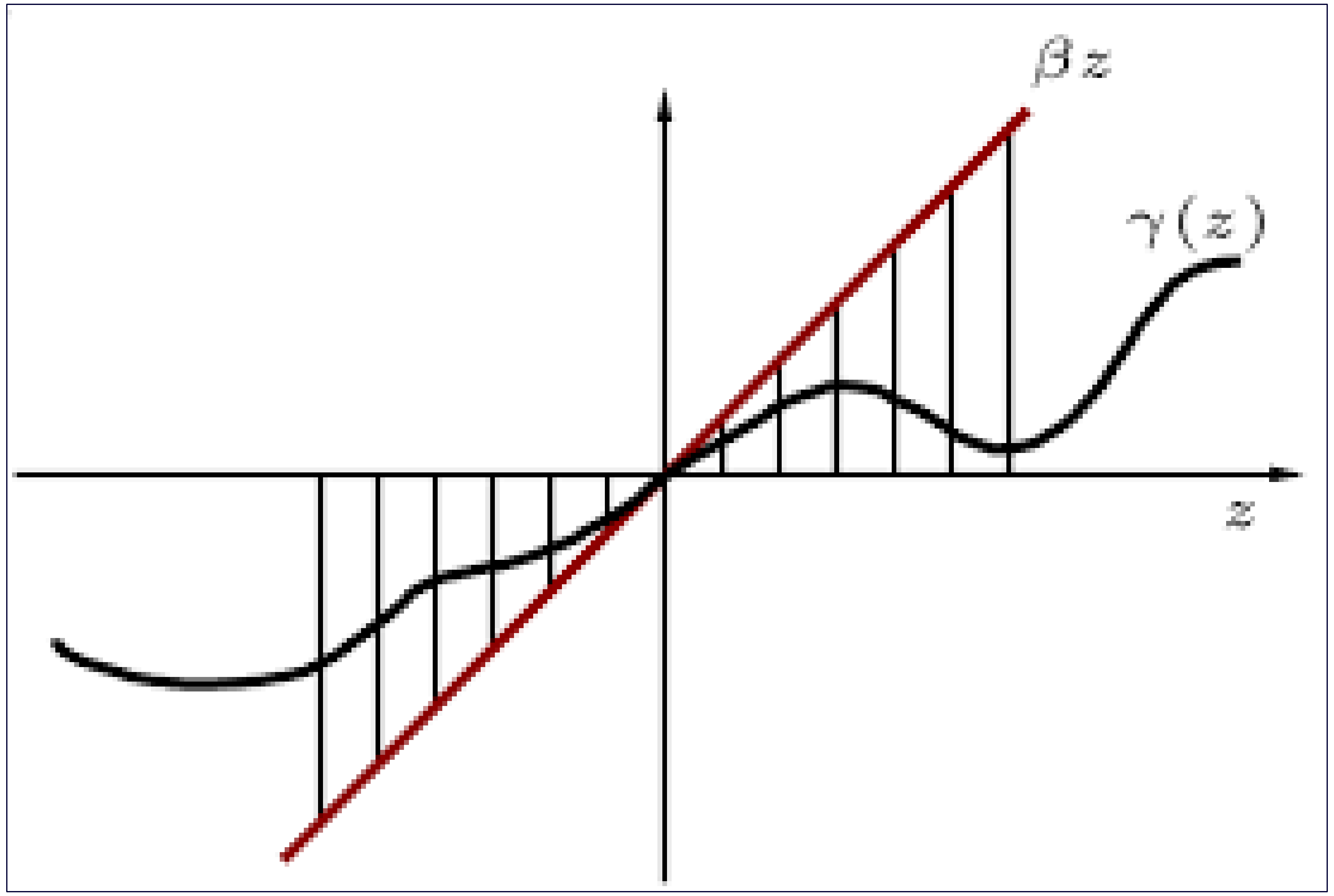

Towards Non-quadratic Absolute Stability Theory

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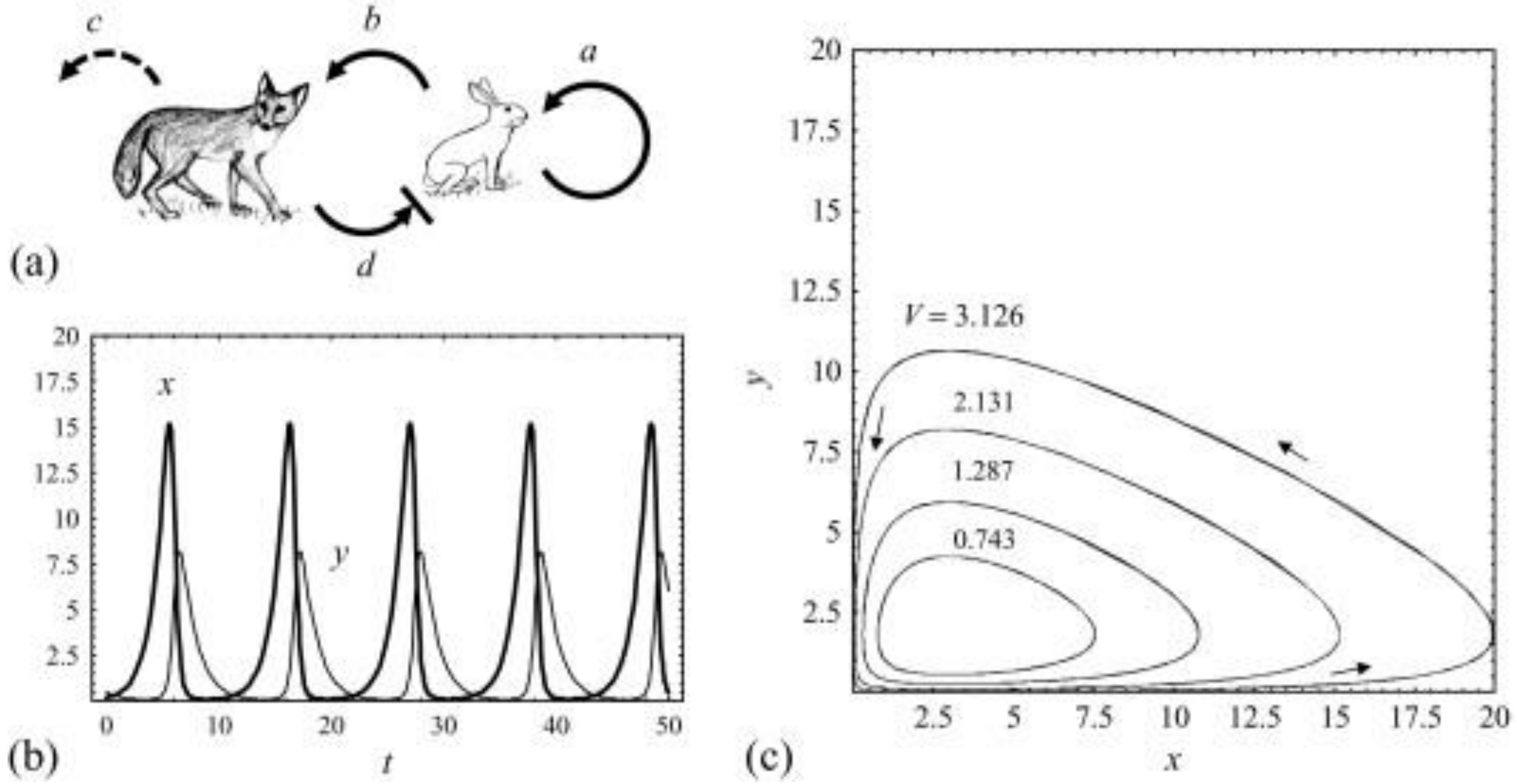
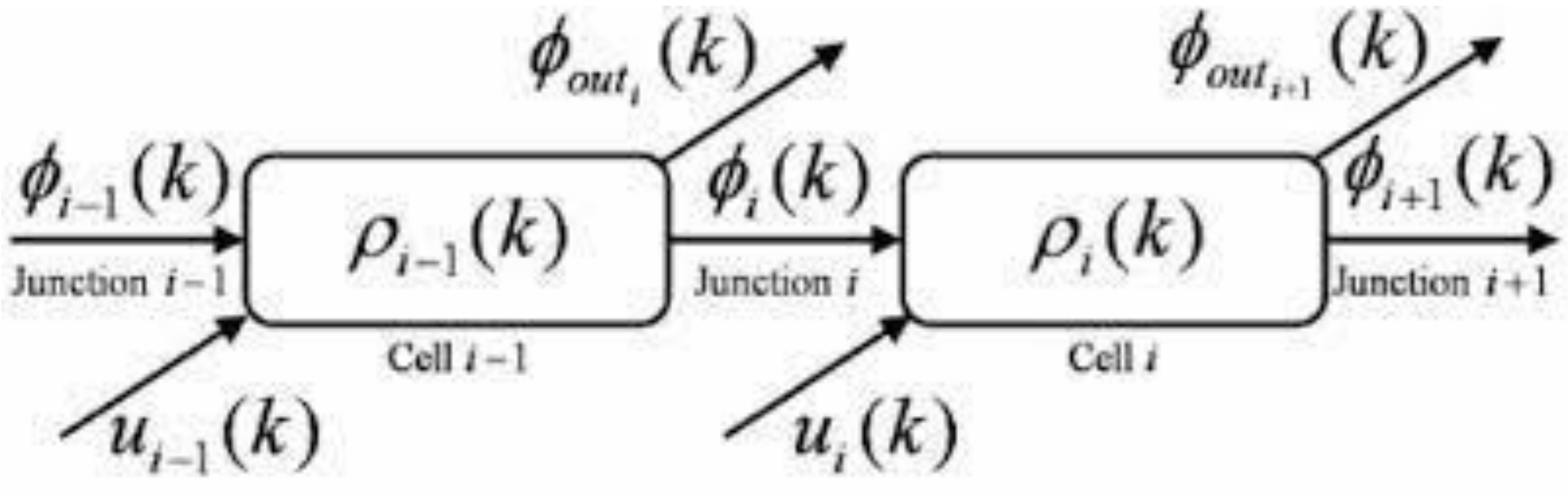
Plan of The Talk

- **Non-quadratic Lyapunov functions: Motivation**
- **Lur’e systems and classical S-lemma**
- **General norms as Lyapunov functions: Extended S-lemma**
- **Absolute stability in non-Euclidean norms.**

Non-Quadratic Lyapunov Functions





