Contraction Theory for Optimization, Control, and Neural Networks

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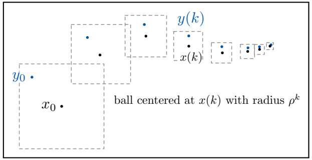
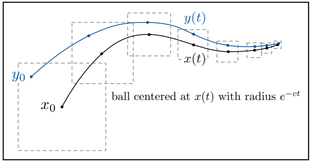
Tutorial Session on **Contraction Theory in Control, Optimization, and Learning**
Speakers: **Samuel Coogan** (Georgia Institute of Technology), **Emiliano Dall'Anese** (Boston University), **Ian Manchester** (University of Sydney), and **Giovanni Russo** (University of Salerno)
2025 IEEE CDC Tutorial Session, Dec 11, 2025





16:30–17:10
Speaker: Francesco Bullo (UC Santa Barbara)
Title: Introduction to Contraction Theory and Advances in Equilibrium Tracking (40 minutes)

17:10–17:30
Speaker: Emiliano Dall'Anese (Boston University)
Title: Contractivity of Interconnected Continuous- and Discrete-Time Systems (20 minutes)

17:30–17:50
Speaker: Giovanni Russo (University of Salerno)
Title: Contraction in Neural Networks and Biologically Plausible Optimization (20 minutes)

17:50–18:10
Speaker: Samuel Coogan (Georgia Institute of Technology)
Title: Linear Differential Inclusions and Contraction Analysis (20 minutes)

18:10–18:30
Speaker: Ian Manchester (University of Sydney)
Title: Neural Networks Designed with Contraction-Theoretic Guarantees (20 minutes)

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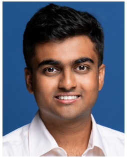

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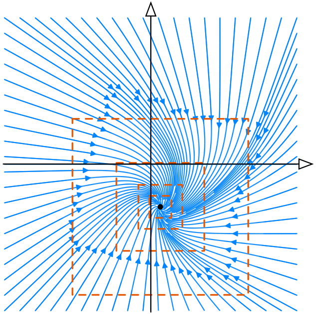
contractivity = robust computationally-friendly stability

fixed point theory + Lyapunov stability theory + geometry of metric spaces



highly-ordered transient and asymptotic behavior, no anonymous constants/functions

search for contraction properties

design engineering systems to be contracting

verify correct/safe behavior via known Lipschitz constants

Contraction theory: historical notes

- **Origins**

  S. Banach. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae*, 3(1):133–181, 1922. doi

- **Dynamics**:

  G. Dahlquist. *Stability and error bounds in the numerical integration of ordinary differential equations*. PhD thesis, (Reprinted in Trans. Royal Inst. of Technology, No. 130, Stockholm, Sweden, 1959), 1958

  S. M. Lozinskii. Error estimate for numerical integration of ordinary differential equations. I. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, 5:52–90, 1958. URL **http://mi.mathnet.ru/eng/ivm2980**. (in Russian)

- **Computation**:

  C. A. Desoer and H. Haneda. The measure of a matrix as a tool to analyze computer algorithms for circuit analysis. *IEEE Transactions on Circuit Theory*, 19(5):480–486, 1972. doi

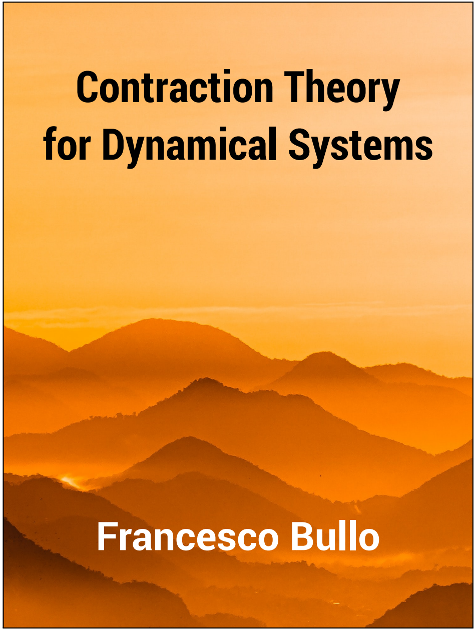
- **Systems and control**:

  W. Lohmiller and J.-J. E. Slotine. On contraction analysis for non-linear systems. *Automatica*, 34(6):683–696, 1998. doi

Contraction conditions without Jacobians

1. **one-sided Lipschitz maps** in: G. Dahlquist. Error analysis for a class of methods for stiff non-linear initial value problems. In G. A. Watson, editor, *Numerical Analysis*, pages 60–72. Springer, 1976. doi and E. Hairer, S. P. Nørsett, and G. Wanner. *Solving Ordinary Differential Equations I. Nonstiff Problems*. Springer, 1993. doi (Section 1.10, Exercise 6)
2. **uniformly decreasing maps** in: L. Chua and D. Green. A qualitative analysis of the behavior of dynamic nonlinear networks: Stability of autonomous networks. *IEEE Transactions on Circuits and Systems*, 23(6): 355–379, 1976. doi
3. no-name in: A. F. Filippov. *Differential Equations with Discontinuous Righthand Sides*. Kluwer, 1988. ISBN 902772699X (Chapter 1, page 5)
4. **maps with negative nonlinear measure** in: H. Qiao, J. Peng, and Z.-B. Xu. Nonlinear measures: A new approach to exponential stability analysis for Hopfield-type neural networks. *IEEE Transactions on Neural Networks*, 12(2):360–370, 2001. doi
5. **dissipative Lipschitz maps** in: T. Caraballo and P. E. Kloeden. The persistence of synchronization under environmental noise. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 461(2059):2257–2267, 2005. doi
6. **maps with negative lub log Lipschitz constant** in: G. Söderlind. The logarithmic norm. History and modern theory. *BIT Numerical Mathematics*, 46(3):631–652, 2006. doi
7. **QUAD maps** in: W. Lu and T. Chen. New approach to synchronization analysis of linearly coupled ordinary differential systems. *Physica D: Nonlinear Phenomena*, 213(2):214–230, 2006. doi
8. **incremental quadratically stable maps** in: L. D'Alto and M. Corless. Incremental quadratic stability. *Numerical Algebra, Control and Optimization*, 3:175–201, 2013. doi

