Contractivity of Interconnected Continuous-time and Discrete-time Systems

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2025 IEEE CDC Tutorial Session on "Contraction Theory in Control, Optimization, and Learning"

Acknowledgments

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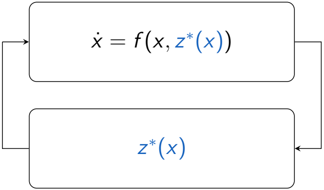
Acknowledgments: AFOSR Awards FA9550-23-1-0740 and FA9550-22-1-0059, NSF Award 2444163

Motivation: optimization-based control



$$\dot{x} = f(x, z^*(x))$$

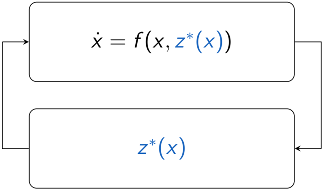
$$z^*(x)$$

$z^*(x)$: sol. of an optimization problem

$$z^*(x) := \arg\min_{u \in \mathcal{U}} C(u, x)$$

$$\text{s. to: } h(u, x) \leq 0$$

Motivation: optimization-based control



$$\dot{x} = f(x, z^*(x))$$

$$z^*(x)$$

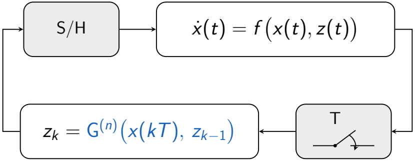
$z^*(x)$: sol. of an optimization problem

But available:

- approximately (computation)
- in a sampled-data fashion (sensing)

Working setup

$$\begin{aligned} \dot{x}(t) &= f(x(t), z(t)) \\ z_k &= \mathsf{G}^{(n)}(x(kT), z_{k-1}) \\ z(t) &= z_k,\ \ t \in [kT, (k+1)T) \end{aligned}$$



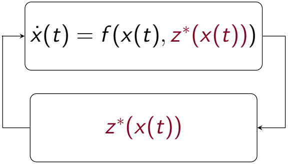
- (Def.) $\mathsf{G}^{(1)}(x,z) = \mathsf{G}(x,z), \ldots\ \mathsf{G}^{(n)}(x,z) = \mathsf{G}(x, \mathsf{G}^{(n-1)}(x,z)),\ n \in \mathbb{Z}_{\geq 0}$

- $\mathrm{Lip}_z(\mathsf{G}) < 1 \Rightarrow$ existence and uniqueness of $z^*(x)$

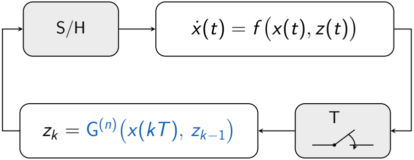
- $\lim_{n\to\infty} \mathsf{G}^{(n)}(x,z) = z^*(x)$ and $z^*(x) = \mathsf{G}(x, z^*(x))$ for any $x \in \mathcal{X}$

Wlog, $f(0,0) = 0$ and $\mathsf{G}(0,0) = 0$.

Connection between the two systems



“Reduced system”



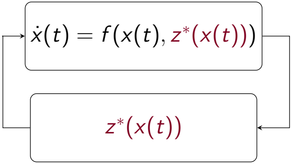
CT-DT interconnection

- $\dot{x}(t) = f(x(t), z^*(x(t)))$ is obtained from the CT-DT interconnection by:

$$n \to \infty \quad \text{and} \quad T \to 0^+$$

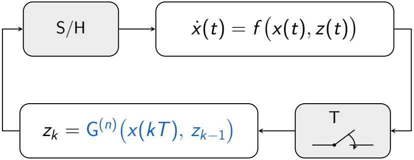
- Inspired by two-time-scale CT systems [Kokotović-Sannuti’68, . . .]

