Linear Differential Inclusions and Contraction Analysis

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Linear Differential Inclusions

A linear differential inclusion (LDI) is a system of the form

$$\dot{x} \in \Omega x, \quad x(0) = x_0$$

where $\Omega \subseteq \mathbb{R}^{n \times n}$ is a set of matrices and $\Omega x = \{Mx \mid M \in \Omega\}$.

Any $t \mapsto x(t)$ satisfying the LDI is called a *trajectory* of the LDI.

Lemma. Given norm $|\cdot|$ on $\mathbb{R}^n$ and corresponding matrix log norm $\mu(\cdot)$ where $\mu(A) := \lim_{h \to 0^+} \frac{\|I + hA\| - 1}{h}$ for induced matrix norm $\|\cdot\|$, and $c \in \mathbb{R}$,

$$\text{if} \quad \mu(M) \leq c \quad \text{for all } M \in \Omega,$$

$$\text{then} \quad |x(t)| \leq e^{ct}|x(0)| \quad \text{for all } t \geq 0.$$

This is essentially a corollary of Coppel's inequality.

Bounding a nonlinearity with an LDI

Consider some differentiable $f : \mathbb{R}^n \to \mathbb{R}^m$ and convex $X \subseteq \mathbb{R}^n$.

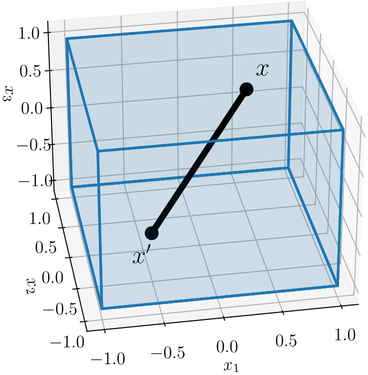
Proposition.[1] If $\mathcal{J} \subseteq \mathbb{R}^{m\times n}$ satisfies

$$\frac{\partial f}{\partial x}(X) \subseteq \mathcal{J},$$

then

$$f(x) - f(x') \in \underbrace{\overline{\text{co}}(\mathcal{J})}_{\text{matrix set}} \underbrace{(x - x')}_{\text{vector}}$$

for every $x, x' \in X$.



Pf. Apply mean value theorem to the line segment connecting x and x'.

[1] Boyd, El Ghaoui, Feron, Balakrishnan, *LMIs in Systems and Control Theory, SIAM*, 1994, page 55

Nonlinear contraction as LDI stability

Combining Lemma and Proposition recovers standard contraction result:

1. Given $\dot{x} = f(x)$ and convex X, let $\mathcal{J} = \dfrac{\partial f}{\partial x}(X)$.
2. For any initial conditions $x_0, x_0' \in X$, set

$$\varepsilon(t) = x(t) - x'(t).$$

   Then the Proposition implies ε satisfies the LDI

$$\dot{\varepsilon} = f(x) - f(x') \in \overline{\mathrm{co}}(\mathcal{J})(x - x') = \overline{\mathrm{co}}(\mathcal{J})\varepsilon.$$

Lemma and convexity of log norm then imply **contraction property**:

If $\sup_{x \in X} \mu\left(\dfrac{\partial f}{\partial x}(x)\right) \leq c$ then $|x(t) - x'(t)| \leq e^{ct}|x_0 - x_0'|$ for any $x_0, x_0' \in X, t \geq 0$.

Some comments on the Jacobian-based LDI interpretation of contraction

- Several other proof techniques possible[2,3], *e.g.*, write $\dot{\varepsilon} = A(t)\varepsilon$, where $A(t) = \int_0^1 \frac{\partial f}{\partial x}(sx(t) + (1-s)x'(t))ds$, apply subadditivity of log norm to conclude $\mu(A(t)) \leq c$.
- Yet another example of contraction by another name: LMI book[4] refers to stability of Jacobian-based LDI as "fading memory" property of the nonlinear system and that "the difference between any two trajectories...converges to zero"

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[2] Aminzare, Sontag, "Contraction methods for nonlinear systems: A brief introduction and some open problems", *IEEE CDC*, 2014

[3] A. Davydov, S. Jafarpour, and F. Bullo, "Non-euclidean contraction theory for robust nonlinear stability," *IEEE TAC*, 2022

[4] Boyd, El Ghaoui, Feron, Balakrishnan, *LMIs in Systems and Control Theory*, *SIAM*, 1994

Why is the LDI interpretation useful?

