction and Equilibrium Tracking Approach, Francesco Bullo

- Possible extension to periodic references/disturbances and harmonic regulation (no time in this presentation)

$$\begin{aligned} \dot{x} &= f(x) + g(x)(u + d(t)) \qquad & d(t+T) &= d(t) \\ e &= h(x) - r(t) & r(t+T) &= r(t) \end{aligned}$$

[Giaccagli, Astolfi Andrieu, Marconi, TAC 2024]

# Conclusions (2)

**Open Problems:**

- Can we relax $\delta$GES with $\delta$GAS to ensure the existence of an equilibrium?

  Kato, Astolfi, Andrieu, Praly, ''Incremental global asymptotic stability equals incremental global exponential stability - but at equilibria'', NOLCOS 2025

- Is the Killing vector condition necessary for global uniform contractions?

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- Can we relax contraction with 2-contraction to ensure the existence of an equilibrium?

  Giaccagli, Lorenzetti, Astolfi, Andrieu, ''PI-control for non-linear systems with multiple equilibria via 2-contraction'', NOLCOS 2025

  what is 2-contraction?

  11:00–11:25: On 2-Contraction and Non-Oscillatory Systems: Some Theory and Applications, David Angeli.

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# Thanks and references

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