Research goals



$\dot{x}(t) = f(x(t), z^*(x(t)))$

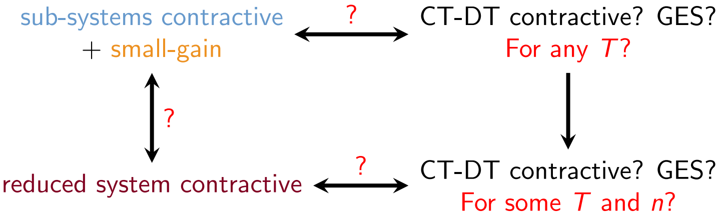
$z^*(x(t))$



$\dot{x}(t) = f\big(x(t), z(t)\big)$

$z_k = \mathsf{G}^{(n)}\big(x(kT),\, z_{k-1}\big)$

How are their contractivity and stability properties related to each other?



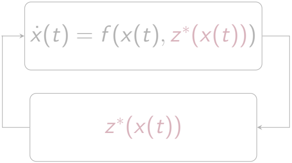
sub-systems contractive
+ small-gain

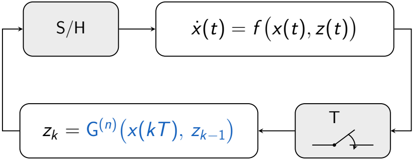
CT-DT contractive? GES?
For any T?

reduced system contractive

CT-DT contractive? GES?
For some T and n?

Small-gain condition





Theorem (Contractivity plus small gain implies DT contractivity) Assume:

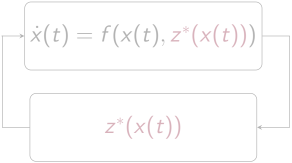
- CT contractivity: $\mathrm{osLip}_x(f) < 0$
- DT contractivity: $\mathrm{Lip}_z(\mathrm{G}) \in (0, 1)$
- Interconnection: $-\mathrm{osLip}_x(f)(1 - \mathrm{Lip}_z(\mathrm{G})) > \mathrm{Lip}_z(f)\mathrm{Lip}_x(\mathrm{G})$

Then, the following holds independent of n and T:

$$\begin{bmatrix} \|x_1(kT) - x_2(kT)\| \\ \|z_1(kT) - z_2(kT)\| \end{bmatrix} \leq \mathcal{A}^k \begin{bmatrix} \|x_1(0) - x_2(0)\| \\ \|z_1(0) - z_2(0)\| \end{bmatrix}, \forall\, k \in \mathbb{Z}_{\geq 0}$$

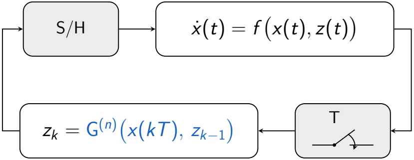
where $\mathcal{A}$ is Shur (and positive).

Contractivity and small-gain condition



$\dot{x}(t) = f(x(t), z^*(x(t)))$

$z^*(x(t))$



S/H

$\dot{x}(t) = f\big(x(t), z(t)\big)$

T

$z_k = \mathrm{G}^{(n)}\big(x(kT),\, z_{k-1}\big)$

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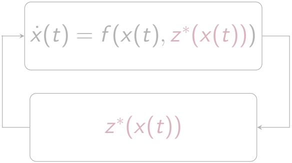
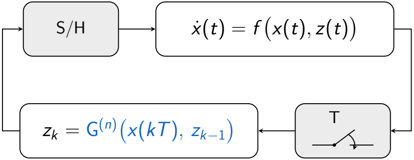
Contractivity and small-gain condition





Theorem (Contractivity plus small gain implies GES) Assume:

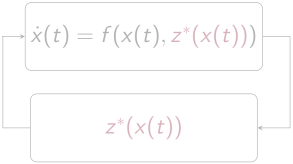
- CT contractivity: $\mathrm{osLip}_x(f) < 0$
- DT contractivity: $\mathrm{Lip}_z(\mathrm{G}) \in (0,1)$
- Interconnection: $-\mathrm{osLip}_x(f)(1 - \mathrm{Lip}_z(\mathrm{G})) > \mathrm{Lip}_z(f)\mathrm{Lip}_x(\mathrm{G})$

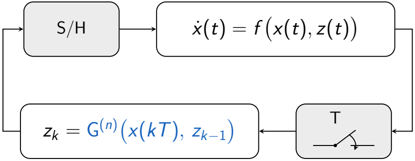
Then, the following holds independent of n and T:

$$\left\| \begin{bmatrix} \|x(t)\| \\ \|z(t)\| \end{bmatrix} \right\|_{2,[\eta]} \leq \kappa e^{ct} \left\| \begin{bmatrix} \|x(0)\| \\ \|z(0)\| \end{bmatrix} \right\|_{2,[\eta]}, \quad \forall\, t \geq 0$$

where $c := \ln(\frac{1+\rho(\mathcal{A})}{2})^{\frac{1}{T}} < 0$ and for some $\kappa > 0$.