Textbook “Contraction Theory for Dynamical Systems”



Contraction Theory for Dynamical Systems, Francesco Bullo, KDP, 1.2 edition, 2024, ISBN 979-8836646806
252 pages and 94 exercises (with solutions)

- **Table of Contents:**

  Examples: neural networks, gradient dynamics, Lur’e systems, traffic networks, diffusively-coupled dynamical systems, and more

- PDF text and slides freely available at **https://fbullo.github.io/ctds**
- paperback and hardcover at: **(link to amazon)**
- 12h recorded minicourse at: **(link to youtube)**
- v1.3 edition, forthcoming in mid 2026

  ”Continuous improvement is better than delayed perfection”
  Mark Twain

Outline

§1. Chapter #1: A tutorial review
- Definitions
- Theorems
- Examples

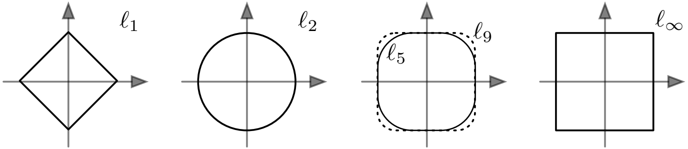
§2. Chapter #2: Equilibrium tracking for optimization-based control
- Equilibrium tracking
- Application to safety filters

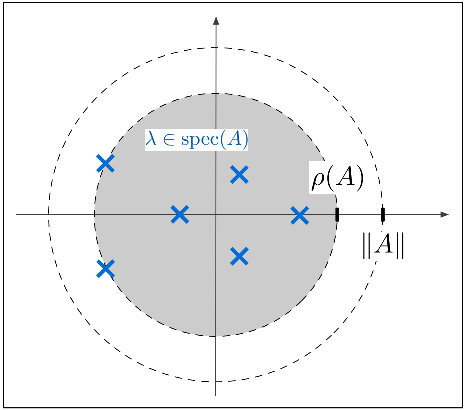
§3. Chapter #3: Contracting stochastic dynamics
- Incremental input and noise to state stability

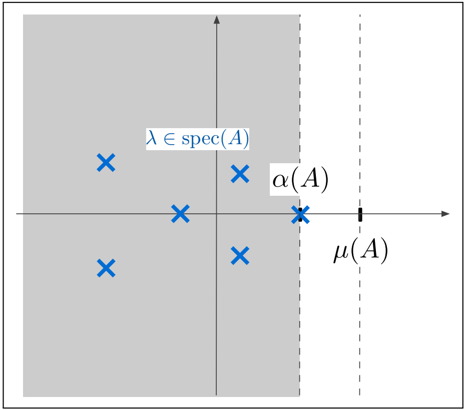
§4. Future work

Induced matrix norms

Vector norm	Induced matrix norm	Induced matrix log norm
$\|x\|_1 = \sum_{i=1}^n \lvert x_i\rvert$	$\|A\|_1 = \max_{j\in\{1,\dots,n\}} \sum_{i=1}^n \lvert a_{ij}\rvert$ $=$ max column "absolute sum" of A	$\mu_1(A) = \max_{j\in\{1,\dots,n\}} \left(a_{jj} + \sum_{i=1,i\neq j}^n \lvert a_{ij}\rvert\right)$ absolute value only off-diagonal
$\|x\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$	$\|A\|_2 = \sqrt{\lambda_{\max}(A^\top A)}$	$\mu_2(A) = \lambda_{\max}\left(\frac{A+A^\top}{2}\right)$
$\|x\|_\infty = \max_{i\in\{1,\dots,n\}} \lvert x_i\rvert$	$\|A\|_\infty = \max_{i\in\{1,\dots,n\}} \sum_{j=1}^n \lvert a_{ij}\rvert$ $=$ max row "absolute sum" of A	$\mu_\infty(A) = \max_{i\in\{1,\dots,n\}} \left(a_{ii} + \sum_{j=1,j\neq i}^n \lvert a_{ij}\rvert\right)$ absolute value only off-diagonal




$\lambda \in \mathrm{spec}(A)$
$\rho(A)$
$\|A\|$


$\lambda \in \mathrm{spec}(A)$
$\alpha(A)$
$\mu(A)$

Discrete-time dynamics and Lipschitz constants

$$x_{k+1} = \mathsf{F}(x_k) \qquad \text{on } \mathbb{R}^n \text{ with norm } \|\cdot\| \text{ and induced norm } \|\cdot\|$$

Lipschitz constant

$$\begin{aligned}\mathsf{Lip}(\mathsf{F}) &= \inf\{\ell > 0 \mid \|\mathsf{F}(x) - \mathsf{F}(y)\| \leq \ell \|x - y\| \quad \text{for all } x, y\} \\ &= \sup_x \|D\mathsf{F}(x)\|\end{aligned}$$