1. **Automated computational tools** using interval analysis for overapproximating $\frac{\partial f}{\partial x}(X)$ and contraction rate c, *e.g.* our tool is `immrax`[5]
2. Illuminates that any strategy for **LDI analysis/synthesis** applies for contraction, *e.g.*, LMI-based control design

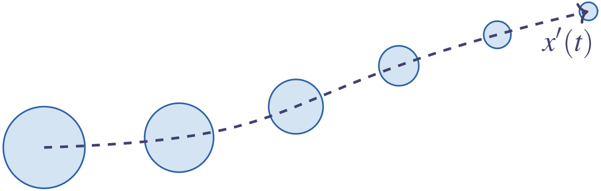
   Example:
   Stabilizing feedback control of error LDI $\implies$ contracting tracking control

3. Points to possibility of **alternative LDIs** that characterize error dynamics (rest of talk)

[5] JAX-based `immrax` using JAX for Python, [Harapanahalli, Jafarpour, Coogan, *ADHS*, 2024]

An observation on contraction to known trajectories

- In many applications, we care about contraction to a known trajectory $x'(t)$, *e.g.*, trajectory tracking, reachability analysis
- In this case, $\dot{\varepsilon} \in \overline{\text{co}}(\mathcal{J})\varepsilon$ remains a valid error inclusion
- However, when $x'(t)$ is known, we can consider *other* error LDIs, *i.e.*, alternative sets Ω such that $\dot{\varepsilon} \in \Omega\varepsilon$ still holds (next slides)



Defining a mixed Jacobian matrix[6]

Definition. For differentiable $f : \mathbb{R}^n \to \mathbb{R}^m$ and $x' \in \mathbb{R}^n$, the ***mixed Jacobian operator*** $M_{x'}$ is given by

$$M_{x'} f : \mathbb{R}^n \times [0,1]^n \to \mathbb{R}^{m \times n}$$

$$(M_{x'} f(x,s))_{ij} = \frac{\partial f_i}{\partial x_j}(x_1, \ldots, x_{j-1}, s_j x_j + (1 - s_j) x_j', x_{j+1}', \ldots, x_n').$$

- The matrix $M_{x'} f(x,s)$ is called the ***mixed Jacobian matrix*** of f at (x,s), since it mixes the arguments to the Jacobian between the point x' and x.

[6] [Coogan, Harapanahalli, *TAC*, 2025]

Mixed Jacobian LDI for bounding a nonlinearity

Consider some differentiable $f : \mathbb{R}^n \to \mathbb{R}^m$ and $X \subseteq \mathbb{R}^n$.

Theorem. For any fixed $x' \in X$, if $\mathcal{M} \subseteq \mathbb{R}^{m \times n}$ satisfies

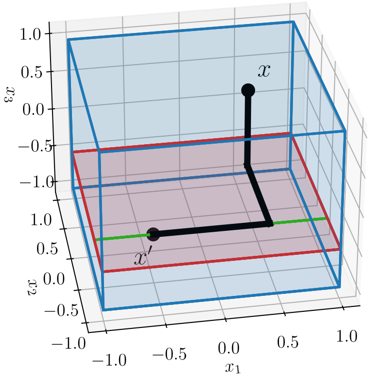
$$M_{x'} f(X, [0,1]^n) \subseteq \mathcal{M},$$

then

$$f(x) - f(x') \in \overline{\text{co}}(\mathcal{M})(x - x')$$

for every $x \in X$.