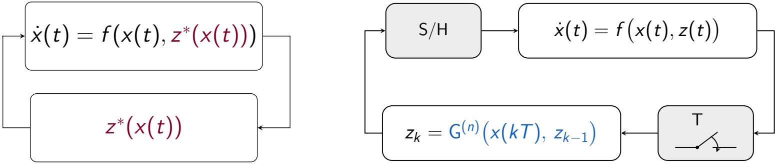
Implication diagram





$\text{osLip}_x(f) < 0$, $\text{Lip}_z(\mathsf{G}) < 1$ + small-gain $\longrightarrow$ CT-DT is T-DTC and GES for any n and T

Contractivity and small-gain condition



$\dot{x}(t) = f(x(t), z^*(x(t)))$

$z^*(x(t))$

S/H

$\dot{x}(t) = f\big(x(t), z(t)\big)$

$z_k = \mathrm{G}^{(n)}\big(x(kT),\ z_{k-1}\big)$

T

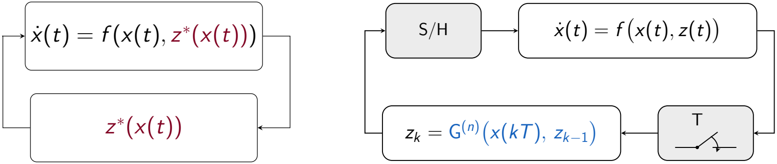
Theorem (Contractivity plus small gain implies RS contractivity) Assume:

- CT contractivity: $\mathrm{osLip}_x(f) < 0$
- DT contractivity: $\mathrm{Lip}_z(\mathrm{G}) \in (0, 1)$
- Interconnection: $-\mathrm{osLip}_x(f)(1 - \mathrm{Lip}_z(\mathrm{G})) > \mathrm{Lip}_z(f)\mathrm{Lip}_x(\mathrm{G})$

Then,

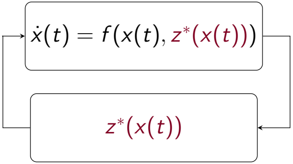
$$\mathrm{osLip}_x(f(x, z^*(x))) < 0.$$

Implication diagram

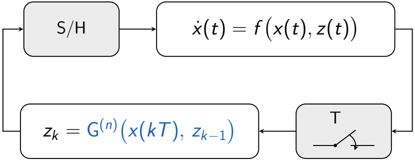


$$\begin{tikzcd} \begin{array}{c}\mathrm{osLip}_x(f) < 0,\ \mathrm{Lip}_z(\mathrm{G}) < 1\\ +\ \text{small-gain}\end{array} \arrow[r] \arrow[d] & \begin{array}{c}\text{CT-DT is } T\text{-DTC and GES}\\ \text{for any } n \text{ and } T\end{array} \\ \begin{array}{c}\mathrm{osLip}_x(f(x, z^*(x))) < 0\\ \mathrm{Lip}_z(\mathrm{G}) < 1\end{array} & \end{tikzcd}$$

Stability from contractivity of the reduced system



Reduced system
$[n = \infty, T \to 0^+]$



CT-DT interconnection
$[n < \infty, T > 0]$

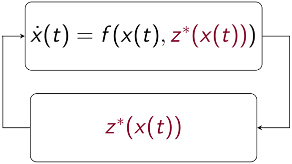
Theorem (RS contractivity implies stability of CT-DT interconnection) Suppose:

- DT sub-system contractivity: $\mathrm{Lip}_z(\mathrm{G}) \in (0, 1)$
- Reduced system contractivity: $\mathrm{osLip}_x(A + Bz^*) < 0$

Then, $\forall\ n \in \mathbb{Z}_{>0}\ \exists\ T(n) > 0$ s.t. for any $T < T(n)$ the CT-DT intercon. is GES.