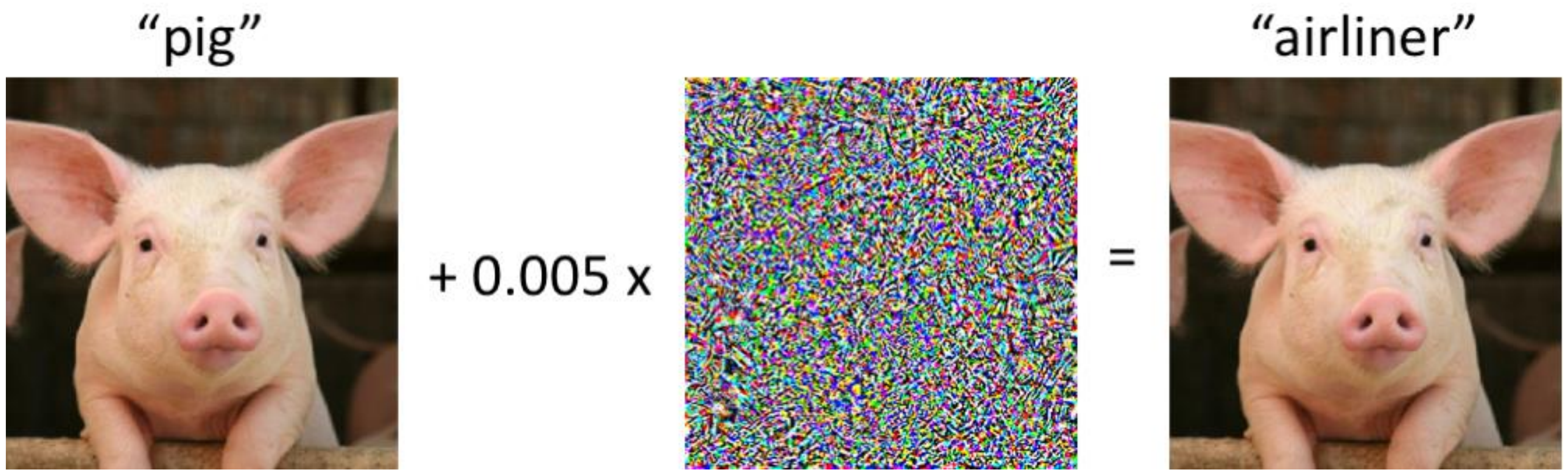
- **Non-quadratic LF** prove to be useful in many applications: Models of populations (Lotka-Volterra etc.), monotone systems, cell transmission models, compartmental systems, LPV systems etc. – **polyhedral and other LFs**
- Still underrepresented; regular design methods are still missing.
- This work: revise one of the oldest application of quadratic LF (Lur'e systems stability/contraction) and expand the design approach to some non-quadratic LF.