Proof. Follows from elementwise application of mean value theorem along each coordinate direction. □



Mixed Jacobian LDI for contraction to known trajectory

Consider $\dot{x} = f(x)$, $X \subset \mathbb{R}^n$, and $x'(t)$ a known trajectory in X.

Theorem. If

$$\sup_{t \geq 0, x \in X, s \in [0,1]^n} \mu\left(M_{x'(t)} f(x,s)\right) \leq c$$

then

$$|x(t) - x'(t)| \leq e^{ct} |x_0 - x_0'| \quad \text{for any } x_0 \in X, t \geq 0.$$

Proof. For any trajectory $x(t)$ in X, the error $\varepsilon(t) = x(t) - x'(t)$ satisfies the LDI

$$\dot{\varepsilon}(t) \in \overline{\text{co}}\left(\mathcal{M}_{x'(t)} f(X, [0,1]^n)\right) \varepsilon(t).$$

□

Comparing mixed Jacobian set $\mathcal{M}$ and Jacobian set $\mathcal{J}$

Suppose exact

$$\mathcal{J} := \frac{\partial f}{\partial x}(X)$$
$$\mathcal{M} := M_{x'}f(X, [0,1]^n).$$

- In general, neither $\mathcal{M} \subseteq \mathcal{J}$ nor $\mathcal{J} \subseteq \mathcal{M}$ is guaranteed
- Thus, no guarantee that using $\mathcal{M}$ is better for contraction analysis, except two special cases:
  1. Using ℓ_1 norm (due to column-wise construction of $M_{x'}$)
  2. Using interval overapproximations (next slide)

Interval overapproximations of mixed Jacobian set $\mathcal{M}$ and Jacobian set $\mathcal{J}$

Corollary. Suppose $X = X_1 \times \cdots \times X_n \subset \mathbb{R}^n$ is an interval and $x' \in X$.

An interval matrix $[\mathcal{M}]$ satisfies $M_{x'}f(X,[0,1]^n) \subseteq [\mathcal{M}]$ if

$$\frac{\partial f_i}{\partial x_j}(X_1,\ldots,X_j,x'_{j+1},\ldots,x'_n) \subseteq [\mathcal{M}]_{ij}.$$

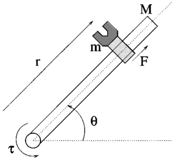
- Compare to: An interval matrix $[\mathcal{J}]$ satisfies $\frac{\partial f}{\partial x}(X) \subseteq [\mathcal{J}]$ if

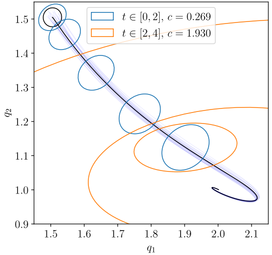
$$\frac{\partial f_i}{\partial x_j}(X_1,\ldots,X_j,X_{j+1},\ldots,X_n) \subseteq [\mathcal{J}]_{ij}.$$

- The smallest interval overapproximations satisfy $[\mathcal{M}] \subseteq [\mathcal{J}]$.
- Reordering states can lead to alternative mixed Jacobian LDIs.

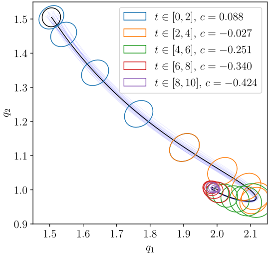
Example: Ellipsoidal reachability

- Four state robot arm[7]
- Ellipsoidal reachability with weighted 2-norm

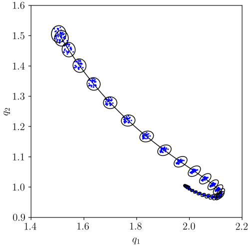




Jacobian Interval Approx.[8]



Mixed Jacobian Approx.