For **scalar map** f, $\quad \mathsf{Lip}(f) = \sup_x |f'(x)|$

For **affine map** $\mathsf{F}_A(x) = Ax + a$

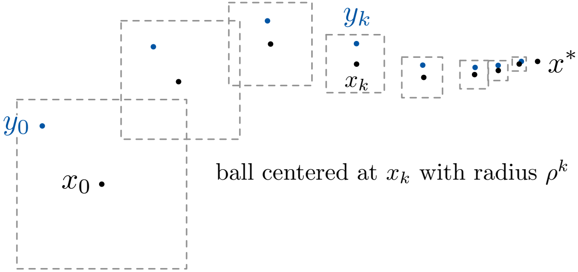
$$\|x\|_{2,P^{1/2}} = (x^\top P x)^{1/2} \qquad \mathsf{Lip}_{2,P^{1/2}}(\mathsf{F}_A) = \|A\|_{2,P^{1/2}} \leq \ell \quad \iff \quad A^\top P A \preceq \ell^2 P$$

$$\|x\|_\infty = \max_i |x_i| \qquad \mathsf{Lip}_\infty(\mathsf{F}_A) = \|A\|_\infty \leq \ell \quad \iff \quad |A|\mathbb{1}_n \leq \ell \mathbb{1}_n$$

Banach contraction theorem for discrete-time dynamics:

If $\rho := \mathrm{Lip}(F) < 1$, then

1. F is **contracting**: $\|x(k) - y(k)\| \leq \rho^k \|x_0 - y_0\|$
2. F has a globally exp stable equilibrium x^*



Continuous-time dynamics and one-sided Lipschitz constants

$$\dot{x} = \mathsf{F}(x) \qquad \text{on } \mathbb{R}^n \text{ with norm } \|\cdot\| \text{ and induced log norm } \mu(\cdot)$$

One-sided Lipschitz constant

$$\mathsf{osLip}(\mathsf{F}) = \sup_x \mu(D\mathsf{F}(x))$$

For **scalar map** f, $\mathsf{osLip}(f) = \sup_x f'(x)$

For **affine map** $\mathsf{F}_A(x) = Ax + a$

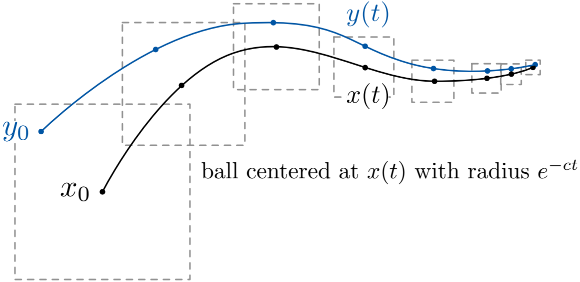
$$\mathsf{osLip}_{2,P}(\mathsf{F}_A) = \mu_{2,P}(A) \leq \ell \qquad \iff \qquad A^\top P + AP \preceq 2\ell P$$

$$\mathsf{osLip}_{\infty,\eta}(\mathsf{F}_A) = \mu_{\infty,\eta}(A) \leq \ell \qquad \iff \qquad a_{ii} + \sum_{j\neq i} |a_{ij}|\eta_i/\eta_j \leq \ell$$

Banach contraction theorem for continuous-time dynamics:

If $-c := \mathrm{osLip}(\mathsf{F}) < 0$, then

1. F is **infinitesimally contracting**: $\|x(t) - y(t)\| \;\leq\; \mathrm{e}^{-ct}\|x_0 - y_0\|$
2. F has a globally exp stable equilibrium x^*



Outline

Properties of contracting dynamics

0. initial conditions are forgotten, and
   monotonic decrease (no overshoot) in distance between trajectories
1. two canonical Lyapunov functions
2. robustness properties
   - bounded input, bounded output (iss)
   - finite input and noise state stability
   - robustness margin wrt unmodeled dynamics
   - robustness margin wrt delayed dynamics
3. modularity and interconnection properties
4. accurate numerical integration and equilibrium point computation
5. periodic input, periodic output

Property #1: Canonical Lyapunov functions

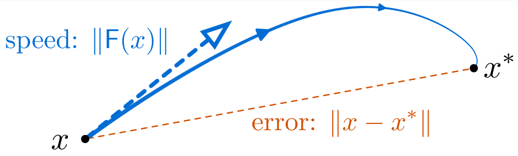
Given vector field F with $\operatorname{osLip}(\mathsf{F}) = -c < 0$ and equilibrium point $x^\star$, define

$$x \mapsto \|x - x^\star\| \qquad \text{and} \qquad x \mapsto \|\mathsf{F}(x)\|$$

Then

$$\|x(t) - x^\star\| \;\leq\; \mathrm{e}^{-ct}\|x_0 - x^\star\| \qquad \textbf{(error)}$$

$$\|\mathsf{F}(x(t))\| \;\leq\; \mathrm{e}^{-ct}\|\mathsf{F}(x_0)\| \qquad \textbf{(speed)}$$



Property #2: Robustness with respect to unmodeled dynamics

$$\dot{x} = \mathsf{F}(x) + \Delta(x)$$

- **contractivity**: $\mathrm{osLip}(\mathsf{F}) \leq -c < 0$
- **bounded disturbance**: $\mathrm{osLip}(\Delta) \leq d < c$

Then

1. $\mathsf{F} + \Delta$ is strongly contracting with rate $c - d$
2. the unique equilibria x^*_{F} of F and $x^*_{\mathsf{F}+\Delta}$ of $\mathsf{F} + \Delta$ satisfy

$$\|x^*_{\mathsf{F}} - x^*_{\mathsf{F}+\Delta}\| \leq \frac{\|\Delta(x^*_{\mathsf{F}})\|}{c - d}$$