F. Bullo**, Contraction Theory for Dynamical Systems,** Kindle Direct Publishing, Edition 1.1, Mar 1, 2023

F. Bullo**, Lectures on Network Systems,** Kindle Direct Publishing, Edition 1.6, Jan 1, 2022

Contraction in ”Non-Standard” Norms: Resilience

Resilience and regularization of CNNs, RNNs, GNNs to adversarial attacks, efficient Lipschitz constants

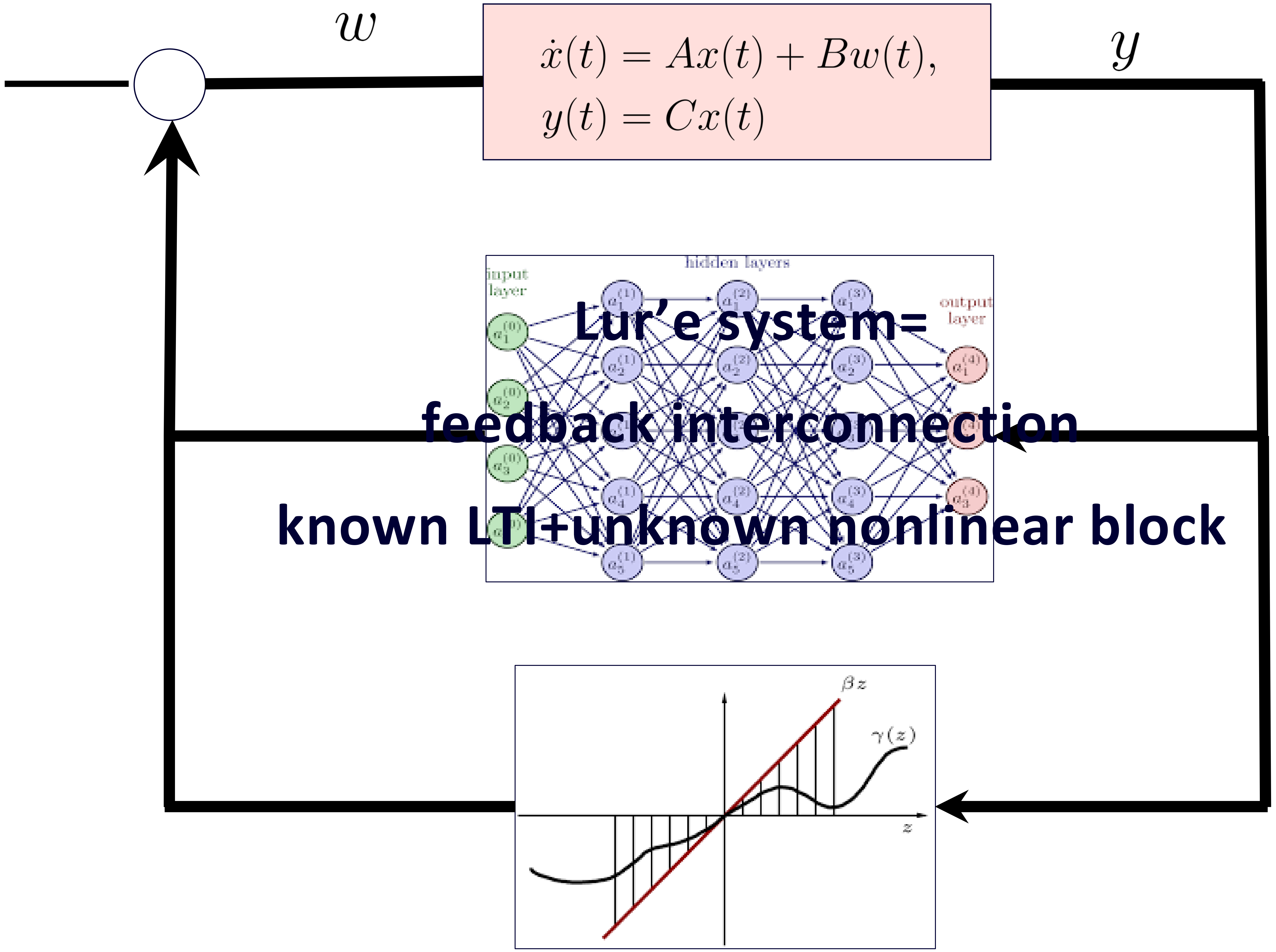


Slight perturbation of an image leads to incorrect classification: “adversarial examples”

- **Noises are naturally characterized in non-Euclidean norms (standard or weighted l_∞)**
- **Lipschitz constants are also mainly interesting in l_∞**
- **Computation of Lipschitz constants can be much more efficient than for the Euclidean norm by (LP vs. SDP). Algorithms to compute fixed-points of neural nets, tight convergence bounds in l_1 and l_∞**
- **Topological equivalence of norms on a space of a huge dimension is misleading: comparison factors are too large.**

Liang, Huang**, Large Norms of CNN Layers Do Not Hurt Adversarial Robustness//***2021,AAAI Conference on Artificial Intelligence,35(10),8565-8573.*
Davydov et al., **Non-Euclidean Contraction Analysis of Continuous-Time Neural Networks, IEEE TAC 2025 (online)**
Jafarpour et al., **Robust Implicit Networks via Non-Euclidean Contractions**//Adv. Neural Information Processing Systems 34, 2021

Lur'e Problem (1957): The Launchpad for Robust Nonlinear Control



Anatolii I. Lur'e (1901-1980)

- **Absolute stability** – **GAS of $x = 0$ for any nonlinear element from a given class**
- **Absolute contractivity – contractivity for any nonlinear element from a given class**

  Class – usually, very broad (sector, slope or other quadratic constraint)

Lur'e, Postnikov, Some Nonlinear Problems in the Theory of Automatic Control**, Her Majesty Stationary Office, 1957**

Absolute Stability: Two Main Tracks in 1960-1990s

Common Lyapunov stability certificates for the state space models

Lur'e, Barbashin, Persidskii, Kalman, Rosenwasser, Yakubovich, Barabanov, Narendra,…

$$V(x) = x^\top H x \qquad \left[+\int_0^y \varphi(s)ds \quad \text{(additional term, optional)}\right]$$

- **Lyapunov condition $\dot{V} < 0$:** quadratic constraints **+ S-Procedure**
- **Existence of the LF --** frequency-domain condition via KYP lemma (old) or **LMIs** (modern).
- **Depending on a problem, entails** GAS, contractivity, output stability, instability, oscillations.

