Linear Differential Inclusions for Computational Contraction Theory

Contraction Workshop @ CDC 2024

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An LDI Perspective to Contraction Analysis (Part 1)

A linear differential inclusion (LDI) is of the form

$$\dot{x} \in \Omega x, \quad x(0) = x_0,$$

where $x \in \mathbb{R}^n$, $\Omega \subseteq \mathbb{R}^{n\times n}$ is a set of matrices.

Proposition 1 (Exponential Stability of LDIs [1])

If there exists $c \in \mathbb{R}$ such that $\mu(A) \leq c$ for every $A \in \Omega$, then

$$|x(t)| \leq e^{ct}|x(0)|$$

for any trajectory $t \mapsto x(t)$ of the LDI.

[1] C. Desoer and M. Vidyasagar, *Feedback systems: Input-output properties*. *SIAM*, 1975.

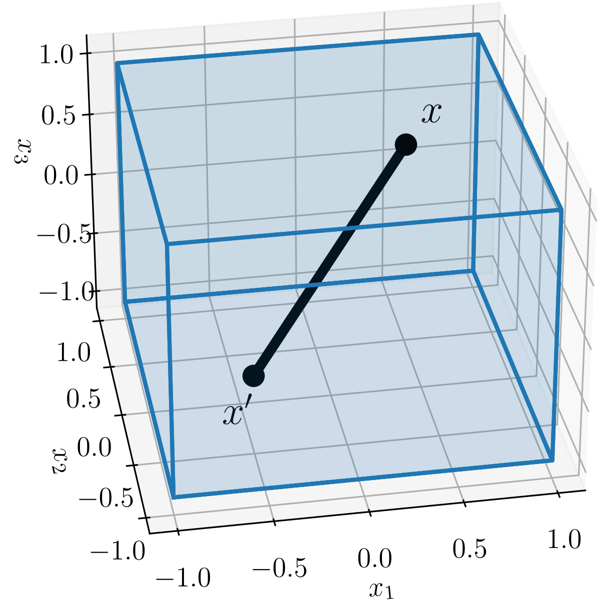
An LDI Perspective to Contraction Analysis (Part 2)

Consider some differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$.

Proposition 2 (Jacobian Linear Inclusion [2])

If $\{\frac{\partial f}{\partial x}(x) : x \in \mathbb{R}^n\} \subseteq \mathcal{J}$, then for any $x, x' \in \mathbb{R}^n$.

$$f(x) - f(x') \in \overline{\text{co}}(\mathcal{J})(x - x').$$



Intuition: Apply mean value theorem to the line segment connecting x and x'

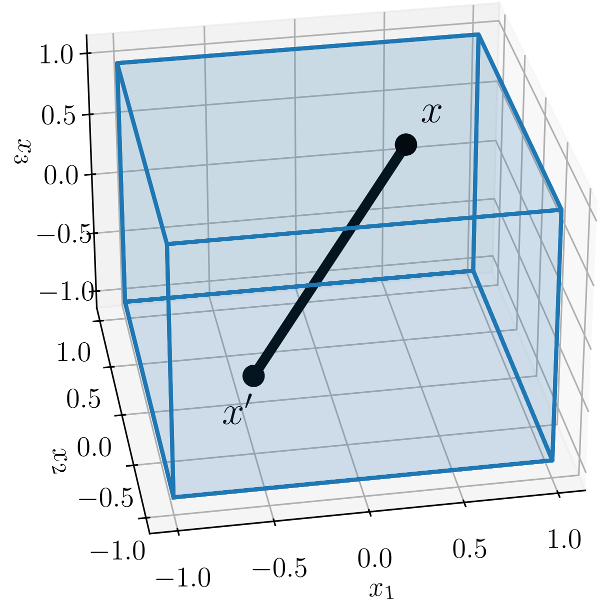
[2] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory. SIAM*, 1994.

An LDI Perspective to Contraction Analysis (Part 3)

Consider $\dot{x} = f(x)$ for C^1 smooth f. Set $\varepsilon = x - x'$.