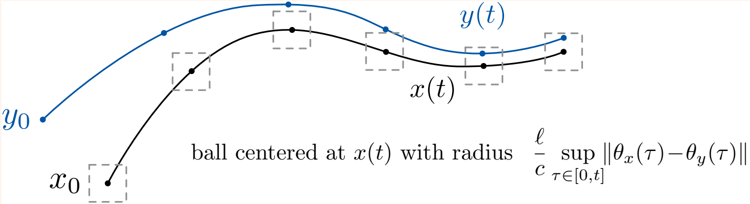
Property #3: Robustness with respect to inputs

$$\dot{x} = \mathsf{F}(x, \theta(t))$$

- **contractivity wrt x**: $\mathrm{osLip}_x(\mathsf{F}) \le -c < 0$, uniformly in θ
- **Lipschitz wrt θ**: $\mathrm{Lip}_\theta(\mathsf{F}) \le \ell$, uniformly in x

Then **incremental ISS property**:

$$\|x(t) - y(t)\| \;\le\; \mathrm{e}^{-ct}\|x_0 - y_0\| \;+\; \frac{\ell}{c}(1 - \mathrm{e}^{-ct}) \sup_\tau \|\theta_x(\tau) - \theta_y(\tau)\|$$



Property #4: Network Contraction Theorem. Consider interconnected subsystems

$$\dot{x}_i = \mathsf{F}_i(x_i, x_{-i}), \qquad \text{for } i \in \{1, \dots, n\}$$

satisfying

- **contractivity wrt x_i**: $\quad \mathrm{osLip}_{x_i}(\mathsf{F}_i) \leq -c_i < 0, \quad$ uniformly in x_{-i}
- **Lipschitz wrt x_j, $j \neq i$**: $\quad \mathrm{Lip}_{x_j}(\mathsf{F}_i) \leq \ell_{ij}, \quad$ uniformly in x_{-j}
- the Lipschitz constants matrix $\Gamma = \begin{bmatrix} -c_1 & \dots & \ell_{1n} \\ \vdots & & \vdots \\ \ell_{n1} & \dots & -c_n \end{bmatrix}$ is **Hurwitz**

Then **interconnected system** is contracting with rate $|\alpha(\Gamma)|$

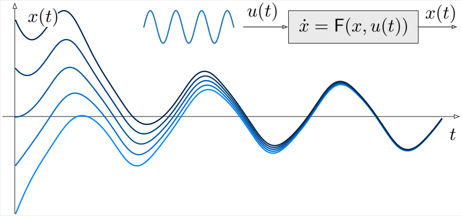
Property #5: Euler Discretization Theorem

Given arbitrary norm $\|\cdot\|$ and differentiable $\mathsf{F} : \mathbb{R}^n \to \mathbb{R}^n$,

Equivalent statements

1. $\dot{x} = \mathsf{F}(x)$ is infinitesimally contracting
2. there exists $\alpha > 0$ such that $x_{k+1} = x_k + \alpha \mathsf{F}(x_k)$ is contracting

Property #6: Entrainment in systems with periodic time-dependence



For time-varying vector field $\mathsf{F}(t, x)$

1. $\mathrm{osLip}_x(\mathsf{F}) \leq -c < 0$, uniformly in t
2. F is T-periodic in t

Then

1. there exists a unique periodic solution $x^* : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ with period T
2. for every initial condition x_0,

$$\|x(t, x_0) - x^*(t)\| \leq \mathrm{e}^{-ct}\|x_0 - x^*(0)\|$$

Outline

Example contracting systems

1. **gradient descent flows** under strong convexity assumptions
   (proximal, primal-dual, distributed, Hamiltonian, saddle, pseudo, best response, etc)
2. **Lur'e systems** under assumptions on nonlinearity and LMI conditions
   (Lipschitz, incrementally passive, monotone, conic, etc)
3. **neural network dynamics** under assumptions on synaptic matrix
   (recurrent, implicit, reservoir computing, etc)
4. **interconnected systems** under contractivity and small-gain assumptions (TAC, review)
   (Hurwitz Metzler matrices, network small-gain theorem, etc)
5. **data-driven learned models (imitation learning)**
6. **incremental ISS systems**
7. **feedback linearizable systems with stabilizing controllers**

Example #1: Gradient descent for strongly convex function

Given differentiable ν-strongly convex $f : \mathbb{R}^n \to \mathbb{R}$, **gradient descent dynamics**

$$\dot{x} = \mathsf{F}_{\mathsf{G}}(x) := -\nabla f(x)$$

F_{G} is infinitesimally contracting wrt $\|\cdot\|_2$ with rate ν

Property #7: Kachurovskii's Theorem: For differentiable $f : \mathbb{R}^n \to \mathbb{R}$, equivalent statements:

1. f is **strongly convex** with parameter ν (and minimum x^*)
2. $-\nabla f$ is **ν-strongly infinitesimally contracting** (with equilibrium x^*)

R. I. Kachurovskii. Monotone operators and convex functionals. *Uspekhi Matematicheskikh Nauk*, 15(4):213–215, 1960

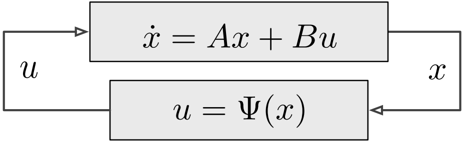
Example #1 (cont'd): Optimization-based contracting dynamics