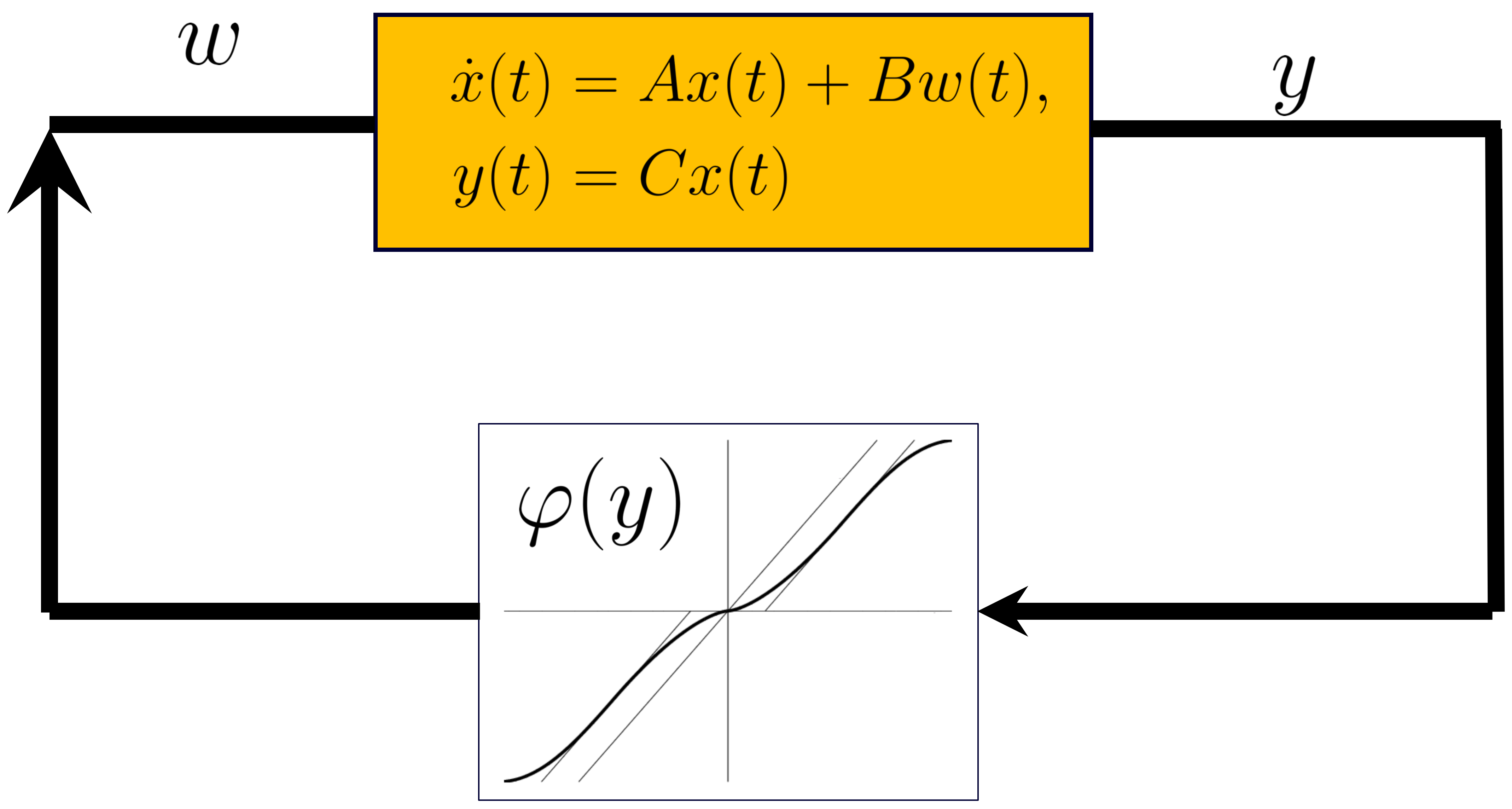
Integral operators in L2 space

Popov, Desoer, Vidyasagar, Yakubovich, Zames, Falb, Rasvan, Megretskii, Rantzer,…

General Volterra equations (also, infinite-dimensional)
Stability in integral L_2 norm, little information about transient process
Conditions in frequency domain: hard to verify in MIMO case

Vasile Mihai Popov
(born 1928)

S-Procedure: A Simple Example



$$0 \leq \frac{\varphi(y_1) - \varphi(y_2)}{y_1 - y_2} \leq \varkappa$$

S-procedure: implication (?) holds if

$$2(\Delta x)^\top H(A\Delta x + B\Delta w) + \varepsilon\|x\|_2^2 \leq \tau \Delta w(\varkappa^{-1}\Delta w - C\Delta x)$$

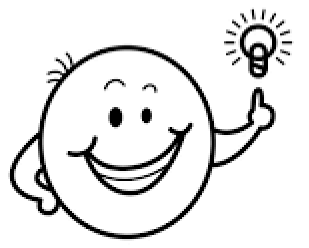
for some $\tau \geq 0$ and all variables

$$\Delta w \in \mathbb{R}, \Delta x \in \mathbb{R}^n.$$

It is an LMI on $H \succ 0, \tau \geq 0$

Try to prove contractivity using a quadratic (incremental) LF

$$V(\Delta x) = (\Delta x)^\top H(\Delta x) \Longrightarrow \dot{V} = 2(\Delta x)^\top H(A\Delta x + B\Delta w)$$



Lyapunov condition for contractivity: how to get it?