[with explicit transient estimates] [local version of the result available too]

Stability from contractivity of the reduced system



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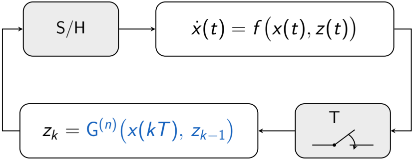


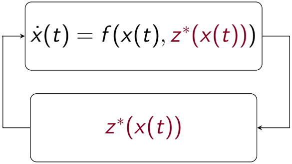
CT-DT interconnection
$[n < \infty, T > 0]$

- *Simply*: If one cannot implement $z^*(x)$, pick n, then sample "fast enough"
- T_G sec to perform $\mathrm{G}(x_k, z_{k-1})$; then

$$(nT_G + \texttt{sensing} + \texttt{actuation}) < T(n)$$

- Classical online optimization setup: $T_G + \texttt{sensing} + \texttt{actuation} < T(1)$

Implication diagram



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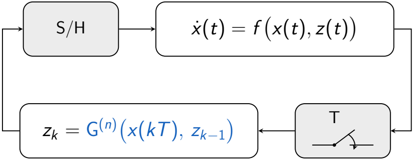

S/H

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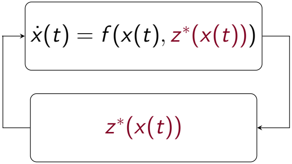
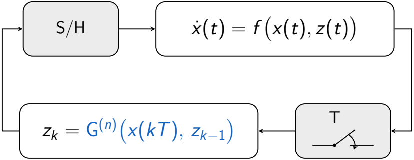
Implication diagram





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\end{tikzcd}$$

Details

Some (ugly) details:

$$T(n) = \frac{1}{\xi} \log \left(\frac{\xi(1 - [\mathrm{Lip}_z(\mathrm{G})]^n)}{C_2(n) + C_1/\zeta} + 1 \right)$$

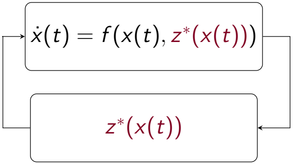
with $\xi := -\mathrm{osLip}_x(f(x, z^*(x))$ and

$$C_1 := \frac{\mathrm{Lip}_z(f)\mathrm{Lip}_x(\mathrm{G})}{1 - \mathrm{Lip}_z(\mathrm{G})} \left(\mathrm{Lip}_x(f) + \frac{\mathrm{Lip}_z(f)\mathrm{Lip}_x(\mathrm{G})}{1 - \mathrm{Lip}_z(\mathrm{G})} \right),$$
$$C_2(n) := [\mathrm{Lip}_z(\mathrm{G})]^n \frac{\mathrm{Lip}_x(\mathrm{G})\mathrm{Lip}_z(f)}{1 - \mathrm{Lip}_z(\mathrm{G})}.$$

but (useful) observations:

- $T(n)$ is strictly increasing w.r.t n and bounded above
- For fixed $\mathrm{Lip}_z(f)$, $\mathrm{Lip}_z(\mathrm{G})$, $\mathrm{Lip}_x(\mathrm{G})$, $T(n)$ increases with the increasing of ξ
- $T(\infty)$ corresponds to a classical sample-data setup

Main applications



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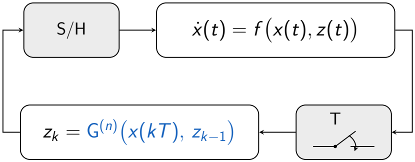

S/H

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T

Model predictive control (MPC) → Time distributed or real-time

[Diehl et al'05], [Liao-McPherson et al'20]

Feedback optimization → Online feedback optimization

[Colombino et al'20], [Hauswirth et al'20]

CBF and CLF controllers → "Early termination" of QP solvers

[Ames et al'14], [Allibhoy-Cortes'23]