Many convex optimization problems can be solved with contracting dynamics

$$\dot{x} = \mathsf{F}(x, \theta)$$

	Convex Optimization	Contracting Dynamics
Unconstrained	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$	$\dot{x} = -\nabla_x f(x, \theta)$
Constrained	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ s.t. $\quad x \in \mathcal{X}(\theta)$	$\dot{x} = -x + \mathrm{Proj}_{\mathcal{X}(\theta)}(x - \gamma \nabla_x f(x, \theta))$
Composite	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta) + g(x, \theta)$	$\dot{x} = -x + \mathrm{prox}_{\gamma g_\theta}(x - \gamma \nabla_x f(x, \theta))$
Equality	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ s.t. $\quad Ax = b(\theta)$	$\dot{x} = -\nabla_x f(x, \theta) - A^\top \lambda,$ $\dot{\lambda} = Ax - b(\theta)$
Inequality	$\min_{x \in \mathbb{R}^n} \quad f(x, \theta)$ s.t. $\quad Ax \leq b(\theta)$	$\dot{x} = -\nabla f(x, \theta) - \gamma^{-1} A^\top \mathrm{relu}\big(Ax + \gamma\lambda - b(\theta)\big),$ $\dot{\lambda} = -\gamma\lambda + \mathrm{relu}\big(Ax + \gamma\lambda - b(\theta)\big)$

Example #2: Systems in Lur'e form



For $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times n}$, **nonlinear system in Lur'e form**

$$\dot{x} = Ax + B\Psi(x) \quad =: \mathsf{F}_{\mathsf{Lur'e}}(x)$$

where $\Psi : \mathbb{R}^m \to \mathbb{R}^m$ is described by an **incremental multiplier matrix** M

For $P = P^\top \succ 0$, following statements are equivalent:

1. $\mathsf{F}_{\mathsf{Lur'e}}$ infinitesimally contracting wrt $\|\cdot\|_{2,P}$ with rate $\eta > 0$ for each Ψ described by M,
2. $\exists\lambda \geq 0$ such that $\begin{bmatrix} PA + A^\top P + 2\eta P & PB \\ B^\top P & \mathbb{0}_{m\times m} \end{bmatrix} + \lambda M \preceq 0$

Example #2: Regularized MPC (linear systems, convex input costs)

Given $x(k+1) = Ax(k) + Bu(k)$, the MPC optimization problem is:

$$\min_{u(k),\dots,u(k+H-1)} \Big(\sum_{h=0}^{H-1} \|x(k+h)\|_Q^2 + \|u(k+h)\|_R^2 + \underbrace{\mathbf{V}(u_{k+h})}_{\text{regularization}} \Big) + \underbrace{\|x(k+H)\|^2_{Q_{\text{terminal}}}}_{\text{control Lyapunov function}}$$

If input cost $\mathbf{V}$ is twice differentiable and $0 \preceq \mathrm{Hess}(\mathbf{V}(u)) \preceq \Theta$ for all u, then

1. contractivity with factor $\eta < 1$ if there exist $P \succ 0$ and diagonal $\Lambda \succeq 0$ satisfying:

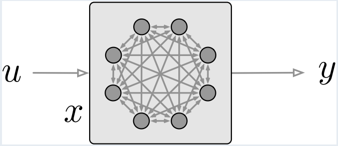
$$\begin{bmatrix} A^\top PA - \eta^2 P & A^\top PB\Pi_1 \\ \Pi_1^\top B^\top PA & \Pi_1^\top B^\top PB\Pi_1 \end{bmatrix} + \begin{bmatrix} 0 & I \\ -C & -D \end{bmatrix}^\top \begin{bmatrix} 0 & \Lambda \otimes I \\ \Lambda \otimes I & -2\Lambda \otimes \Theta^{-1} \end{bmatrix} \begin{bmatrix} 0 & I \\ -C & -D \end{bmatrix} \preceq 0$$

   for projection matrix Π_1 and appropriate C and D

2. there exists $0 \preceq \Theta \preceq \theta_{\text{max}} I$ such that LMI is solvable

R. Shima, A. Gokhale, A. Davydov, and F. Bullo. Regularized model predictive control: Contractivity and applications, Sept. 2025. Working Document

Example #3: Firing-rate networks for implicit ML via ℓ_∞



$$\dot{x} = -x + \Phi(Ax + Bu + b) \qquad \text{(recurrent NN)}$$

$$x = \Phi(Ax + Bu + b) \qquad \text{(implicit NN)}$$

$$x_{k+1} = (1-\alpha)x_k + \alpha\Phi(Ax_k + Bu + b) \qquad \text{(Euler discrt.)}$$

If

$$\mu_\infty(A) < 1 \qquad \left(\text{i.e., } a_{ii} + \sum_{j\neq i} |a_{ij}| < 1 \text{ for all } i\right)$$

- **recurrent NN is infinitesimally contracting** with rate $1 - \mu_\infty(A)_+$
- **implicit NN is well posed**
- **Euler discretization is contracting** at $\alpha^* = (1 - \min_i(a_{ii})_-)^{-1}$

Outline

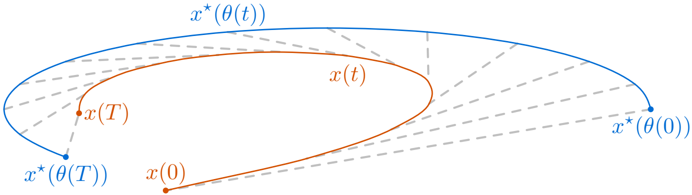
Parametric and time-varying convex optimization

1. **parametric contracting dynamics for parametric convex optimization**

$$\min \mathcal{E}(x,\theta) \iff \dot{x} = \mathsf{F}(x,\theta) \rightsquigarrow x^\star(\theta)$$