Proposition 2: $\dot{\varepsilon} = f(x) - f(x') \in \overline{\text{co}}(\mathcal{J})(x - x') = \overline{\text{co}}(\mathcal{J})\varepsilon$

Proposition 1 + Convexity: $\sup_{J \in \mathcal{J}} \mu(J) \leq c \implies |\varepsilon(t)| \leq e^{ct}|\varepsilon(0)|$



Result is the usual norm bound. For any two trajectories $t \mapsto x(t), x'(t)$,

$$|x(t) - x'(t)| \leq e^{ct}|x(0) - x'(0)|$$

- [3], [4]: Fundamental Theorem + Coppel + Gronwall yields the result using sub-additivity of μ, integrating the line segment connecting x and x':

$$f(x) - f(x') = \left(\int_0^1 Df(sx(t) + (1-s)x'(t))\,\mathrm{d}s\right)(x - x').$$

[3] Z. Aminzare and E. D. Sontag, "Contraction methods for nonlinear systems: A brief introduction and some open problems," *IEEE CDC*, 2014.

[4] A. Davydov, S. Jafarpour, and F. Bullo, "Non-euclidean contraction theory for robust nonlinear stability," *IEEE TAC*, 2022.

Why is this Equivalent Viewpoint Useful?

Theoretical Infinitesimal Linearization (closed-form) $\Longleftrightarrow$ Computational Error LDI Analysis (set)

Computational Advantage: Automatic, parallelizable and differentiable constructions of overapproximating interval Jacobian sets using modern tools: automatic differentiation and interval analysis [5]

```python
import immrax as irx
import jax.numpy as jnp

f = lambda x : jnp.array([x[0]**2, x[1]**3 + jnp.sin(x[0])]) # Dynamics
J = irx.jacM(f) # autodiff + interval analysis
print(J(irx.interval([-1., -1.], [1., 1.]))) # Interval Overapprox of Jacobian
```

Can apply any strategy from LDI analysis [2], including LMI-based control design.

Example: Stable feedback control design on error LDI $\Longrightarrow$ contracting feedback tracking control

[5] A. Harapanahalli, S. Jafarpour, and S. Coogan, "immrax: A parallelizable and differentiable toolbox for interval analysis and mixed monotone reachability in JAX," *IFAC ADHS*, 2024

[2] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory. SIAM*, 1994.

Mixed Jacobian LDI: Contraction to Known Trajectories [6]

In many applications, a nominal trajectory is fixed: reachable set computation, tracking control design, training, etc.
An elementwise application of the mean value theorem obtains a different inclusion: potential improvement for fixed x'.

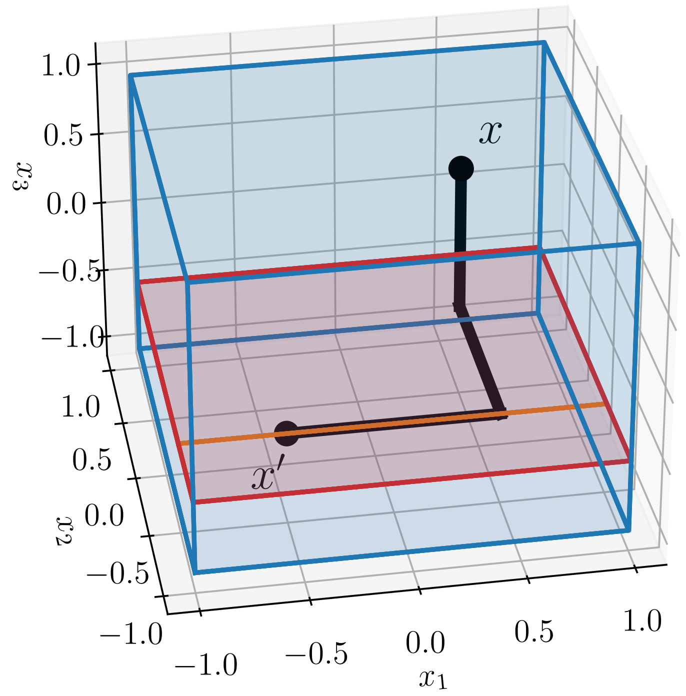
Mixed Jacobian LDI

Fixing a point $x' \in \mathbb{R}^n$, an element-wise application of mean value yields