Slope inequality!

$$\dot{V} + \varepsilon\|\Delta x\|_2^2 \leq 0 \qquad \text{(A)}$$

(?) ⟵

$$\Delta w(\varkappa^{-1}\Delta w - \Delta y) \leq 0, \quad \Delta y = C\Delta x \qquad \text{(B)}$$

S-Lemma: S-Procedure is "Lossless".

Vladimir A. Yakubovich 1926-2012

The Yakubovich S-Lemma:

(?) holds for all variables

$$\Delta w \in \mathbb{R}, \Delta x \in \mathbb{R}^n$$

if and only if $\tau \geq 0$ exists satisfying the inequality

S-procedure finds **all** quadratic LFs helping to derive contraction from the single slope constraint.

S-procedure:

implication (?) holds if

$$2(\Delta x)^\top H(A\Delta x + B\Delta w) + \varepsilon\|x\|_2^2 \leq \tau \Delta w(\varkappa^{-1}\Delta w - C\Delta x)$$

for some $\tau \geq 0$ and all variables

$$\Delta w \in \mathbb{R}, \Delta x \in \mathbb{R}^n.$$

It is an LMI on $H \succ 0, \tau \geq 0.$

The idea of S-procedure is inspired by Lagrange multipliers.
There is an explicit relation between S-lemma and the Lagrange duality.

$$\dot{V} + \varepsilon\|\Delta x\|_2^2 \leq 0 \quad \overset{?}{\Longleftarrow} \quad \begin{aligned} \Delta w(\varkappa^{-1}\Delta w - \Delta y) &\leq 0, \\ \Delta y &= C\Delta x \end{aligned}$$

Further Developments of S-Lemma

- S.V. Gusev, A.L. Likhtarnikov, **Kalman-Popov-Yakubovich lemma and the S-procedure: A historical essay**, Automation and Remote Control, 67 (2006), pp. 1768–1810

- I. Polik and T. Terlaky, **A survey of the S-lemma**, SIAM Review, 49 (2007), pp. 371–418

$$x^\top Q_1 x \leq 0, \ldots, x^\top Q_n x \leq 0 \Longrightarrow x^\top P x \leq 0$$

- **Holds if there exist multipliers $\tau_1 \geq 0, \tau_2 \geq 0, \ldots, \tau_n \geq 0$ such that** $P \leq \sum_j \tau_j Q_j$

- **For usual quadratic forms, it is SDP feasibility. Can also be validated for integral functionals on L_2.**

- **The "only if" part is a rather exceptional property:**

a) n=1 (Yakubovich);
b) n=2 (Hermitian forms on a complex space (Yakubovich, Fradkov);
c) integral functionals on L_2 (Megretsky, Treil, Yakubovich, Matveev etc.) – n arbitrary.

Non-Quadratic Lyapunov Functions in Absolute Stability

- Special class of Non-quadratic LFs – squared norms $V(x) = \|x\|^2$
- General results on absolute stability of differential inclusions and switching systems (E.S. Pyatnitskii et al., Barabanov N.E.): existence of the non-Euclidean LF of this structure, approximations by polyhedral norms.

- Davydov et al. work - New insights about use of such LFs for contraction and stability

✓ **Derive the Lyapunov condition $\dot{V} < 0$ from "pseudo-quadratic" constraint + S-Lemma**

✓ **Existence of LF: conditions involving the log-norm ("matrix measure"). For each specific norm, convex optimization problem.**

✓ **Efficient stability/contractivity conditions in non-Euclidean norm, e.g., l_1 and l_∞ (important in machine learning, e.g., robustness certificates for NN, also for systems with conserved quantities, distributed iterations)**

Alexander Davydov, Saber Jafarpour, Francesco Bullo, Non-Euclidean Contraction Theory for Robust Nonlinear Stability, IEEE TAC 2022

Technical Prelimianries - I

Norm on a finite-dimensional space $\|\cdot\|$ → **Operator norm on matrices**

$$\|A\| = \max_{\|x\|=1} \|Ax\|$$

Compatible "weak pairing" (WP) -------------- **Log-norm ("measure") on matrices**

$$[x,x] = \|x\|^2, \forall x \qquad [\cdot,\cdot]$$

$$\mu(A) = \lim_{h\to 0+} \frac{\|I + hA\| - 1}{h}$$

a *weak pairing* on $\mathbb{R}^n$ is a map $[\![\cdot,\cdot]\!] : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ satisfying

(i) (Subadditivity and continuity of first argument) $[\![x_1 + x_2, y]\!] \leq [\![x_1, y]\!] + [\![x_2, y]\!]$, for all $x_1, x_2, y \in \mathbb{R}^n$ and $[\![\cdot,\cdot]\!]$ is continuous in its first argument,

(ii) (Weak homogeneity) $[\![\alpha x, y]\!] = [\![x, \alpha y]\!] = \alpha\, [\![x, y]\!]$ and $[\![-x, -y]\!] = [\![x, y]\!]$, for all $x, y \in \mathbb{R}^n$, $\alpha \geq 0$,

(iii) (Positive definiteness) $[\![x, x]\!] > 0$, for all $x \neq 0_n$,

(iv) (Cauchy-Schwarz inequality) $|[\![x, y]\!]| \leq [\![x, x]\!]^{1/2} [\![y, y]\!]^{1/2}$, for all $x, y \in \mathbb{R}^n$.