Improved Mixed Jacobian[9]

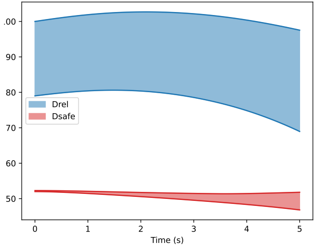
[7] Angeli, Sontag, Wang, "A characterization of integral input-to-state stability", *IEEE TAC*, 2002

[8] C. Fan, J. Kapinski, X. Jin, and S. Mitra, "Simulation-driven reachability using matrix measures," *ACM TECS*, 2017

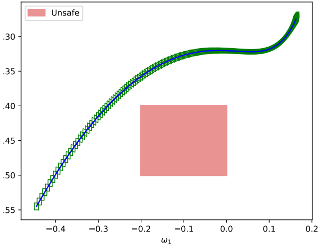
[9] Harapanahalli, Coogan, "Parametric Reachable Sets Via Controlled Dynamical Embeddings" (WeC16.6)

ARCH-COMP neural-network controlled nonlinear system benchmarks[10]

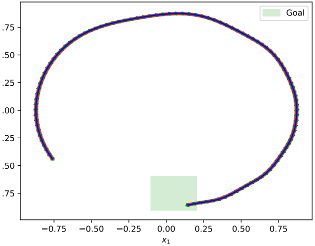
Benchmark	Instance	CORA	CROWN-Reach	JuliaReach	NNV	immrax
ACC	safe-distance	3.091	2.525	0.638	26.751	**0.066**
AttitudeControl	avoid	3.418	3.485	5.728	–	**0.507**
TORA	reach-tanh	3.166	7.486	0.357	63.523	**0.020**
TORA	reach-sigmoid	6.001	5.293	0.458	118.312	**0.023**



ACC



AttitudeControl



TORA (sigmoid)

[10] https://cps-vo.org/group/ARCH/FriendlyCompetition

Conclusions and acknowledgements

1. Linear Differential Inclusions (LDIs) is a (underutilized) vantage point for contraction analysis
2. Introduced mixed Jacobian matrix for alternative LDIs for contraction to known trajectories
3. LDIs are particularly amenable to interval analysis for automated computational tools (e.g., `immrax`)

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for papers and code

`pip install immrax`

Thank you!

Bounding a nonlinearity with an LDI (Proof)

Proposition. If $\frac{\partial f}{\partial x}(X) \subseteq \mathcal{J}$ then $f(x) - f(x') \in \overline{\text{co}}(\mathcal{J})(x - x')$

Pf. Fix $\ell \in \mathbb{R}^m$ and $x, x' \in X$, consider $\gamma : [0,1] \to X$, $\gamma(s) = sx + (1-s)x'$. By mean value theorem, there exists $s' \in (0,1)$ such that

$$\ell^T \big(f(\gamma(1)) - f(\gamma(0))\big) = \ell^T \frac{\partial f}{\partial x} \big(\gamma(s')\big) \big(\gamma(1) - \gamma(0)\big).$$

Since $\gamma(s) \in X$ by convexity, $\frac{\partial f}{\partial x} \big(\gamma(s')\big) \in \mathcal{J}$. Thus,

$$\ell^T \big(f(x) - f(x')\big) \leq \sup_{J \in \overline{\text{co}}(\mathcal{J})} \ell^T J(x - x'),$$

implying $f(x) - f(x')$ belongs to every halfspace containing $\overline{\text{co}}(\mathcal{J})(x - x')$. Since $\overline{\text{co}}(\mathcal{J})(x - x')$ is closed and convex, it equals the intersection of these halfspaces. □

Mixed Jacobian matrix is preferred under ℓ_1 norm

Theorem. Let $X \subseteq \mathbb{R}^n$ be an interval and $x' \in X$. Then

$$\sup_{x \in X,\, s \in [0,1]^n} \mu_1\left(M_{x'} f(x,s)\right) \leq \sup_{x \in X} \mu_1\left(Df(x)\right).$$