$$\begin{aligned}(M_{x'}f(x,s))_{ij} \\ := \frac{\partial f_i}{\partial x_j}(x_1, \dots, x_{j-1}, s_j x_j + (1-s_j)x'_j, x'_{j+1}, \dots, x'_n)\end{aligned}$$

$$M_{x'}f(\mathbb{R}^n, [0,1]^n) \subseteq \mathcal{M} \implies f(x) - f(x') \in \overline{\mathrm{co}}(\mathcal{M})(x - x'),$$

for any $x \in \mathbb{R}^n$.



[6] A. Harapanahalli, and S. Coogan, "A linear differential inclusion for contraction analysis to known trajectories," *arXiv preprint*, 2024

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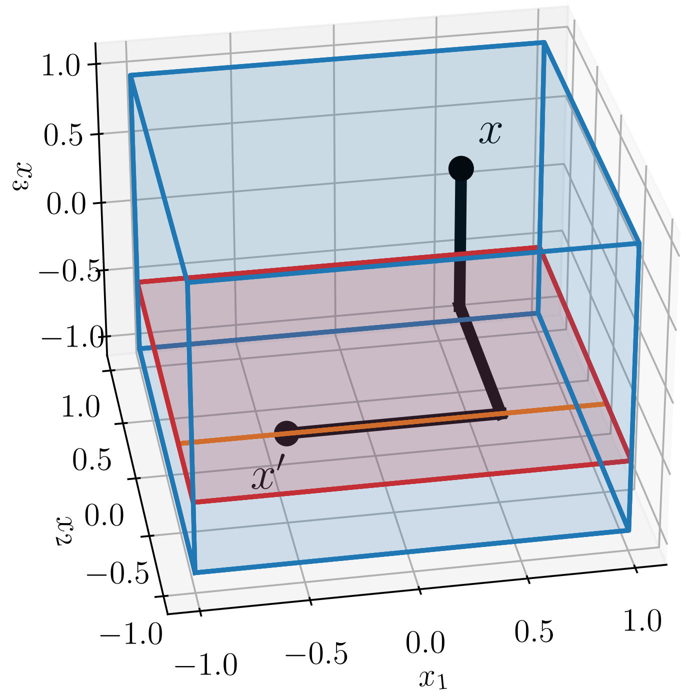

Proposition 1 + Convexity: $\sup_{M \in \mathcal{M}} \mu(M) \leq c \implies |\varepsilon(t)| \leq e^{ct} |\varepsilon(0)|$