Compatible WP is non-unique

Can always (and henceforth will be) chosen in such a way that

$$\mu(A) = \sup_{|x|=1} [Ax, x]$$

Derivative of the (differentiable) curve's norm

$$\frac{d}{dt}\|x(t)\|^2 = [\dot{x}(t), x(t)] \quad \text{a.e.}$$

Technical Prelimianries - II

Norm	Weak pairing	Log norms and Lumer's equality
$\|x\|_2 = \sqrt{x^\top x}$	$[\![x,y]\!]_2 = x^\top y$	$\mu_2(A) = \frac{1}{2}\lambda_{\max}(A + A^\top) = \max_{\|x\|_2=1} x^\top A x$
$\|x\|_p = \left(\sum_i \|x_i\|^p\right)^{1/p}$, $1 < p < \infty$	$[\![x,y]\!]_p = \|y\|_p^{2-p}(y \circ \|y\|^{p-2})^\top x$	$\mu_p(A) = \max_{\|x\|_p=1} (x \circ \|x\|^{p-2})^\top A x$
$\|x\|_1 = \sum_i \|x_i\|$	$[\![x,y]\!]_1 = \|y\|_1 \operatorname{sign}(y)^\top x$	$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \Big(a_{jj} + \sum_{i\neq j} \|a_{ij}\|\Big) = \sup_{\|x\|_1=1} \operatorname{sign}(x)^\top A x$
$\|x\|_\infty = \max_i \|x_i\|$	$[\![x,y]\!]_\infty = \max_{i\in I_\infty(y)} x_i y_i$	$\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \Big(a_{ii} + \sum_{j\neq i} \|a_{ij}\|\Big) = \sup_{\|x\|_\infty=1} \max_{i\in I_\infty(x)} (Ax)_i x_i$

TABLE 1

Table of norms, weak pairings, and log norms for ℓ_2, ℓ_p for $p \in (1,\infty)$, ℓ_1, and ℓ_∞ norms. We adopt the shorthand $I_\infty(x) = \{i \in \{1,\dots,n\} \mid |x_i| = \|x\|_\infty\}$.

Easy to find: p=1,2, ∞

Quite difficult to find for a general p, leads to the non-convex optimization problem.

The operator p-norm computation is known to be NP hard (seems to be an open problem for log-norms).

“Pseudo-Quadratic” S-Lemma.

Suppose that one wishes to prove the following implication with “pseudo-quadratic” forms:

$$[Px, x] \leq 0 \quad \overset{?}{\Longleftarrow} \quad [Qx, x] \leq 0$$

<u>Lemma.</u> Let the WP [,] and the log-norm μ correspond to the same vector norm $\|\cdot\|$.

The implication (?) is valid for each vector *x* if a parameter $\tau \geq 0$ exists such that $\mu(P - \tau Q) \leq 0.$

<u>Proof</u>: Without loss of generality (since both inequalities are homogeneous), $\|x\| = 1.$

$$[Px, x] \leq [Px - \tau Qx, x] + [\tau Qx, x] \leq \mu(P - \tau Q) + \tau [Qx, x] \leq \tau [Qx, x].$$

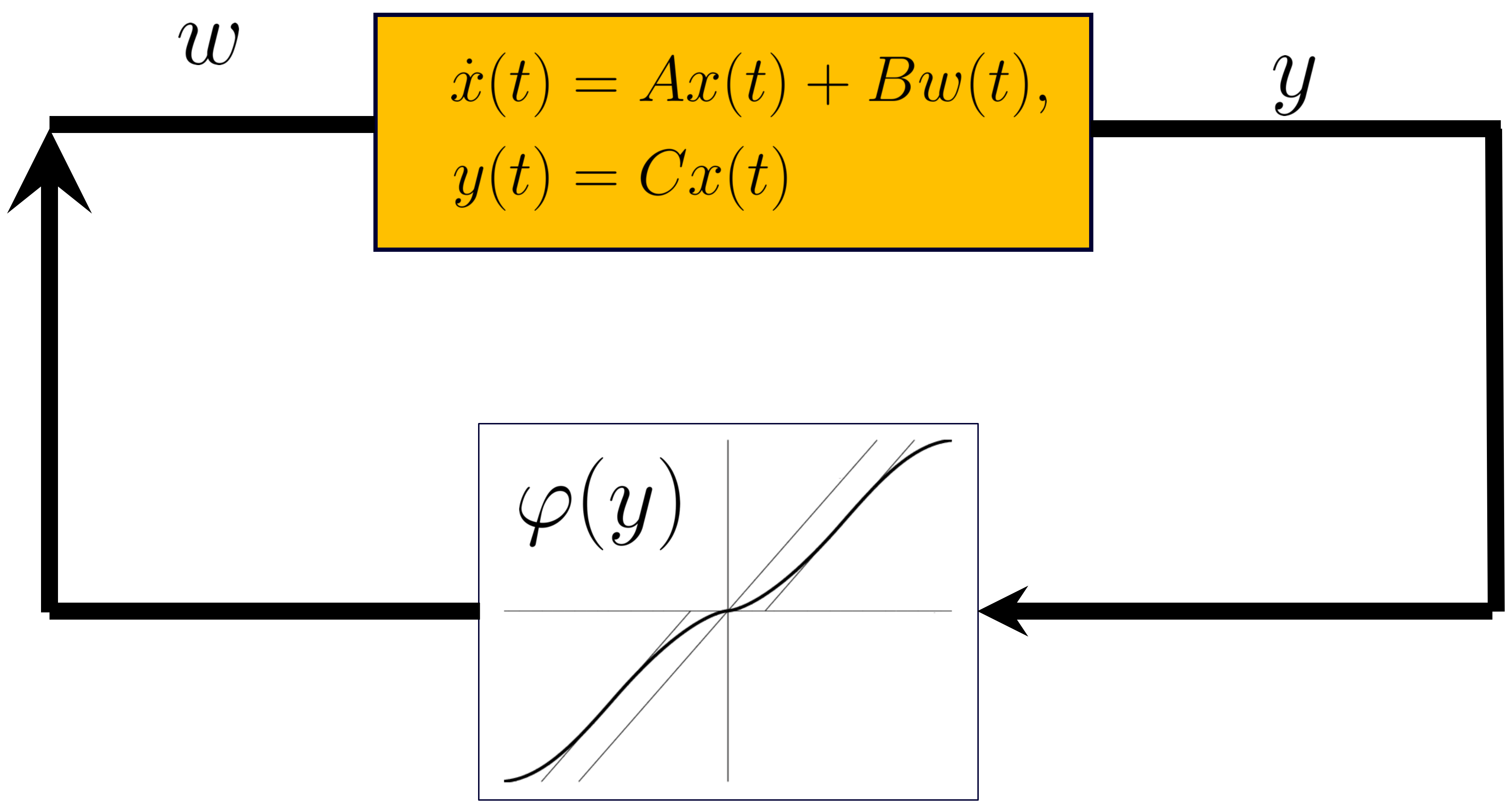
"Pseudo-Quadratic" S-Lemma: Some Remarks

$$x^\top Px = [Px, x] \leq 0 \quad \Longleftarrow \quad x^\top Qx = [Qx, x] \leq 0$$

$$\mu(P - \tau Q) \to \inf \quad \text{s.t. } \tau \geq 0$$

- **S-lemma in the general case reduces to optimization problem (the implication holds if inf <= 0)**
- **The problem is convex, because the log-norm is a convex function**
- **The oracle computing the function (and gradient, if needed) is of high complexity, except for special norms, e.g. l_1 , l_2, l_∞ .**
- **Can be trivially generalized to the case of multiple "pseudo-quadratic constraints" (several multipliers arise)**
- **Leads to new absolute stability and absolute contraction criteria in non-Euclidean norms.**

S-procedure with a Non-Euclidean Norm



$$0 \le \frac{\varphi(y_1) - \varphi(y_2)}{y_1 - y_2} \le \varkappa$$

$$V(\Delta x) = \|\Delta x\|_{p,R}^2 = \|R\Delta x\|_p^2, \ \det R \ne 0.$$

Program: repeat the trick with S-lemma.