MPC

- MPC with horizon of N intervals of length Δ
- Discretize system as $x_{k+1} = \Phi x_k + \Psi u_k, \quad \Phi := e^{A\Delta}, \; \Psi := \int_0^\Delta e^{A(\Delta-\tau)} B d\tau$
- Example of MPC:

$$z^*(x) := \arg \min_{u \in \mathbb{R}^{m(N-1)}, \; x \in \mathbb{R}^{nN}} \sum_{i=1}^{N-1} \left(\|x_i\|_Q^2 + \|u_i\|_R^2 + \|x_N\|_P^2\right) + b_i(x_i)$$

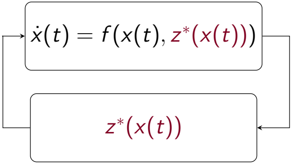
$$\text{s. to :} \qquad x_{i+1} = \Phi x_i + \Psi u_i, \qquad i = 1, \ldots, N-1$$

$$x_1 = x$$

where

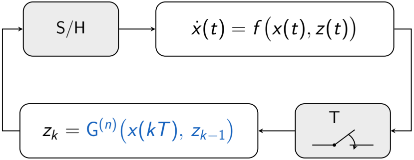
$$b_i(x_i) = \gamma \sum_{i=1}^{N} \left\| \max\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 10 \\ 3 \end{bmatrix} - x_i \right\} \right\|^2 + \gamma \sum_{i=1}^{N} \left\| \max\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, x_i - \begin{bmatrix} -10 \\ -3 \end{bmatrix} \right\} \right\|^2$$

Batch and online MPC



Batch, continuous MPC

$$z^*(x(t)) := \arg \min_{u \in \mathbb{R}^{m(N-1)}} \tilde{f}(z; (x(t)))$$



Example of online MPC

$$\mathsf{G}(x, z) = z - \alpha \nabla_z \tilde{f}(z; x)$$
$$z_k = \mathsf{G}^{(n)}(x(kT), z_{k-1})$$

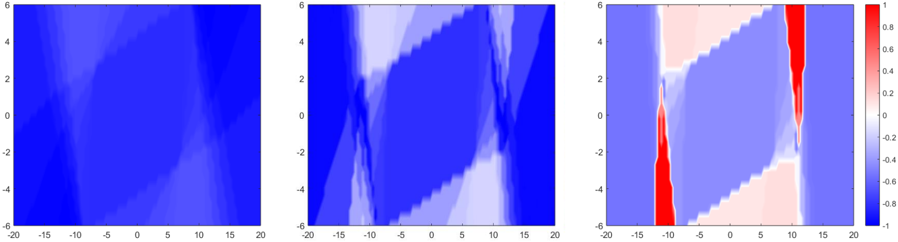
Other examples:

$\mathsf{G}(x, u)$: Approx. Newton iteration [Diehl et al'05]

$\mathsf{G}(x, u, \lambda)$: Primal-dual iteration [Liao-McPherson et al'20]

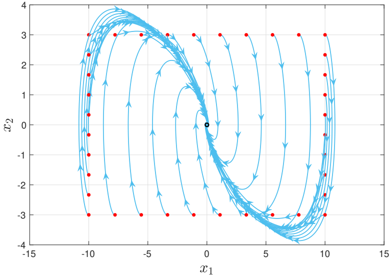
Contractivity of the continuous MPC

Contour plot of $\mu_{2,P}(A + B\Pi_1 J_{z^*}(x))$

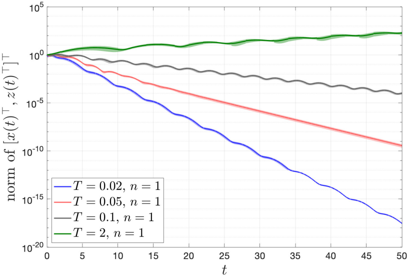


$\gamma = 1$ (left), $\gamma = 10$ (center), and $\gamma = 100$ (right).

Stability of the online MPC



Phase portrait of of the online MPC when $n = 1$ and $T = 0.02$



Statistics for 100 randomly picked, i.i.d. initial conditions $x(0), z(0)$.

Concluding remarks

- Interconnection of CT and DT systems

$$\begin{tikzcd}
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\end{tikzcd}$$

- Application to optimization-based control of physical systems
  - DT control due to sampling and computation
  - Application to MPC: fix n and sample fast
  - Stability of single-step MPC

Thank you!

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