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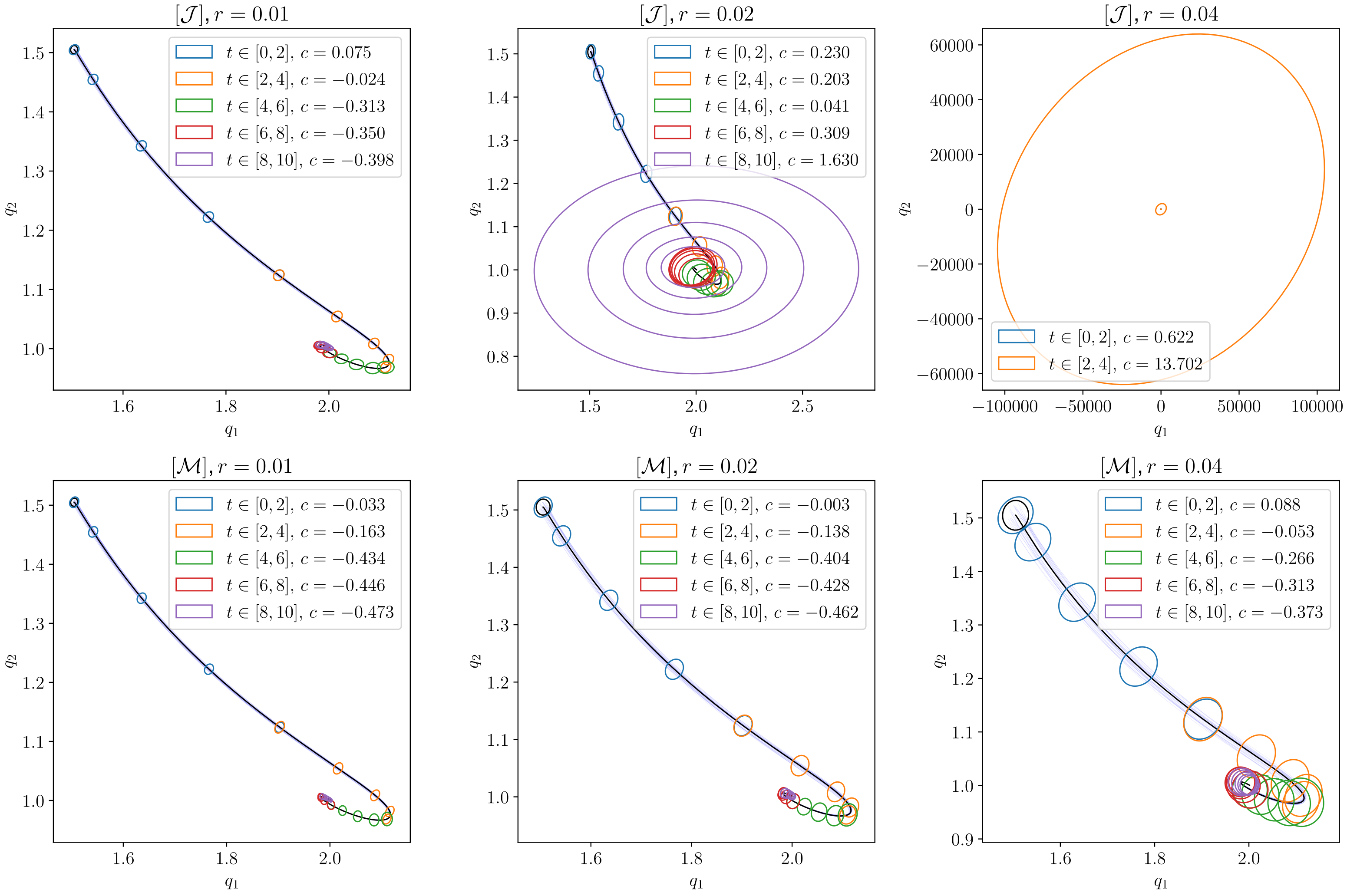
Application: Reachable Sets Using Matrix Measures

Reachable Set Computation: [7]
Simulate nominal $t \mapsto x'$, upper bound logarithmic norm around x', bloat/shrink a norm ball.

Interval Overapproximations: [8]
Use interval overapproximations of the Jacobian to overapproximate logarithmic norm.

Automated: `immrax` automatically computes interval $[\mathcal{M}]$ matrices, SDP searches for $\|\cdot\|_{2,P^{1/2}}$ norms.

Novelty: Compare directly to x', not arbitrary trajectories. *Strict Improvement*: $[\mathcal{M}] \subseteq [\mathcal{J}]$



4 state robot arm model, projection onto q_1-q_2 pictured

[7] J. Maidens and M. Arcak, "Reachability analysis of nonlinear systems using matrix measures," *IEEE TAC*, 2014.
[8] C. Fan, J. Kapinski, X. Jin, and S. Mitra, "Simulation-driven reachability using matrix measures," *ACM TECS*, vol. 17, no. 1, pp. 1–28, 2017.