2. **contracting dynamics for time-varying strongly-convex optimization**

$$\min \mathcal{E}\big(x,\theta(t)\big) \iff \dot{x} = \mathsf{F}\big(x,\theta(t)\big) \rightsquigarrow x^\star\big(\theta(t)\big)$$



$x(t)$ = our controlled trajectory

$x^\star(\theta(t))$ = target trajectory

For parameter-dependent vector field $\mathsf{F} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^n$ and differentiable $\theta : \mathbb{R}_{\geq 0} \to \Theta \subset \mathbb{R}^d$

$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t))$$

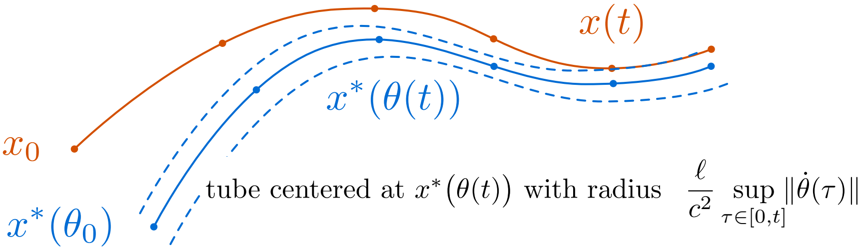
- **contractivity wrt x**: $\mathsf{osLip}_x(\mathsf{F}) \leq -c < 0$
- **Lipschitz wrt θ**: $\mathsf{Lip}_\theta(\mathsf{F}) \leq \ell$

Equilibrium tracking

error : $$\|x(t) - x^\star(\theta(t))\| \leq e^{-ct}\|x_0 - x^\star(\theta_0)\| + \frac{\ell}{c^2} \sup_{\tau\in[0,t]} \|\dot{\theta}(\tau)\|$$

speed : $$\|\mathsf{F}(x(t), \theta(t))\| \leq e^{-ct}\|\mathsf{F}(x_0, \theta_0)\| + \frac{\ell}{c} \sup_{\tau\in[0,t]} \|\dot{\theta}(\tau)\|$$

Equilibrium tracking and tube invariance



$$\dot{x}(t) = \mathsf{F}(x(t), \theta(t))$$
$$x^\star(\theta(t)) = \text{equilibrium trajectory}$$

Time-varying contracting dynamics with feedforward prediction

$$\dot{x}(t) = \mathsf{F}(x(t),\theta(t)) - \underbrace{\big(D_x\mathsf{F}(x(t),\theta(t))\big)^{-1} D_\theta\mathsf{F}(x(t),\theta(t))}_{\text{differentiable } \mathsf{F}}\,\dot{\theta}(t)$$

Asymptotically-exact equilibrium tracking

error : $\|x(t) - x^\star(\theta(t))\| \leq \frac{1}{c}\mathrm{e}^{-ct}\|\mathsf{F}(x_0,\theta_0)\| \overset{\ell_x = \mathsf{Lip}_x(\mathsf{F})}{\leq} \frac{\ell_x}{c}\mathrm{e}^{-ct}\|x_0 - x^\star(\theta_0)\|$

speed : $\big\|\mathsf{F}\big(x(t),\theta(t)\big)\big\| \leq \mathrm{e}^{-ct}\big\|\mathsf{F}\big(x_0,\theta_0\big)\big\|$

Discretized equilibrium tracking for parametrized dynamics $\dot{x} = \mathsf{F}(x, \theta(t))$

contraction rate c, Lipschitz constants ℓ_x and ℓ_θ

pick step size α so that $\gamma = \mathsf{Lip}(I_n + \alpha F) < 1$

define *discrete time* $t_k = \alpha k$

$$\text{forward Euler :} \quad x_{k+1} \leq x_k + \alpha \mathsf{F}\big(x_k, \theta(t_k)\big)$$

$$\text{error :} \quad \|x_k - x^\star(\theta(t_k))\| \leq \gamma^k \|x_0 - x^\star(\theta_0)\| + \frac{\ell_\theta}{c(1-\gamma)} \sup_{k>0} \frac{\|\theta(t_{k+1}) - \theta(t_k)\|}{t_{k+1} - t_k}$$

Z. Marvi, F. Bullo, and A. G. Alleyne. Discrete control barrier proximal dynamics: Quantized multi-actuator and sampled-data systems. *Automatica*, 2025a. Submitted

Outline

Application to safety filters and control barrier functions

Given $\dot{x} = F(x) + G(x)u$ with nominal controller $u_{\text{nom}}(x)$

Safe control design: render forward invariant safe set $\{x \in \mathbb{R}^n \mid h_i(x) \geq 0, \;\; i \in \{1, \dots, k\}\}$

Safety filter (parametric QP with linear inequalities)

$$\begin{aligned} u^\star(x) = \operatorname{argmin} \quad & \|u - u_{\text{nom}}(x)\|_2^2 \\ \text{s.t.} \quad & \dot{h}_i(x, u) \geq -\alpha(h_i(x)), \quad i \in \{1, \dots, k\} \qquad \text{(safety constraints)} \\ & \|u\|_\infty \leq \overline{u} \qquad \text{(actuator constraints)} \end{aligned}$$

High-performance constrained control methods rely on **online optimization**. However,

- even for a QP, complexity grows **cubically** with decision variables,
- per-step optimization creates a **scalability** bottleneck.