- **Write the slope restriction as a "pseudo-quadratic" constraint**
- **Write the Lyapunov condition as a "pseudo-quadratic" constraint**
- **Apply S-lemma**
- **Principal difference: extended state-control vector needs to be used!**
- **For simplicity, let *R=I*.**

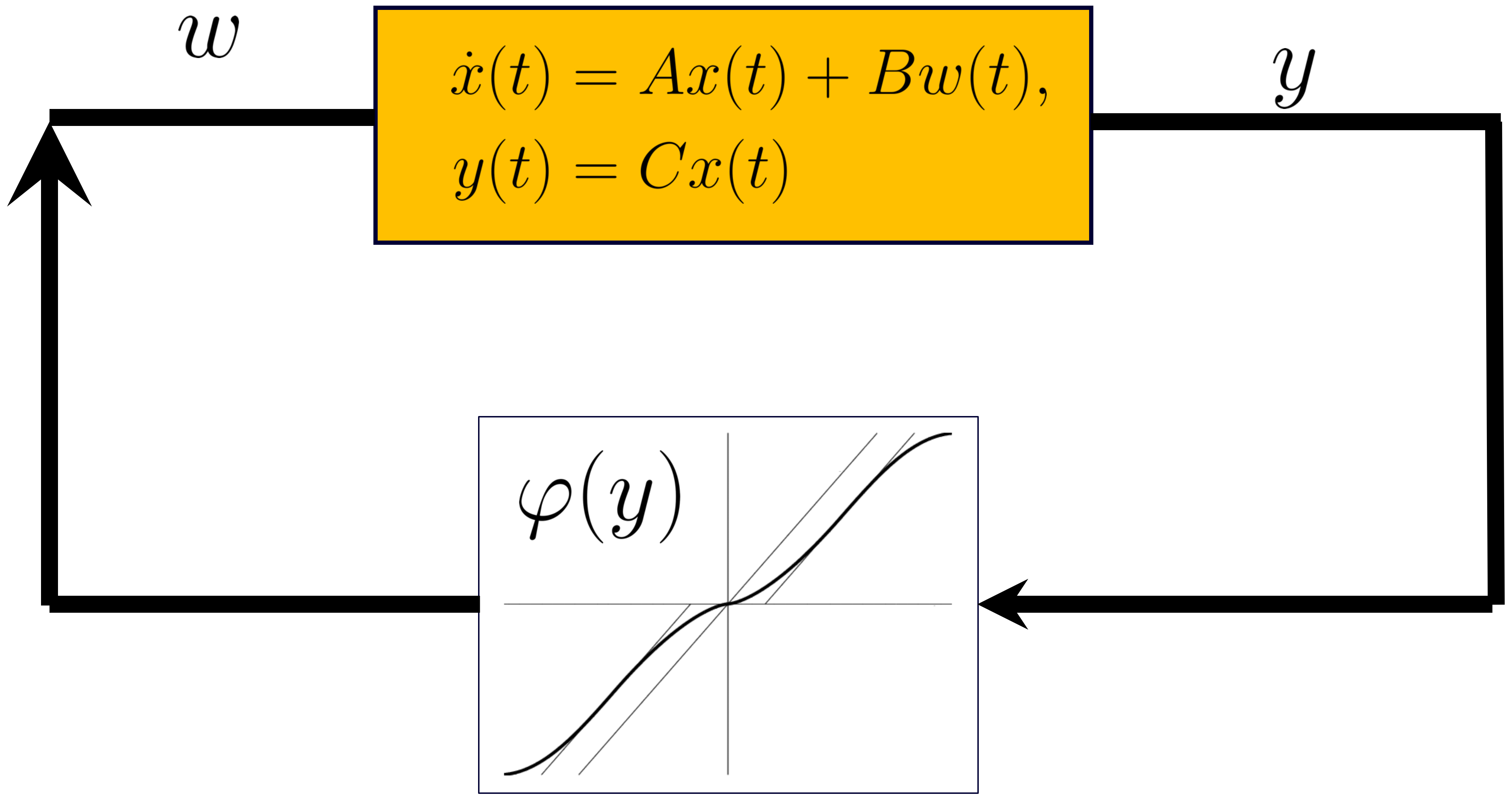
It remains to apply S-lemma!

$$\dot{V} \le -c\|\Delta x\|_p^2 \iff$$
$$\iff [A\Delta x + c\Delta x + B\Delta w, \Delta x]_p \le 0$$
$$\iff \left[\begin{pmatrix} A + cI & B \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta w \end{pmatrix}, \begin{pmatrix} \Delta x \\ \Delta w \end{pmatrix}\right]_p \le 0$$

?

$$0 \le \frac{\Delta w}{\Delta y} \le \varkappa \iff$$
$$\left[\begin{pmatrix} 0 & 0 \\ -C & \varkappa^{-1} \end{pmatrix}\begin{pmatrix} \Delta x \\ \Delta w \end{pmatrix}, \begin{pmatrix} \Delta x \\ \Delta w \end{pmatrix}\right]_p \le 0$$

Applying S-Lemma: Contraction in l_p Norm



$$0 \le \frac{\varphi(y_1) - \varphi(y_2)}{y_1 - y_2} \le \varkappa$$

Theorem.