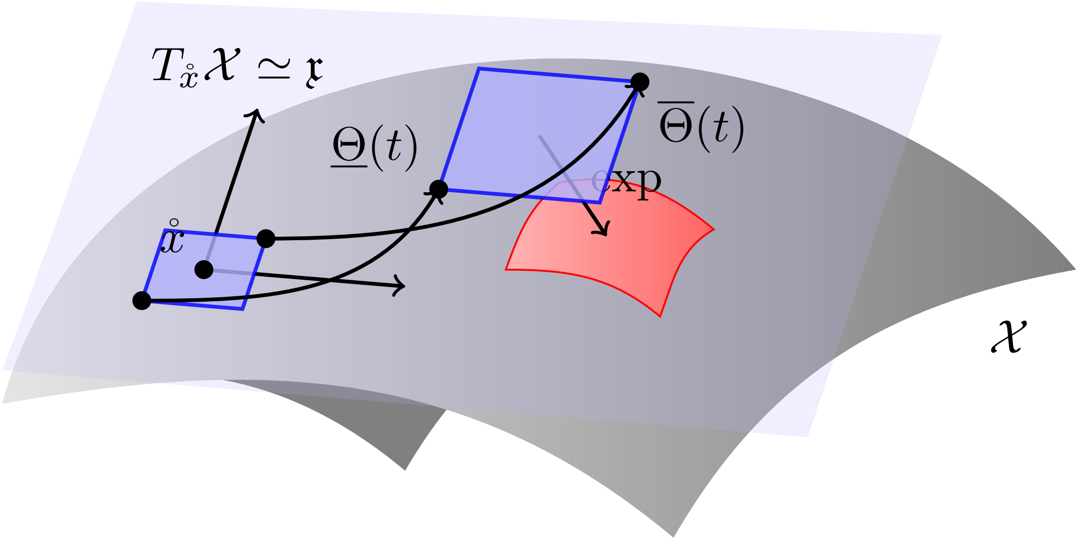
Conclusions

- An LDI encompassing the error dynamics recovers the norm-based matrix measure contraction bound
- When comparing to a known trajectory of the system, the mixed Jacobian LDI potentially provides better results compared to using the normal Jacobian set

For all the details, please see the preprint

https://arxiv.org/pdf/2411.11587

Thank you for your attention!
Presenting "Efficient Reachable Sets on Lie Groups Using Lie Algebra Monotonicity and Tangent Intervals" tomorrow:
MoA20.2, 10:20 - 10:40, Suite 9



References

[1] C. Desoer and M. Vidyasagar, *Feedback systems: Input-output properties*. Society for Industrial; Applied Mathematics, 1975.

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[4] A. Davydov, S. Jafarpour, and F. Bullo, "Non-euclidean contraction theory for robust nonlinear stability," *IEEE Transactions on Automatic Control*, vol. 67, no. 12, pp. 6667–6681, 2022.

[5] A. Harapanahalli, S. Jafarpour, and S. Coogan, "Immrax: A parallelizable and differentiable toolbox for interval analysis and mixed monotone reachability in JAX," *IFAC-PapersOnLine*, vol. 58, no. 11, pp. 75–80, 2024, doi: https://doi.org/10.1016/j.ifacol.2024.07.428. Available: https://www.sciencedirect.com/science/article/pii/S2405896324005275

[6] A. Harapanahalli and S. Coogan, "A linear differential inclusion for contraction analysis to known trajectories." 2024. Available: https://arxiv.org/abs/2411.11587

[7] J. Maidens and M. Arcak, "Reachability analysis of nonlinear systems using matrix measures," *IEEE Transactions on Automatic Control*, vol. 60, no. 1, pp. 265–270, 2014.

[8] C. Fan, J. Kapinski, X. Jin, and S. Mitra, "Simulation-driven reachability using matrix measures," *ACM Transactions on Embedded Computing Systems (TECS)*, vol. 17, no. 1, pp. 1–28, 2017.