Design approach: design **contracting solver** with approximation error $\|u(t) - u_{\text{nom}}\|$

Approach #1: Projected gradient for relaxed constraints

1. relax safety constraints into logarithmic barriers in cost function

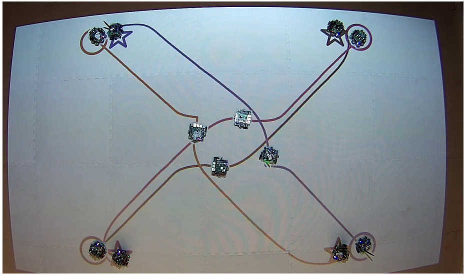
$$\mathcal{E}_\eta(u, x) = \|u - u_{\mathsf{nom}}(x)\|_2^2 - \eta \sum_{i=1}^{k} \log\Big(\nabla h_i(x)^\top \big(F(x) + G(x)u\big) + \alpha\big(h_i(x)\big)\Big)$$

2. adopt projected gradient dynamics

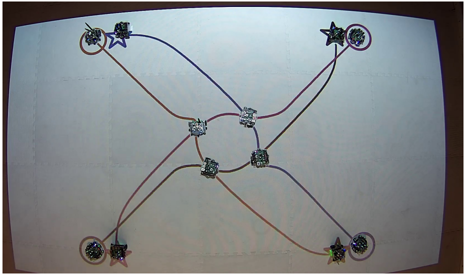
$$\dot{u} = -u + \mathrm{Proj}_{\|u\|_\infty \leq \overline{u}}\Big(u - \nabla_u \mathcal{E}_\eta\big(u, x\big)\Big) + \mathrm{FeedForward}_\eta(u(t), x(t))$$

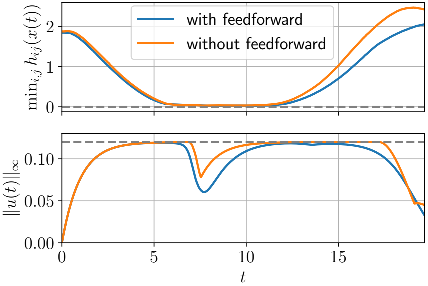
3. discretize using forward Euler

No feedforward



With feedforward





Videos:

- experiment without feedforward
- experiment with feedforward

Code: github link

contracting systems as controllers =
promising approach to optimization-based control

Approach #2: Control barrier proximal dynamics

Assume

1. $\dot{x} = F(x) + Bu$ and
2. $\mathcal{C}$ is convex described by affine $h(x) = Hx - h_0$

Safety filter (parametric QP with linear inequalities)

$$u^\star(x) = \operatorname{argmin} \quad \|u - u_{\mathsf{nom}}(x)\|_2^2 \qquad \text{(performance)}$$
$$\text{s.t.} \quad Au \leq b(x) \qquad \text{(safety \& actuator constraints)}$$

Control Barrier Proximal Dynamics (CBPD)

$$\dot{u} = -u + u_b(x) - \frac{1}{\gamma} A^\top \operatorname{relu}(Au - b(x) + \gamma\lambda)$$
$$\dot{\lambda} = -\gamma\lambda + \operatorname{relu}(Au - b(x) + \gamma\lambda)$$

Time-scaling for reduced tracking error

Given contracting controller $\tau \dot{u} = F(u, x)$, contraction rate c and Lipschitz constant ℓ_x

$$\mathrm{osLip}_u\left(\frac{F}{\tau}\right) = \frac{-c}{\tau}, \quad \mathrm{Lip}_x\left(\frac{F}{\tau}\right) = \frac{\ell_x}{\tau}$$

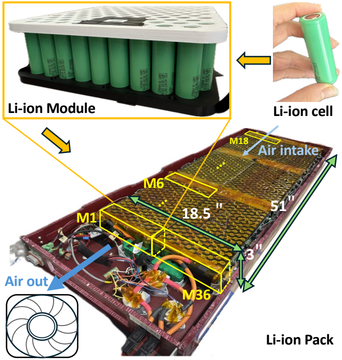
With initially safe controller (i.e., zero transient error)

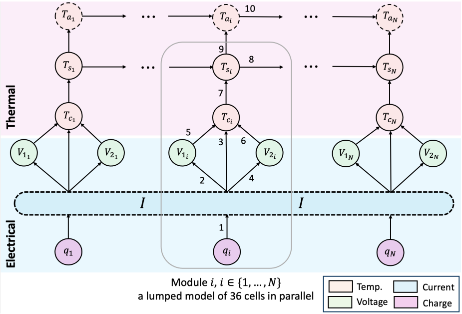
$$\|u(t) - u^\star(x(t))\| \leq \tau \frac{\ell_x}{c^2} \|\dot{x}(t)\| = \tau \bar{\delta}(t)$$

CBPD-based safety guarantee CBPD Controller renders the safe set invariant with an **arbitrarily small** violation safety margin bounded by

$$m = \frac{\tau}{\alpha} K \bar{\delta}$$

Tracking error translates to safety margin.
Design variables 1) time scale τ (i.e, solver rate) 2) CBF parameter α



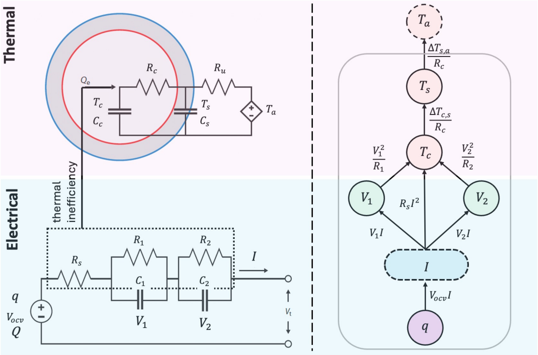


System description: The UMN solar EV battery pack has 36 series modules (M1–M36), each with 36 parallel Li-ion cells (1296 total).