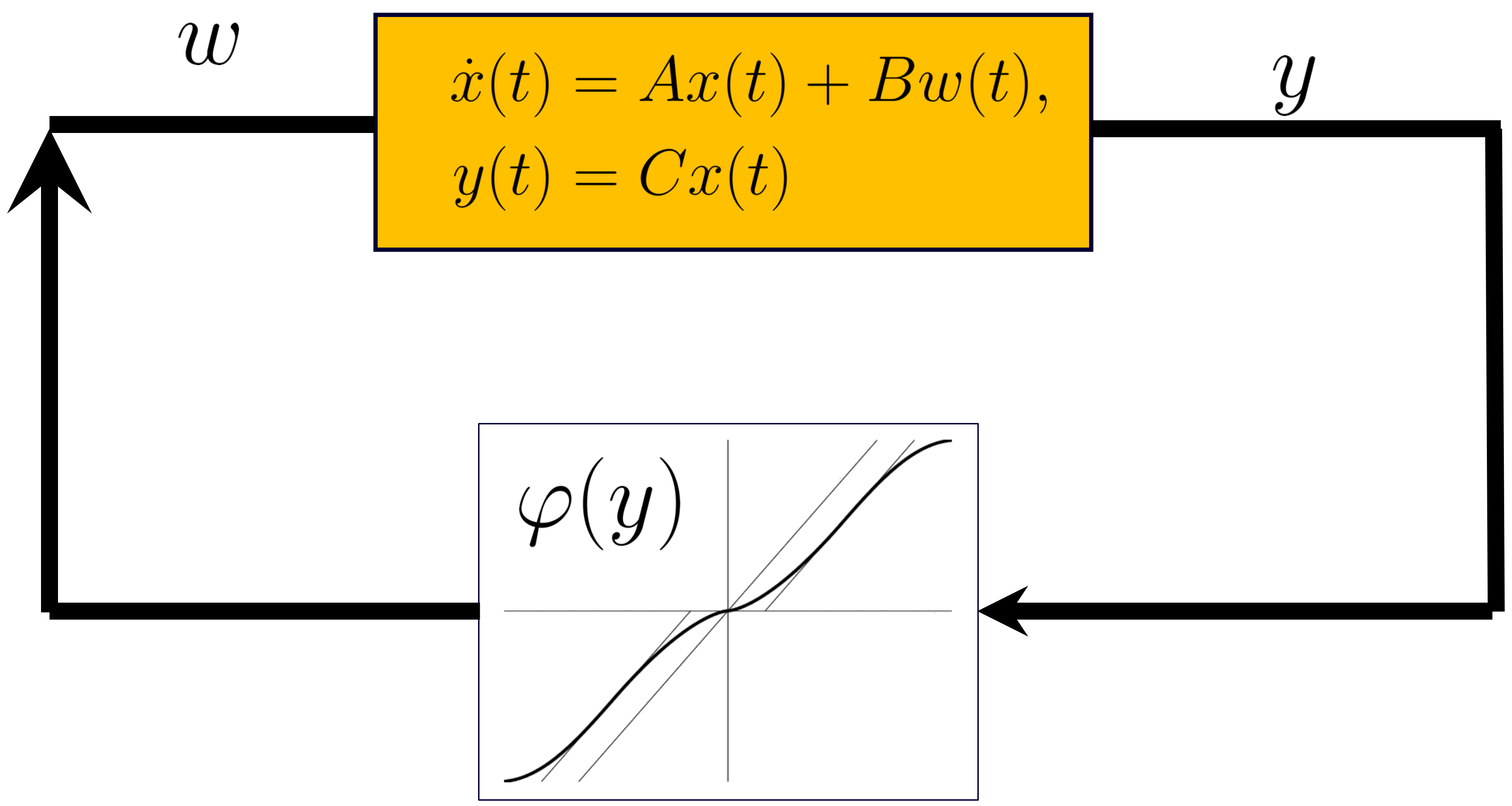
The system is contractive with respect to the norm $\|\Delta x\|_p$ and rate *c* if $\tau \geq 0$ exists such that

$$\mu_p \begin{pmatrix} A + cI & RB \\ \tau C & -\tau \varkappa^{-1} \end{pmatrix} \le 0$$

Remarks:

- This constraint is **convex!**
- However, the computation of the log-norm is troublesome (optimization with a **computationally expensive oracle**).
- Easily computable for:
  - **p=1 or ∞**
  - **p=2: classical LMIs.**

More General Result: Contraction in Weighted l_p Norm



$$0 \leq \frac{\varphi(y_1) - \varphi(y_2)}{y_1 - y_2} \leq \varkappa$$

Theorem. The system is contractive with respect to the (non-Euclidean) norm

$$\|\Delta x\|_{p,R} = \|R\Delta x\|_p$$

and rate *c* if *$\tau \geq 0$* exists such that

$$\mu_p \begin{pmatrix} RAR^{-1} + cI & RB \\ \tau CR^{-1} & -\tau\varkappa^{-1} \end{pmatrix} \leq 0$$

Remarks:

- For a **fixed** weight matrix, the condition is convex. For a variable weight, the constraint is non-convex!
- Sometimes, the problem reduces to **quasi-convex optimization (**Davydov, Proskurnikov, Bullo, IEEE TAC 2025**)**
  - p=1 or ∞ + diagonal weight matrix *R*
  - p=2: classical LMIs
- Example (in the journal paper): strong Kalman conjecture for positive systems. If $A, A - \varkappa BC$ are Hurwitz and Metzler, then the *R= R(p)* exists for all p ***(contraction is any norm is guaranteed).***

Conclusions and Future Works

Journal Paper: A.V. Proskurnikov, A. Davydov, F. Bullo**,** *The Yakubovich S-Lemma Revisited: Stability and Contractivity in Non-Euclidean Norms*, SICON 2023

- Many details omitted in this talk
- Some examples

Journal Paper: A.V. Proskurnikov, A. Davydov, F. Bullo**,** Non-Euclidean contraction analysis of continuous-time neural networks, TAC 2025

- Applications to contractivity and Lipschitz constant computation for RNNs
- Special norms: Weighted l_1 and l_∞

Topics for future research:

- **Efficient computational algorithms: how to validate log-norm criteria?**
- **Complicated nonlinearities:**
- **NNs, dynamical blocks, optimization solvers (control allocation logic, MPC etc.)**
- **Does non-quadratic norms work better?**

  Can we outperform classical stability criteria with non-quadratic norms?


thank you
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