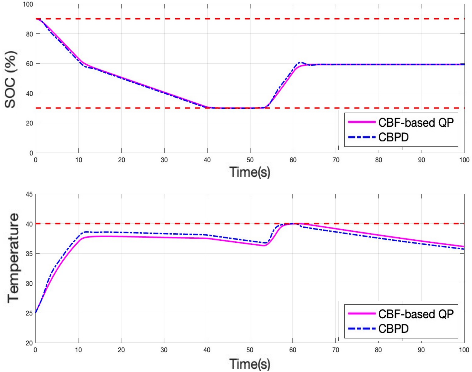
Model description: Validated conservation-based graph model: each module is a lumped model, with thermal connections capturing spatial variations across the pack.

Simulation results for single battery: CBPD vs QP

- Lithium-ion cell control with state of charge (SOC) and temperature constraints.
- Conservation-based electro-thermal model, vertices: system states and edges: power flow.



(left) cell equivalent circuit, (right) conservation-based graph model



Evolution of SOC and temp. by applying: (a) CBF, and (b) CBPD.

- Similar dynamics and safety for both controllers.
- CBPD is 1–2 orders of magnitude faster.

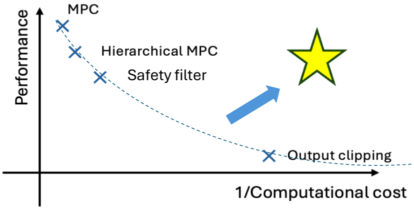
Max deviation	Runtime ratio
3%	3700%

Table: CBF vs CBPD comparison

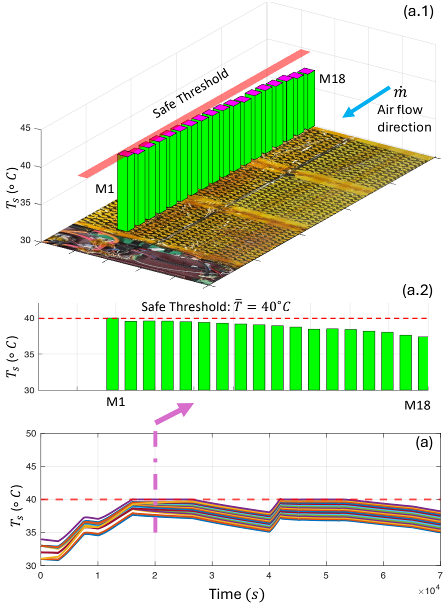
Objective: Keeping all module temperatures below a safe threshold with minimal control effort.

$$\min I_f^2 \ \text{s.t.} \ \max_{i \in \{1,\dots,N\}} \{T_{s,i}\} \leq \bar{T}$$

Comp. cost comparable to a switched PI controller (Per-call runtime: PI = 3.095 ms, CBPD = 3.171 ms).



CBPD's scalability enables spatially accurate thermal management in battery packs.



Safe temperatures across all modules.

Outline

Summary

§1. Chapter #1: A tutorial review

- Definitions
- Theorems
- Examples

§2. Chapter #2: Equilibrium tracking for optimization-based control

- Equilibrium tracking
- Application to safety filters

§3. Chapter #3: Contracting stochastic dynamics

- Incremental input and noise to state stability

§4. Future work

Ongoing and future work

theory:

1. equilibrium tracking
   discrete-time, stochastic, distributed, internal model principle
2. local, weak, k-, periodic, and other generalizations of contractivity
3. local contractivity, invariant sets, and region of attraction

examples & applications:

1. optimization-based control: contractivity of *MPC*
   Lure-based approaches to global and local contractivity
2. catalog of contracting dynamics with sharp Lipschitz estimates
   discrete-time and discretized solvers: *Newton-Raphson*, *interior point*, etc
3. *hybrid integral control action* for nonlinear overshoot regulation

Examples of contracting dynamics:

- V. Centorrino, A. Gokhale, A. Davydov, G. Russo, and F. Bullo. Euclidean contractivity of neural networks with symmetric weights. *IEEE Control Systems Letters*, 7:1724–1729, 2023. doi

Applications to machine learning:

- S. Jafarpour, A. Davydov, A. V. Proskurnikov, and F. Bullo. Robust implicit networks via non-Euclidean contractions. In *Advances in Neural Information Processing Systems*, Dec. 2021. doi
- S. Jaffe, A. Davydov, D. Lapsekili, A. K. Singh, and F. Bullo. Learning neural contracting dynamics: Extended linearization and global guarantees. In *Advances in Neural Information Processing Systems*, 2024. doi

Application to neuroscience:

- V. Centorrino, A. Gokhale, A. Davydov, G. Russo, and F. Bullo. Positive competitive networks for sparse reconstruction. *Neural Computation*, 36(6):1163–1197, 2024. doi

Applications to optimization-based control:

- Z. Marvi, F. Bullo, and A. G. Alleyne. Control barrier proximal dynamics: A contraction theoretic approach for safety verification. *IEEE Control Systems Letters*, 8:880–885, 2024. doi
- Y. Chen, F. Bullo, and E. Dall'Anese. Sampled-data systems: Stability, contractivity and single-iteration suboptimal MPC. *IEEE Transactions on Automatic Control*, 2025. doi. Submitted
- Z. Marvi, F. Bullo, and A. G. Alleyne. Air cooled battery pack thermal management via control barrier proximal dynamics. *IEEE Transactions on Control Systems Technology*, June 2025b. Submitted
- A. Davydov, V. Centorrino, A. Gokhale, G. Russo, and F. Bullo. Time-varying convex optimization: A contraction and equilibrium tracking approach. *IEEE Transactions on Automatic Control*, 70(11): 7446–7460, 2025. doi